Summary of lecture 1 (I/II)

Formalized how we can measure the size of signals and systems using various norms.

Signal size
\[
\|z\|_2^2 = \int_{-\infty}^{\infty} |z(t)|^2 dt = \int_{-\infty}^{\infty} z^T(t)z(t) dt,
\]

\[
\|z\|_{\infty} = \sup |z(t)|.
\]

The gain of system \( S \), where \( y = S(u) \):
\[
\|S\| = \sup_u \frac{\|y\|_2}{\|u\|_2} = \sup_u \frac{\|S(u)\|_2}{\|u\|_2}
\]

The gain of a linear stable SISO system \( G \) was shown to be
\[
\|G\| = \sup_\omega |G(i\omega)|.
\]
“The direction of the input signal” 5(27)

Introduce $||G||_{\infty} = \sup_{\omega} |G(i\omega)|$. Compared to the SISO case we also have to maximize with respect to “the direction of $u$” (recall that this is taken case of via $\sigma(G(i\omega))$).

We have $||y||_2 \leq ||G||_{\infty}||u||_2$ with equality if $u(t) = u^* \cos(\omega t)$ where $\omega$ is the frequency for which $|G(i\omega)|$ is maximized and the direction of $u^*$ is the same as (i.e. parallell to) the largest singular eigenvector of $G^*(i\omega)G(i\omega)$.

Hence, $||S|| = ||G||_{\infty}$.

Example – a heat exchanger (I/III) 6(27)

Assume (for simplicity): $f_C = f_H = f$ is constant.

Inputs: $u_1 = T_{C_i}$ and $u_2 = T_{H_i}$.

States: $x_1 = T_C$ and $x_2 = T_H$.

Using the following numerical values $f = 0.01$ (m$^3$/min), $\beta = 0.2$ (m$^3$/min) and $V_H = V_C = 1$ (m$^3$), results in

$$
\dot{x} = \begin{pmatrix}
-0.21 & 0.2 \\
0.2 & -0.21
\end{pmatrix} x + \begin{pmatrix}
0.01 & 0 \\
0 & 0.01
\end{pmatrix} u,
$$

$y = x$.

Example – a heat exchanger (II/III) 7(27)

Example – a heat exchanger (III/III) 8(27)
**Stability**

A system is **input-output stable** if its gain is finite.

A solution to a differential equation is stable if a small perturbation of the initial condition give small, possibly a vanishing effect.

For linear systems, stability is a system property, all solutions have the same stability properties.

The stability theory for multivariable linear systems is the same as in the basic courses. In this course we will also analyze stability for nonlinear systems, which leads to Lyapunov theory (chapter 12).

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**Stability – the small gain theorem (I/II)**

Two stable systems $S_1$ and $S_2$ which are connected according to the figure below results in a closed loop system that is stable if

$$\|S_1\| \cdot \|S_2\| < 1.$$

Note that the small gain theorem is valid both for linear and nonlinear systems. For linear systems the criterion is simplified to

$$\|S_2 S_1\| < 1.$$

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**Poles (of multivariable transfer functions $G(s)$)**

The **poles** are given by the eigenvalues (counted with multiplicity) of the system matrix $A$ of a minimal (controllable and observable) state-space realization of the system.

What if $G(s)$ is given?

The pole polynomial is the **least common divider** of all minors of $G(s)$. The system poles are then given by the zeros of the pole polynomial.

Note that this also provides a way of finding the order of a minimal realization! The order of the system = the number of poles (counted with multiplicity).

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**Zeros (of multivariable transfer functions $G(s)$)**

**Definition** (quadratic $G(s)$): The zero polynomial is given by the numerator of $\det(G(s))$ (after it has been normalized to have the pole polynomial as its denominator).

**Definition** (general $G(s)$): The zero polynomial is the greatest common divider of the numerators of the maximal minors (after they have been normalized to have the pole polynomial as its denominator).

