1. Summary of lecture 5
2. General properties
3. Linearization and stationary points
4. Phase portraits
\( H_2 \) and \( H_\infty \) synthesis:

- Make \( W_u G_{wu}, W_S S, W_T T \) small.
- \( H_2 \): Minimize \( \int \left( |W_u G_{wu}|^2 + |W_S S|^2 + |W_T T|^2 \right) d\omega \).
- \( H_\infty \): Set an upper bound for \( |W_u G_{wu}|, |W_S S|, |W_T T| \) for all \( \omega \).
- Results in algebraic Riccati equations.

\( H_2, H_\infty \) synthesis – pros and cons:

(+): Directly handles the specifications on \( S, T \) and \( G_{wu} \).
(+): Let us know when certain specifications are impossible to achieve (via \( \gamma \)).
(+): Easy to handle several different specifications (in the frequency domain).
(-): Can be hard to control the behaviour in the time domain in detail.
(-): Often results in complex controllers (number of states in the controller = number of states in \( G, W_u, W_S, W_T \)).
Linear multivariable controller synthesis summary:

1. Perform an RGA analysis
2. Use simple SISO controllers of PID type if the RGA analysis indicates that it might be possible.
3. Otherwise make use of LQ, MPC or $H_2/H_\infty$ synthesis.

DC motor controlled using a lead controller. We want to control the motor angle. The saturation

$$u = \text{sat}(\tilde{u}) = \begin{cases} 
\tilde{u} & |\tilde{u}| \leq 1, \\
1 & \tilde{u} > 1, \\
-1 & \tilde{u} < -1,
\end{cases}$$

renders the system nonlinear.
Step responses for two different amplitudes of the reference $r$ signal.

**Blue:** Amplitude 1

**Red:** Amplitude 5 (scaled with $1/5$)

**Conclusion:** The step response is amplitude dependent. If the system would have been linear the two step responses would have coincided.

Both the ramp and the sine responses are roughly the same as for a linear system.
DC motor with a saturated control signal (IV/V) 9(17)

Red: $r$. Blue: $y$. Green: $y$ when $r$ is a ramp (same as on the previous slide).

Something happens here: There is no sine present in the response and the ramp error has increased...
This violates the superposition principle and the frequency fidelity!

DC motor with a saturated control signal (V/V) 10(17)

Red: before the saturation ($\tilde{r}$). Blue: after the saturation ($u$).
We can approximate a nonlinear system

\[ \dot{x} = f(x, u), \quad y = h(x, u), \]

by linearizing the system around an equilibrium (stationary) point 
\((x_0, u_0)\). Intuitively this amounts to approximating the system by a
flat hyperplane (straight line in the scalar case).

Let \( \Delta x(t) = x(t) - x_0 \), \( \Delta u(t) = u(t) - u_0 \), \( \Delta y(t) = y(t) - y_0 \).

A Taylor expansion (only keeping the linear terms) results in

\[
\begin{align*}
\frac{d}{dt} \Delta x &= \left[ \frac{\partial f(x_0, u_0)}{\partial x} \Delta x + \frac{\partial f(x_0, u_0)}{\partial u} \Delta u \right], \\
\Delta y &= \left[ \frac{\partial h(x_0, u_0)}{\partial x} \Delta x + \frac{\partial f(x_0, u_0)}{\partial u} \Delta u \right].
\end{align*}
\]

### Phase portraits for linear systems

- **Sign**: Is the solution moving towards the origin or away from
  the origin (along the eigenvector)?
- **Relative size**: “fast” and “slow” eigenvectors, which is
  dominating the solution behaviour for \( t \approx 0 \) and \( t \gg 0 \)?
- **Complex/real**: Complex conjugated eigenvalues results in
circles and spirals.

**Cases to consider:**

1. Two distinct real valued eigenvalues implies two eigenvectors.
2. One multiple eigenvalue implies one eigenvector.
3. Complex eigenvalues.
### Two distinct eigenvalues with the same sign

The solution is \( x(t) = c_1e^{\lambda_1 t}v_1 + c_2e^{\lambda_2 t}v_2 \).

**Stable node:** For eigenvalues \( \lambda_1 < \lambda_2 < 0 \). The first term dominates for small \( t \), the second term dominates for large \( t \).

**Unstable node:** For eigenvalues \( 0 < \lambda_1 < \lambda_2 \). Also here, the first term dominates for small \( t \), the second term dominates for large \( t \).

### Two distinct eigenvalues with opposite sign

For eigenvalues \( \lambda_1 < 0 < \lambda_2 \) (with corresponding eigenvectors \( v_1, v_2 \)) the solution is

\[
x(t) = c_1e^{\lambda_1 t}v_1 + c_2e^{\lambda_2 t}v_2.
\]

Trajectories close to \( v_1 \) will approach the origin. \( v_1 \) is called the **stable eigenvector**.

Trajectories close to \( v_2 \) will move away from the origin. \( v_2 \) is called the **unstable eigenvector**.
A multiple eigenvalue

For multiple eigenvalues $\lambda_1 = \lambda_2$.

Stable node (unstable: change direction).

Stable star node (unstable: change direction).

Two examples in 3D

Example of a generalization to 3D.

Left: Focus + one real eigenvalue.

Right: Three real eigenvalues.
**Equilibrium points:** An equilibrium point is a point $x_0, u_0$ where the system is in rest, i.e. $f(x_0, u_0) = 0$. Also referred to as stationary points.

**Linearization:** Find a Taylor expansion of the nonlinear system around an equilibrium point and only keep the linear parts. This means that we are approximating the system using a flat hyperplane.

**Phase plane:** A two dimensional state space that is simple to visualize graphically.

**Phase portraits:** A plot where one state variable is plotted against another state variable.

**Limit cycle:** A limit cycle is a closed trajectory in phase space having the property that at least one other trajectory spirals into it either as time approaches $\pm$ infinity.