Final exam: Automatic Control II (Reglerteknik II, 1TT495)

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Instructions

The solutions to the problems can be given in Swedish or in English, except for Problem 2 that must be solved in English.

Problem 1 is an alternative to the homework assignment. (In case you choose to hand in a solution to Problem 1 you will be accounted for the best performance of the homework assignments and Problem 1.)

Solve each problem on a separate page.

Write your code on every page.

Provide motivations for your solutions. Vague or lacking motivations may lead to a reduced number of points.

Aiding material: Textbooks in automatic control (such as ‘Reglerteori – flervariabler och olinjära metoder’, ‘Reglerteknik – Grundläggande teori’, and others), mathematical handbooks, collection of formulas (formelsamlingar), textbooks in mathematics, calculators, copies of OH transparencies. Note that the following is not allowed: Exempelsamling med lösningar.

Good luck!
Problem 1

Consider the system
\[
\begin{align*}
x(t + 1) &= \begin{pmatrix} 0.5 & 0 \\ 0 & 2 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} u(t) \\
y(t) &= \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} x(t)
\end{align*}
\]

(a) Is the system controllable? 2 points
(b) Is the system observable? 2 points
(c) Is the system controllable from the input \(u_1\)? 2 points

Problem 2

Consider the system \(G_e(s)\) controlled with the regulator \(H_e(s)\) according to the block diagram.

\[
\Sigma \rightarrow H_e \rightarrow G_e \rightarrow -1
\]

In this case, the system is a simple DC motor. Set
\[
G_e(s) = \frac{1}{s(s + 1)}, \quad H_e(s) = K
\]

(a) For which values of \(K\) is the closed-loop system stable? 2 points

(b) Assume the system is computer-controlled, so that the closed loop system actually can be represented as in the block diagram below. Let the sampling interval be equal to \(h\).
Determine the discrete-time transfer function $G_d(z)$ that corresponds to sampling $G_c(s)$. Set $\alpha = e^{-h}$.  

**Hint.** It may be an advantage to first split $G_c(s)$ as a sum of two first order systems: $G_c(s) = 1/s - 1/(s+1)$.

(c) Assume that the discrete-time regulator is a proportional regulator, so that $H_d(q) = K$. Show that the closed-loop system can be unstable for some values of $K$.  

**Problem 3**

Consider a random process in discrete-time $y(t)$ with covariance function 
\[ r(\tau) = 5a^{|\tau|}, \quad (0 < a < 1) \]
(a) Assume that we want to make prediction $k$ steps ahead of the process. A very simple estimator of the $y(t+k)$ from data available at time $t$ would be to take 
\[ \hat{y}(t+k) = y(t) \]
that is to take the last available measurement. How large will the prediction error variance 
\[ V = E[y(t+k) - \hat{y}(t+k)]^2 \]
become?  

(b) A somewhat more sophisticated predictor would be to multiply the last available measurement with a constant, say 
\[ \hat{y}(t+k) = \alpha y(t) \]
Determine how the prediction error variance depends on the parameter $\alpha$, and how it can be minimized with respect to $\alpha$.  

(c) Determine the spectrum of the process.  

(d) Use the spectrum to derive a time-domain model of the process on state-space form 
\[ x(t+1) = Ax(t) + Bv(t) \]
\[ y(t) = Cx(t) + e(t) \]
and specify also the variances of $v(t)$ and $e(t)$.  

**Problem 4**

Consider a first order system given on polynomial form 
\[ (1 + aq^{-1})y(t) = bq^{-1}u(t) + (1 + aq^{-1})e(t) \]
where $e(t)$ is white noise, with zero mean and variance $\lambda^2$. Assume that $|e| < 1$ holds.
(a) Show that the system can be written in standard state-space form as follows

\[
\begin{align*}
x(t + 1) &= \begin{pmatrix} -a & 1 \\ 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} b \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} 1 \\ c \end{pmatrix} v(t) \\
y(t) &= \begin{pmatrix} 1 & 0 \end{pmatrix} x(t)
\end{align*}
\]

How are \( v \) and \( e \) related? \hspace{1cm} 2 points

(b) Assume both states can be measured. Determine the optimal feedback

\[ u(t) = -Lx(t) \]

that minimizes \( E \sum y^2(t) \). \hspace{1cm} 4 points

(c) Determine the Kalman filter (the predictor form) giving \( \hat{x}(t|t - 1) \). \hspace{1cm} 5 points

Problem 5

Consider a first order process \( x(t) \) observed with some noise \( e(t) \). This may be modelled as

\[ \begin{align*}
\dot{x} &= -ax + v \\
y &= x + e
\end{align*} \]

where \( v \) and \( e \) are white noise processes, with intensities \( r_1 \) and \( r_2 \), respectively. Also, assume that \( a > 0 \).