**Maximal minor (maximal underdeterminant)**: A determinant of maximal size.
### The closed loop system: Important signals

1. **u**, control signal
2. **z**, the quantity we want to control
3. **r**, reference signal (i.e., what we want **z** to be)
4. **y**, output signal
5. disturbances
   - **w**_u_, input disturbance
   - **w**, output disturbance
   - **n**, measurement disturbance

"The canonical block diagram"

Open look system:

\[
\begin{align*}
  z(t) &= G u(t) + w(t), \\
  y(t) &= z(t) + n(t).
\end{align*}
\]

Feedback (linear systems):

\[
    u(t) = F_r r(t) - F_y y(t) + w_u(t).
\]

### The closed loop system: Important transfer functions

Transfer functions:

\[
\begin{align*}
  G_c &= (I + GF_y)^{-1} G_r \\
  S &= (I + GF_y)^{-1} \\
  S_u &= (I + F_y G)^{-1} \\
  T &= (I + GF_y)^{-1} G_y
\end{align*}
\]

Relations among signals:

\[
\begin{align*}
  z &= G_r r + S w - T n + G S_u w_u \\
  u &= S_u F_r r - S_u F_y (w + n) + S_u w_u
\end{align*}
\]

### Stability of the closed loop system

Consider the close loop system as a system with two input signals (**w**_u_, **w**) and two output signals (**u**, **y**). For now **r** = **n** = 0.

\[
\begin{bmatrix}
  y \\
  u
\end{bmatrix} =
\begin{bmatrix}
  GS_u & S \\
  S_u & -S_u F_y
\end{bmatrix}
\begin{bmatrix}
  w_u \\
  w
\end{bmatrix}
\]

(1)

If **G** and **F_y** are represented by controllable and observable state-space descriptions it can be shown that the closed loop system in (1) is also controllable and observable. If we check all 4 transfer functions

\[
\begin{align*}
  GS_u, S, S_u, S_u F_y
\end{align*}
\]

we will find all poles for the systems (and we can then (for example) check the stability).

### Internal stability

From the example we saw that we have to be careful with pole-zero cancellations.

**Definition (internal stability):** The closed loop systems is said to be *internally stable* if the following four transfer functions are stable (after possible cancellations), as well as **F_r**,

\[
\begin{align*}
  w_u(ty) &\to u(t) : S_u = (I + F_y G)^{-1}, \\
  w_u(t) &\to y(t) : S G = (I + GF_y)^{-1} G, \\
  w(t) &\to u(t) : -S_u F_y = -(I + F_y G)^{-1} F_y, \\
  w(t) &\to y(t) : S = (I + GF_y)^{-1}.
\end{align*}
\]
Neglecting disturbances we have
\[ z_0 = (I + G_0F_y)^{-1}G_0Fr \]
for the true system \( G_0 \) and
\[ z = (I + GF_y)^{-1}GF_r \]
for the model \( G \).

Let us study the relationship between \( z_0 \) and \( z \) by introducing \( \Delta z \) and \( S_0 \) according to
\[ z_0 = (I + \Delta z)z, \]
\[ S_0 = (I + G_0F_y)^{-1}. \]

The sensitivity function \( S_0 \) (for the true system) describes how the relative model error \( \Delta G \) is transformed into a relative output signal error \( \Delta z \) according to
\[ \Delta z = S_0\Delta G. \]

In practice \( G_0 \) is unknown, implying that \( S_0 \) has to be approximated using
\[ S = (I + GF_y)^{-1}. \]

Interpretation: \( S \) is the gain from model error to signal error.

What model errors \( \Delta G \) can be allowed without endangering the stability of the closed loop system?

The small gain theorem can be used to prove that the closed loop system remains stable if
\[ \|\Delta GT\|_\infty < 1, \]
which in turn is fulfilled if
\[ |T(i\omega)| < \frac{1}{|\Delta G(i\omega)|}, \forall \omega \]

Formulation of the control problem in words
“Choose a controller such that the controlled variable resembles the reference signal as close as possible, despite disturbances, measurement errors and model errors, while at the same time not making use of too large control signals.”

