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Summary of lecture 9 (I/II)

Characteristics of an optimal control problem:
1. Infinite dimensional optimization problem (chose $u(t)$ for $0 \leq t \leq t_f$).
2. The cost function is related to the control signal via the system dynamics.
3. Constraints on the control signal.
4. Constraints on the states at the start and end.
5. The final time $t_f$ is unknown and part of the problem.

Theorem [the maximum principle]: Assume that the optimization problem

$$\min_{u(t)} \phi(x(t_f))$$
$$\dot{x}(t) = f(x(t), u(t)),$$
$$u(t) \in U, \quad 0 \leq t \leq t_f,$$
$$x(0) = x_0.$$

has a solution $u^*(t), x^*(t)$. The it must hold that

$$\min_{u \in U} \lambda^T(t)f(x^*(t), u) = \lambda^T(t)f(x^*(t), u^*(t)), \quad 0 \leq t \leq t_f,$$

where $\lambda(t)$ fulfills

$$\dot{\lambda}(t) = -f_x(x^*(t), u^*(t))^T \lambda(t), \quad \lambda(t_f) = \phi_x(x^*(t_f))^T.$$
An extended “imaginary” system $G_W$

\[ z_1 = W_u G_w u, \quad z_2 = -W_T w, \quad z_3 = W_S w \]

When we close the system using $u = -F_y y$ we have:

- $z_1 = W_u G_w u$
- $z_2 = -W_T w$
- $z_3 = W_S w$

\[ \mathcal{H}_2 \text{ control} \]

Definition: The $\mathcal{H}_2$-norm of the system $y = G(p)u$ is given by

\[ \|G\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(i\omega)|^2 d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr} (G^*(i\omega)G(i\omega)) d\omega. \]

Design criterion $\mathcal{H}_2$-design: Choose the controller such that

\[ V = \|W_S S\|_2^2 + \|W_T T\|_2^2 + \|W_u G_w u\|_2^2 \]

is minimized.

We proved that an equivalent problem is given by minimizing

\[ \|z\|_2^2 = \|Mx\|_2^2 + \|u\|_2^2, \]

which is a LQG problem!

\[ \mathcal{H}_\infty \text{ control} \]

Definition: The $\mathcal{H}_\infty$-norm of the system $y = G(p)u$ is given by

\[ \|G\|_\infty = \sup_u \frac{\|y\|_2}{\|u\|_2} = \sup_\omega \sigma(G(i\omega)) \]

Design objective $\mathcal{H}_\infty$-design: Find the controller that minimize

\[ \|G_{ec}\|_\infty = \max_\omega \sigma(G_{ec}(i\omega)). \]

This is a hard (non-convex) problem, instead we search for controllers that satisfies

\[ \|G_{ec}\|_\infty < \gamma, \]

by solving a sequence of Riccati equations.

Basic limitations and conflicts

- All signal amplitudes have limits.
- $S(s) + T(s) = 1$.
- For stable systems we derived Bode’s integral theorem,

\[ \int_0^\infty \log |S(i\omega)| d\omega = 0. \]

- Compromise between $S$ and $T$ (Bode’s relation)
- Poles in the RHP (unstable poles)
- Zeros in the RHP (Non-minimum phase zeros)
- Bounded inputs
- Time delays (steal phase)
- Model uncertainties (robustness)
A Lyapunov function $V(x)$ “measures the distance to the goal”:

- Let $V(x)$ denote a (generalized) distance from $x$ to an equilibrium point $x_0$.
- The distance must remain positive until the system has arrived in the equilibrium point $x_0$.
  \[ V(x) > 0, \quad x \neq x_0, \quad V(x_0) = 0. \]
- The distance must decrease until the final destination is reached,
  \[ \frac{d}{dt}V(x(t)) = V_x(x(t)) \dot{x}(t) = V_x(x(t)) f(x(t)) < 0, \quad x(t) \neq x_0. \]
- If the system “diverge”, this must be clearly visible
  \[ V(x) \to \infty, \quad |x| \to \infty. \]

Theorem: If a Lyapunov function $V$ satisfying
\[ V_x(x(t)) f(t) < 0, \quad x \neq x_0, \quad V(x) \to \infty \quad \text{as} \quad |x| \to \infty \]
can be found, then the equilibrium point $x_0$ is globally asymptotically stable.

The tricky part is to find the Lyapunov function!

Summary for the exam (in one slide)

- Signal sizes and gains, singular values
- Small gain theorem and the circle criterion
- Computing poles and zeros for transfer matrices
- Block schedule calculations for MIMO systems
- Stability – internal stability
- Basic limitations and conflicts – general understanding
- The pairing problem and RGA
- Decentralized and decoupled control
- IMC, $H_2$ and $H_{\infty}$ controllers
- Computing and using linearizations
- Understanding and using phase portraits
- Lyapunov stability
- Describing functions

Summary for life

What should you remember from automatic control?

- The principles: Feedback (and feed forward)
- Stability and instability: That they exist
- The possibilities: Use automatic control where it has never been used before.

The TSTF-principle: Try Simple Things First
We have seen many examples where automatic control has been successfully used.

<table>
<thead>
<tr>
<th>Automatic control is used almost everywhere</th>
<th>Tack!</th>
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<tbody>
<tr>
<td>To summarize it is fair to say that automatic control is used almost everywhere, but it is often hidden.</td>
<td>&quot;Automatic control is the art of getting things to behave as you want.&quot;</td>
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<tr>
<td>Automatic control is sometimes refereed to as the “hidden science”.</td>
<td>&quot;Tack för mig och lycka till med allt ni tar er för framöver!!&quot;</td>
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Automatic Control III, Lecture 10 – Summary of the course
T. Schön, 2013