(a) Determine the spectrum and the variance of the signal \( x(t) \). \hspace{1cm} 2 points

(b) As the signal \( x(t) \) is not measured directly, consider estimating it with an observer

\[ \hat{x} = -a\hat{x} + K(y - \hat{x}) \]

Show that this can be written as

\[ \hat{x}(t) = G(p)y(t) \]

and determine the transfer function operator \( G(p) \), and sketch the character of its Bode plot, that is sketch how \( |G(i\omega)| \) varies with \( \omega \). Let \( K \) be a fixed parameter. \hspace{1cm} 2 points

(c) Examine the estimation error \( \tilde{x} = x - \hat{x} \) as a function of \( v \) and \( e \). Show that the error can be written in the form

\[ \tilde{x}(t) = H_1(p)v(t) + H_2(p)e(t) \]

and determine the two transfer function operators \( H_1(p) \) and \( H_2(p) \). \hspace{1cm} 2 points

(d) Determine the variances of the two error terms in part (c), that is the variances of \( H_1(p)v(t) \) and of \( H_2(p)e(t) \). \hspace{1cm} 2 points
(e) Determine the observer gain $K$ that minimizes the total error variance

$$V(K) = E[H_1(p)v(t)]^2 + E[H_2(p)e(t)]^2$$

2 points

(f) Determine the Kalman filter related to the state space model set up in this problem. What is the variance $Ex^2(t)$ when the Kalman filter is used?

2 points
Problem 1

(a) As $B$ is nonsingular, the controllability matrix has full rank. The system is controllable.

(b) As $C$ is nonsingular, the observability matrix has full rank. The system is observable.

(c) As the element $B_{11} = 0$ and the system is in diagonal form, the state $x_1$ is not controllable. The system is not controllable using $u_1$.

Problem 2

(a) The closed loop characteristic equation becomes

$$1 + G_c(s)H_c(s) = 0 \Rightarrow s^2 + s + K = 0$$

The poles will lie in the left half plane for all positive values of $K$.

(b)

$$G_c(s) = \frac{1}{s(s + 1)} = \frac{1}{s} - \frac{1}{s + 1}$$

The sampled-data system can be written as follows. Using $\alpha = e^{-h}$,

$$G_d(z) = \frac{h}{z - 1} \frac{1 - \alpha}{z - \alpha} = \frac{hz - h\alpha - (z - 1)(1 - \alpha)}{(z - 1)(z - \alpha)} = \frac{z(h - 1 + \alpha) + (1 - \alpha - h\alpha)}{(z - 1)(z - \alpha)}$$

(c) The closed loop characteristic equation becomes

$$1 + G_d(z)H_d(z) = 0$$

leading to

$$(z - 1)(z - \alpha) + K[z(h - 1 + \alpha) + (1 - \alpha - h\alpha)] = 0$$

or yet

$$z^2 + z (-1 - \alpha + K(h - 1 + \alpha)) + \alpha + K(1 - \alpha - h\alpha)$$

A necessary condition for stability is

$$|a_1| < 2, \ |a_2| < 1$$

Both these two inequalities are violated if $K$ is large enough.
Problem 3

(a) In this case

\[ V = E [y(t + k) - \hat{y}(t + k)]^2 = E [y(t + k) - y(t)]^2 = E y^2(t + k) + E y^2(t) - 2E y(t + k)y(t) = r(0) + r(0) - 2r(k) = 10 - 10a^k = 10(1 - a^k) \]

(b) In this case

\[ V(\alpha) = E [y(t + k) - \hat{y}(t + k)]^2 = E [y(t + k) - \alpha y(t)]^2 = r(0) + \alpha^2 r(0) - 2\alpha r(k) \]

Minimizing \( V(\alpha) \) with respect to \( \alpha \) gives that the best value of \( \alpha \) is \( \alpha = r(k)/r(0) \), and

\[ \min_{\alpha} V(\alpha) = r(0) - r^2(k)/r(0) = 5 - 5a^{2k} = 5(1 - a^{2k}) \]

which is always smaller than \( 10(1 - a^k) \).