We now have a machinery to formalize these words. Recall the following relationships:
\[ e = (I - G_c)r - Sw + Tn, \]
\[ u = G_{ru}r + G_{wu}(w + n), \]
\[ \Delta z = S_0\Delta G, \]
\[ \|\Delta GT\|_\infty < 1. \]
Formalizing the control problem (specifications) 21(27)

1. $|I - G_c|$ small $\Rightarrow$ The controlled variable follows the reference signal.
2. $S$ small $\Rightarrow$ Small influence of model errors and process disturbances.
3. $T$ small $\Rightarrow$ Small influences of measurement errors and to make sure that model errors do not jeopardize the stability.
4. $G_{ru}$ and $G_{wu}$ small $\Rightarrow$ Control signal must not be too large.

Important to note that these requirements are in conflict. For example:

\[ S + T = I, \quad G_c = GG_{ru}. \]

Design specifications in the time domain (I/III) 22(27)

Design criterion 1: Chose a controller so that $M$, $e_0$, $T_r$ and $T_s$ for the step response of $G_c$ is smaller than some given values.

The step response of the sensitivity function $S$ can be examined in the same way to ensure that system disturbances are removed.

Design specifications in the time domain (II/III) 23(27)

Recall the error coefficients from the basic course. The first error coefficient, the static error coefficient is given by

\[ e_0 = \lim_{t \to \infty} e(t) = I - G_c(0). \]

Assume $F_y = F_r$. This means that $I - G_c = I - T = S$, which in turn implies that

\[ e_0 = S(0). \]

Usual design requirement: $e_0 = S(0) = 0$.

Design specifications in the time domain (III/III) 24(27)

Design criterion 2: Choose the controller such that $G_c$ and $S$ are equal to given transfer functions. (Note: Internal Model Control, (IMC) in Chapter 8)

Design criterion 3: Choose the controller such that the poles of $G_c$ and $S$ are placed within given areas. (Note: Recall pole placement in the basic course)

Design criterion 4: Choose the controller such that

\[ \|e\|^2_{Q_1} + \|u\|^2_{Q_2} \]

is minimized. (Note: linear quadratic controllers (LQG))
Design spec. in the frequency domain (I/II)

Requirements:

\[ |S(i\omega)| \leq |W^{-1}(i\omega)|, \quad |T(i\omega)| \leq 1, \quad |\Delta G(i\omega)|, \quad \forall \omega. \]

In the multivariable case

\[ \|WS\|_\infty \leq 1, \quad \|WT\|_\infty \leq 1, \quad \|WruGru\| \leq 1. \]

Design criterion 5: Choose the controller such that

\[ \|WS\|_\infty \leq 1, \quad \|WT\|_\infty \leq 1, \quad \|WruGru\| \leq 1. \]

Note: This leads to \( \mathcal{H}_\infty \)-controllers (Chapter 10).

Design criterion 6: Choose the controller such that

\[ V = \|WS\|_2^2 + \|WT\|_2^2 + \|WruGru\|_2^2 \]

is minimized. Note: This leads to \( \mathcal{H}_2 \)-controllers (Chapter 10).

Design spec. in the frequency domain (II/II)

A few concepts to summarize lecture 2

**Minor**: The determinant of a quadratic matrix of \( A \) obtained by crossing out rows and columns in \( A \).

**Pole of a multivariable system**: The pole polynomial is the least common divider of all minors of \( G(s) \). The system poles are then given by the zeros of the pole polynomial.

**Zero of a multivariable system**: The zero polynomial is the greatest common divider of the numerators of the maximal minors (after they have been normalized to have the pole polynomial as its denominator).

**Internal stability**: The concept of internal stability allows us to assess the stability of the closed loop system, without “missing any unseen modes”.

**Robustness**: What model errors \( \Delta G \) can be allowed without endangering the stability of the closed loop system?