Welcome to Automatic Control III!!

*Lecture 1 – Introduction and linear systems theory*

Thomas Schön
Division of Systems and Control
Department of Information Technology
Uppsala University.

Email: thomas.schon@it.uu.se,
Phone: 018 - 471 2594.

---

What is automatic control all about?

"Automatic control is the art of getting things to behave as you want."

---
Almost all abbreviations in the sales brochures hide control systems.

- ABS (Anti-lock Braking System, controlling the brake force)
- ESC (Electronic Stability Control, controlling course stability)
- ACE (Active Cornering Enhancement, controlling shock absorbers in curves)
- TCS (Traction Control System, controlling road grip)
- ACC (Adaptive Cruise Control, controlling speed/distance)
- ANC (Active Noise Control, controlling sound)
- ...

A robot arm is relatively weak and oscillates strongly when moving. Automatic control is necessary to achieve speed and accuracy.
We have reached the limits of how large mirrors can be made. Large telescopes are built with lots of small mirrors which are then continuously controlled so that the image is in focus (called adaptive optics).

- Controlling the signal strength between the mobile phone and the base station
- Traffic control
### Example – medical technology

- pacemaker
- insuline pumps
- anaesthesia

### Example – Aerospace

- Stabilization
- Cruise control
- Altitude control
- Navigation
- Weapon systems
- Quadrotors

(www.youtube.com/watch?v=3CR5y8qZf0Y)
The control problem

z: Control entity
r: Reference signal
w: Process disturbance
n: Measurement error
u: Control signal
y: Measurement signal

We want \( r - z \) to be small (i.e. \( z \) should follow \( r \) nicely) despite \( w, n \) and model uncertainties. At the same time we want \( u \) to take on reasonable values.

Automatic control is used almost everywhere

1. Improves already existing technical systems
2. Central area for many high-tech companies
3. Thankful area filled with many fun applications.
4. Very interesting mathematics

To summarize it is fair to say that Automatic Control is used almost everywhere, but it is often hidden.

Automatic control is sometimes refereed to as the "hidden science".
What is new in this course?
1. More on multivariable systems
2. Systematic design methods
3. Fundamental bounds on control performance
4. Nonlinear dynamics
5. Implementation aspects
Examination (passing the course (grade 3)):
- Two compulsory homework assignments:
  - Homework 1 (HW1) is solved in groups of up to four students. Each group hands in one solution in the form of a written report.
  - Homework 2 (HW2) is solved individually. Every student hands in his/her solution as an individual report. Identical reports/solutions will be rejected.
- An oral exam that must all be passed. The oral exam is based on the home work assignments.

Examination (for grades 4 and 5):
- Besides fulfilling the requirements for grade 3 you have to take an additional written exam (you can bring the textbook).

Outline – entire course
1. Lecture 1: Introduction and linear multivariable systems
2. Lecture 2–5: Multivariable linear control theory
   a) systems theory, closed loop system
   b) Basic limitations
   c) Controller structures and control design
   d) $\mathcal{H}_2$ and $\mathcal{H}_\infty$ loop shaping
3. Lecture 6–8: Nonlinear control theory
   a) Linearization and phase portraits
   b) Lyapunov theory and the circle criterion
   c) Describing functions
4. Lecture 9: Optimal control
5. Lecture 10: Summary and repetition
### Outline lecture 1

1. Introduction
2. What is automatic control?
3. Course administration
4. Systems theory – linear multivariable systems
   - a) Signal size
   - b) Gain
   - d) The frequency function and singular values
   - e) Stability and the small gain theorem

### Multivariable vs. scalar

The following concepts generalizes straightforwardly from SISO to multivariable systems:
- The weight function is now a matrix
- The transfer function is now a matrix
- Stability
- State space formulation and the solution of the state equation
- Controllability and observability

The following requires some thought:
- Gain and plotting the gain
- Poles and zeros
- Turning $G(s)$ into a state space description
For a linear transformation $y = Ax$, the operator norm ("the gain") is defined as how much larger the norm of $y$ can (maximally) become compared to the norm of $x$:

$$|A| = \sup_{x \neq 0} \frac{|y|}{|x|} = \sup_{x \neq 0} \frac{|Ax|}{|x|}$$

You can think of $|A|$ as the "gain" of the transformation $A$.