(c) The spectrum becomes directly from the definition

\[ \phi(\omega) = \sum_{k=-\infty}^{\infty} r(k)e^{-ik\omega} \]

\[ = \sum_{k=0}^{\infty} r(k)e^{-ik\omega} + \sum_{k=-\infty}^{0} r(k)e^{-ik\omega} = \sum_{k=0}^{\infty} r(k)e^{-ik\omega} + \sum_{k=0}^{0} r(k)e^{-ik\omega} \]

\[ = \sum_{k=0}^{\infty} 5(a^{-i\omega})^k + \sum_{k=-\infty}^{0} 5(a^{-1} e^{-i\omega})^k - 5 \]

\[ = \frac{5}{1 - ae^{-i\omega}} + \frac{5}{1 - ae^{i\omega}} - 5 \]

\[ = \frac{5}{(1 - ae^{-i\omega})(1 - ae^{i\omega})} \left[ 1 - ae^{i\omega} + 1 - ae^{-i\omega} - 1 - a^2 + ae^{-i\omega} + ae^{i\omega} \right] \]

\[ = \frac{5(1 - a^2)}{1 + a^2 - 2a \cos(\omega)} \]

(d) The process is apparently an AR(1) process, and can be represented as a first order system with

\[ A = a, \quad C = 1, \quad R_1 = 5(1 - a^2), \quad C = 1, \quad R_2 = 0 \]
Problem 4

(a) The transfer function operators from \( u(t) \) and \( v(t) \) to \( y(t) \) appear from the calculations

\[
y(t) = (1 \ 0) \begin{pmatrix} q + a & -1 \\ 0 & q \end{pmatrix}^{-1} \begin{pmatrix} b \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} 1 \\ c \end{pmatrix} v(t)
\]

\[
= (1 \ 0) \frac{1}{q(q + a)} \begin{pmatrix} q & 1 \\ 0 & q + a \end{pmatrix} \begin{pmatrix} b \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} 1 \\ c \end{pmatrix} v(t)
\]

\[
= \frac{1}{q(q + a)} \begin{pmatrix} q & 1 \end{pmatrix} \begin{pmatrix} b \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} 1 \\ c \end{pmatrix} v(t)
\]

\[
= \frac{b}{(q + a)} u(t) + \frac{q + c}{q(q + a)} v(t)
\]

This describes precisely the given system if \( v(t) = e(t + 1) \).

(b) In this case \( Q_1 = C^T C \), \( Q_2 = 0 \). The Riccati equation becomes

\[
S = A^T S A + C^T C - A^T S B \left( B^T S B \right)^{-1} B^T S A
\]

Set

\[
S = \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix}
\]

Spelling out the Riccati equation elementwise leads to

\[
\begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} = \begin{pmatrix} -a & 1 \\ 1 & 0 \end{pmatrix} s_{11} \begin{pmatrix} -a & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

\[
- \begin{pmatrix} -a & 1 \\ 1 & 0 \end{pmatrix} \frac{s_{11}}{s_{11}} \begin{pmatrix} -a & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

The feedback gain becomes

\[
L = (B^T S B)^{-1} B^T S A
\]

\[
= \frac{bs_{11}}{b^2 s_{11}} \begin{pmatrix} -a & 1 \end{pmatrix} = \frac{1}{b} \begin{pmatrix} -a & 1 \end{pmatrix}
\]

(c) Set

\[
P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}
\]

The Riccati equation is in this case

\[
P = APA^T + N \lambda^2 N^T - APC^T \left( CPC^T \right)^{-1} CPA^T
\]

\[
= \begin{pmatrix} -a & 1 \\ 0 & 0 \end{pmatrix} P \begin{pmatrix} -a & 0 \\ 1 & 0 \end{pmatrix} + \lambda^2 \begin{pmatrix} 1 \\ c \end{pmatrix} \begin{pmatrix} 1 & c \end{pmatrix} - \begin{pmatrix} -a & 1 \\ 0 & 0 \end{pmatrix} P \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{1}{p_{11}} \begin{pmatrix} 1 & 0 \end{pmatrix} P \begin{pmatrix} -a & 0 \\ 1 & 0 \end{pmatrix}
\]
\[
= \lambda^2 \begin{pmatrix} \frac{1}{c} \\ 1 & c \end{pmatrix} \begin{pmatrix} 1 & c \\ \end{pmatrix} \\
+ \begin{pmatrix} \frac{1}{0} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 1 \\ \end{pmatrix} \begin{pmatrix} \frac{1}{p_{11}} \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \end{pmatrix} \begin{pmatrix} -a \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \\
= \lambda^2 \begin{pmatrix} \frac{1}{c} \\ 1 & c \end{pmatrix} \begin{pmatrix} 1 & c \end{pmatrix} \\
+ \begin{pmatrix} \frac{1}{0} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 1 \\ \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} p_{22} - p_{12}^2 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} -a \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} p_{22} - p_{12}^2/p_{11} \end{pmatrix} \\
= \lambda^2 \begin{pmatrix} \frac{1}{c} \\ 1 & c \end{pmatrix} \begin{pmatrix} 1 & c \end{pmatrix} + \begin{pmatrix} \frac{1}{0} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} p_{22} - p_{12}^2/p_{11} \end{pmatrix}
\]

