Summary of lecture 7 (I/III)

- Defined stability of equilibrium (stationary) points; stable, asymptotically stable and globally asymptotically stable.
- Investigated stability of a nonlinear system by studying how the distance to the origin changes over time.
- The above idea lead us into Lyapunov theory.

Summary of lecture 7 (II/III)

A Lyapunov function $V(x)$ "measures the distance to the goal":
- Let $V(x)$ denote a (generalized) distance from $x$ to an equilibrium point $x_0$.
- The distance must remain positive until the system has arrived in the equilibrium point $x_0$, $V(x) > 0, x \neq x_0, V(x_0) = 0$.
- The distance must decrease until the final destination is reached, $\frac{d}{dt} V(x(t)) = V_x(x(t)) x(t) = V_x(x(t)) f(x(t)) < 0, x(t) \neq x_0$.
- If the system "diverge", this must be clearly visible $V(x) \to \infty, |x| \to \infty$. 
Summary of lecture 7 (III/III)

**Theorem:** If a Lyapunov function $V$ satisfying

$$V_x(x(t))f(t) < 0, x \neq x_0, \quad V(x) \to \infty \text{ as } |x| \to \infty$$

can be found, then the equilibrium point $x_0$ is globally asymptotically stable.

The tricky part is to find the Lyapunov function!

We also showed that finding a Lyapunov function for a linear system amounts to solving the **Lyapunov equation**,

$$A^T P + PA = -Q.$$  

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Stability – the small gain theorem

Two stable systems $S_1$ and $S_2$ which are connected according to the figure below results in a closed loop system that is stable if

$$\|S_1\| \cdot \|S_2\| < 1.$$  

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Circle criterion

**Theorem:** [Circle criterion] Assume that $G(s)$ has no poles in the RHP and that $f(0) = 0, k_1 \leq f(y)/y \leq k_2$ for $y \neq 0$. Then the closed loop system is input output stable if the Nyquist curve $G(i\omega)$ does not enter, nor encircle the circle which intersects the negative real axis (perpendicularly) in $-1/k_1$ and $-1/k_2$.

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A simple feedback system

**Unstable!**
A simple feedback system – with saturation

The same system, but now with a saturation (a static nonlinearity) in the loop.

Note the amplitude-stability!

Passing a sine through a static nonlinearity

\[ u = C \sin \omega t \]

\[ w = f(C \sin \omega t) \]

Fourier series expansion of \( w \):

\[ w = \frac{1}{2} A_0(C) + \sum_{n=1}^{\infty} \left( A_n(C) \cos(n\omega t) + B_n(C) \sin(n\omega t) \right) \]

\[ = A_0(C) + \sum_{n=1}^{\infty} A_n(C) \sin(n\omega t + \phi_n(C)) \]

Define the describing function as

\[ Y_f(C) = \frac{A_1(C)e^{i\phi(C)}}{C}, \]

where \(|Y_f(C)|\) is the gain and \(\arg Y_f(C)\) is the phase shift.

Sine through a static nonlinearity \( G(s) \) (I/II)

\[ u = C \sin \omega t \]

\[ w = A_0(C) + \sum_{n=1}^{\infty} A_n(C) \sin(n\omega t + \phi_n(C)) \]

\[ y = A_0(C)G(0) + \sum_{n=1}^{\infty} A_n(C)\left|G(i\omega)\right| \sin(n\omega t + \phi_n(C) + \psi(n\omega)) \]

Recall – Fourier series

A Fourier series decomposes periodic signals into the sum of a (possibly infinite) set of simple oscillating functions, namely sines and cosines (or complex exponentials).

\[ N = 1 \]

\[ N = 2 \]

\[ N = 5 \]

\[ N = 100 \]
**Sine through a static nonlinearity** \( G(s) \) (II/II)

Assume:
- \( A_0 = 0 \) (valid for example if \( f \) is an odd function).
- \( |G(\pm \omega)| < |G(\pm \omega)|, \quad |k| > 1 \), i.e. \( G \) “steep LP filter”.

Then we have

\[
y \approx A_1(C) \left| G(i\omega) \right| \sin(\omega t + \phi_1(C) + \psi(\omega))
\]

where \( \psi(\omega) = \arg G(i\omega) \).

**Follow the sine around the loop** (I/III)

Only keep the fundamental frequency:

\[
u = C \sin \omega t \\
w = A_1(C) \sin(\omega t + \phi_1(C)) \\
y = A_1(C) |G(i\omega)| \sin(\omega t + \phi_1(C) + \psi(\omega)) \\
e = -y
\]

**Follow the sine around the loop** (II/III)

Conditions for oscillation: \( e = u \), i.e.

\[
e = A_1(C) |G(i\omega)| \sin(\omega t + \phi_1(C) + \psi(\omega) + \pi) = C \sin(\omega t) = u
\]

The same amplitude: \( A_1(C) |G(i\omega)| = C \)

The phase is the same, save for \( 2\pi \): \( \phi_1(C) + \psi(\omega) = \pi + n2\pi \).

**Follow the sine around the loop** (III/III)

or, more compactly (phase and amplitude in one equation)

\[
Y_f(C)G(i\omega) = -1
\]

since \( G(i\omega) = |G(i\omega)| e^{i\psi(\omega)} \).
The describing function is given by

\[ Y_f(C) = \frac{A_f(C)e^{i\phi_f(C)}}{C} \]

- **Interpretation**: The “transfer function” for the nonlinearity for a stationary sine (the fundamental frequency). An “amplitude dependent gain”.
- The gain is given by \( |Y_f(C)| \) and the phase shift is given by \( \arg Y_f(C) \).

**Circle criterion**: The circle criterion generalizes the Nyquist criterion to static nonlinearities.

**Describing function**: An approximate method for examining existence of periodic solutions for systems involving a static nonlinearity in the feedback loop.