For a dynamical system $S (y = S(u))$ we can similarly define the gain of the system as

$$\|S\| = \sup_{u} \frac{\|y\|_2}{\|u\|_2} = \sup_{u} \frac{\|S(u)\|_2}{\|u\|_2}$$

"Choose $u$ such that the quotient between the size of the output signal and the input signal is maximized".

As we have seen it is fairly easy to compute the gain for:

1. Static nonlinearity
   - The gain varies with the amplitude.
   - The gain does not vary with the frequency.

2. Linear dynamical systems (scalar)
   - The gain does not vary with the amplitude.
   - The gain varies with the frequency.
How big is $G(i\omega)$ if $G(i\omega)$ is a matrix?

When we have several input and output signals it gets more complicated.

$$G(i\omega) = \begin{pmatrix} G_{11}(i\omega) & G_{12}(i\omega) & \ldots & G_{1m}(i\omega) \\ \vdots & \vdots & \ddots & \vdots \\ G_{p1}(i\omega) & G_{p2}(i\omega) & \ldots & G_{pm}(i\omega) \end{pmatrix}$$

Which Bode diagrams are interesting?

An example with two input signals (I/VII)

Controlling the level $x$ in a tank. The flow to the tank is controlled by the two flows $u_1$ and $u_2$, which both can be either positive or negative.

$$\dot{x} = u_1 + u_2$$

$$G(s) = \begin{pmatrix} 1/s & 1/s \end{pmatrix}$$
Consider a control law where we neglect $u_2 (u_2 = 0)$.

\[ \dot{x} = u_1 + u_2, \quad G(s) = \left( \frac{1}{s} \right) \]

What is the gain if we instead combines the use of $u_1$ and $u_2$?

Combine the use of $u_1$ and $u_2$, i.e., let the pumps cooperate ($u_1 = u_2$). The level adjusts faster.

\[ \dot{x} = u_1 + u_2, \quad G(s) = \left( \frac{1}{s} \right) \]
Let the pumps counter-act each other ($u_1 = -u_2$): the level does not change at all.

$$\dot{x} = u_1 + u_2, \quad G(s) = \left( \frac{1}{s} \right)^2$$

Consider arbitrary combinations (e.g., $u_2 = -5.03u_1$) of the combined use of the pumps.

$$\dot{x} = u_1 + u_2, \quad G(s) = \left( \frac{1}{s} \right)^2$$
Consider arbitrary combinations of the combined use of the pumps.

\[
\dot{x} = u_1 + u_2, \quad G(s) = \left(\frac{1}{s} \quad \frac{1}{s}\right)
\]

“All” combinations leads to “en himla många Bode diagram” ... Leads to a natural question: Can we somehow find the most relevant?

Yes, we can! The most relevant plots are given by the singular values of the transfer matrix. Now, what does this mean?
The singular values are computed using the **singular value decomposition (SVD)**, which states that every matrix $A \in \mathbb{R}^{n \times m}$ can be written as

$$A = U \Sigma V^*,$$

where $U \in \mathbb{R}^{n \times n}$ and $V \in \mathbb{R}^{m \times m}$ are unitary ($UU^* = I$) matrices and $\Sigma \in \mathbb{R}^{n \times m}$ with the singular values on the diagonal and zeros elsewhere,

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n \end{pmatrix}.$$

In MATLAB: $[U, \Sigma, V] = \text{svd}(A)$.

---

The SVD (applicable to any $n \times m$ matrix) is more general than the eigenvalue (applicable to certain square matrices) decomposition, but the two are connected.

Let $A = U \Sigma V^*$. Hence,

$$AA^* = U \Sigma V^* V \Sigma^* U^* = U \begin{pmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n^2 \end{pmatrix} U^*,$$

which is the eigenvalue decomposition of $AA^*$. Hence, $\sigma_i^2$ are the eigenvalues of $AA^*$. 

---
How big is $G(i\omega)$ if $G(i\omega)$ is a matrix? (revisited)

We can now answer this question.