Hence
\[
P = \lambda^2 \begin{pmatrix} \frac{1}{c} \\ 1 & c \end{pmatrix} + \begin{pmatrix} \gamma & \frac{0}{0} \\ \end{pmatrix}
\]

where \(\gamma\) is the remaining unknown to be determined. It holds
\[
\gamma = p_{22} - p_{12}^2/p_{11} \Rightarrow \gamma = \lambda^2 c^2 - \frac{(\lambda^2 c)^2}{\lambda^2 + \gamma}
\]

leading to
\[
\gamma \lambda^2 + \gamma^2 = \lambda^4 c^2 + \lambda^2 c^2 \gamma - \lambda^4 c^2 \Rightarrow \gamma \left[ \gamma + \lambda^2 - \lambda^2 c^2 \right] = 0
\]

with the two solutions
\[
\gamma_1 = 0, \quad \gamma_2 = \lambda^2(c^2 - 1) < 0
\]

and hence \(\gamma_1\) must be chosen. Therefore,
\[
P = \lambda^2 \begin{pmatrix} \frac{1}{c} \\ 1 & c \end{pmatrix}
\]

The Kalman gain becomes
\[
K = APC^T \left(CPC^T\right)^{-1} \\
= \begin{pmatrix} \frac{1}{0} \\ 0 & 1 \end{pmatrix} \lambda^2 \begin{pmatrix} \frac{1}{c} \\ 1 & c \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{1}{\lambda^2} = \begin{pmatrix} \frac{1}{0} \\ 0 & 1 \end{pmatrix} (c - a)
\]

The Kalman filter will be
\[
\hat{x}(t+1|t) = (A - KC)\hat{x}(t|t-1) + Bu(t) + Ky(t) \\
= \begin{pmatrix} -c & 1 \\ 0 & 0 \end{pmatrix} \hat{x}(t|t-1) + \begin{pmatrix} b \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} c - a \\ 0 \end{pmatrix} y(t)
\]

It even follows that \(\hat{x}_2(t+1|t) = 0\). As this applies at all times, we also have
\[
\hat{x}_1(t+1|t) = -c\hat{x}_1(t|t-1) + bu(t) + (c - a)y(t)
\]
Problem 5

(a)

\[ x(t) = \frac{1}{p + a} v(t) \Rightarrow \phi_x(\omega) = \frac{r_1}{\omega^2 + a^2} \]

The variance of \( x(t) \) can be obtained by integrating the spectrum, but even easier by solving a Lyapunov equation,

\[ 0 = -2aP + r_1 \Rightarrow P = E x^2(t) = \frac{r_1}{2a} \]

(b) One gets directly

\[ G(p) = \frac{K}{p + a + K} \]

which is a first order, lowpass, filter. The static gain is \( K/(a + K) \).

(c) Set \( \tilde{x} = x - \dot{x} \). Then

\[
\begin{align*}
\dot{\tilde{x}} &= (-ax + v) - (-a\tilde{x} + Ky - K\tilde{x}) \\
&= (-a - K)\tilde{x} + v - Ke
\end{align*}
\]

It follows that

\[ H_1(p) = \frac{1}{p + a + K}, \quad H_2(p) = \frac{-K}{p + a + K} \]

(d) Using part (a),

\[ E[H_1(p)v(t)]^2 = \frac{r_1}{2(a + K)}, \quad E[H_2(p)v(t)]^2 = \frac{K^2r_2}{2(a + K)} \]

(e)

\[ V(K) = \frac{r_1 + K^2r_2}{2(a + K)} \]

\[ V'(K) = 0 \Rightarrow (a + K) \times 2Kr_2 - (r_1 + K^2r_2) \times 1 = 0 \]

\[ \Rightarrow K^2r_2 + 2Kar_2 - r_1 = 0 \Rightarrow K = -a \pm \sqrt{a^2 + r_1/r_2} = K = -a + \sqrt{a^2 + r_1/r_2} \]

The positive sign is chosen as \( a + K > 0 \) must hold to guarantee stability.

(f) The Riccati equation gives in this case

\[
\begin{align*}
0 &= -aP - Pa + r_1 - P^2/r_2 \\
P^2 + 2ar_2P - r_1r_2 &= 0 \\
P &= -ar_2 + \sqrt{a^2r_2^2 + r_1r_2} \\
K &= P/r_2 = -a + \sqrt{a^2 + r_1/r_2}
\end{align*}
\]

which is the same result as in part (e).