- $|G(i\omega)|$ is the largest singular value of $G(i\omega)$
- $|Y(i\omega)| \leq |G(i\omega)||U(i\omega)|$
- $\|G\|_\infty = $ the largest singular value of $G(i\omega)$ for any $\omega$.
- $\|Y\|_2 \leq \|G\|_\infty \|U\|_2$

Plotting the singular values of $G(i\omega)$ as a function of $\omega$ corresponds to the plot of the amplitude curve for SISO systems.

For multivariable systems the actual gain depends on the direction of the input vector $U(i\omega)$.

Example – a heat exchanger (I/III)

$V_C \frac{dT_C}{dt} = f_C(T_{C_i} - T_C) + \beta(T_H - T_C)$,

$V_H \frac{dT_H}{dt} = f_H(T_{H_i} - T_H) - \beta(T_H - T_C)$. 

$f_C, f_H$ flows, $V_C, V_H$ volumes, $T_C, T_H$ temperatures.
Assume (for simplicity): $f_C = f_H = f$ is constant.

Inputs: $u_1 = T_C$ and $u_2 = T_H$.

States: $x_1 = T_C$ and $x_2 = T_H$.

Using the following numerical values $f = 0.01 \text{ (m}^3/\text{min)}$, $\beta = 0.2 \text{ (m}^3/\text{min)}$ and $V_H = V_C = 1 \text{ (m}^3)$, results in

$$\dot{x} = \begin{pmatrix} -0.21 & 0.2 \\ 0.2 & -0.21 \end{pmatrix} x + \begin{pmatrix} 0.01 & 0 \\ 0 & 0.01 \end{pmatrix} u,$$

$$y = x.$$

G(s) = \frac{0.01}{(s + 0.01)(s + 0.41)} \left( \begin{array}{cc} s + 0.21 & 0.2 \\ 0.2 & s + 0.21 \end{array} \right)$$

Plot the singular values in MATLAB:

```matlab
A = [-0.21 0.2; 0.2 -0.21];
B = 0.01*eye(2); C = eye(2);
D= zeros(2,2);
sigma(A,B,C,D)
```
A system is **input-output stable** if its gain is finite.

A solution to a differential equation is stable if a small perturbation of the initial condition gives small, possibly a vanishing effect.

For linear systems, stability is a system property, all solutions have the same stability properties.

The stability theory for multivariable linear systems is the same as in the basic courses. In this course we will also analyze stability for nonlinear systems, which leads to Lyapunov theory (chapter 12).

---

### Stability – the small gain theorem (I/II)

Two stable systems $S_1$ and $S_2$ which are connected according to the figure below results in a closed loop system that is stable if

$$
\|S_1\| \cdot \|S_2\| < 1.
$$

![Diagram of a closed loop system](image)

Note that the small gain theorem is valid both for linear and nonlinear systems. For linear systems the criterion is simplified to

$$
\|S_2 S_1\| < 1.
$$
### Useful MATLAB commands

- `ss2zp, zp2ss, tf2zp, zp` Transformations between different representations (state space (`ss`), transfer functions (`tf`), zeros and poles (`zp`))
- `tzero` Calculation of zeros (for multivariable systems)
- `lsim, step, impulse` Simulation, step and impulse responses
- `pole, eig, roots` Eigenvalues and poles
- `bode` Frequency function and Bode plot
- `sigma` Computes the singular values of the frequency function
- `obsv, ctrb` Observability and controllability

### A few concepts to summarize lecture 1

**Automatic control:** “the art of getting things to behave as you want.”

**Singular value:** The singular values of a matrix $A$ are given by $\sigma_i = \sqrt{\lambda_i}$, where $\lambda_i$ denotes the eigenvalues of $A^*A$.

**Singular values of the frequency function:** Plotting the singular values of $G(i\omega)$ (for a multivariable system) as a function of $\omega$ corresponds to the plot of the amplitude curve for SISO systems.

**Input-output stability:** A systems is input-output stable if its gain is finite.

**Small gain theorem:** Let the two systems $S_1$ and $S_2$ be connected in a feedback loop, then the closed loop system is input-output stable if $\|S_1\| \cdot \|S_2\| < 1$. 