Close the loop around a linear system $G(s)$ using a static nonlinearity $f(y)$, where

$$f(0) = 0, \quad k_1 \leq \frac{f(y)}{y} \leq k_2,$$

The closed loop system is stable if the Nyquist curve for $G(i\omega)$ does not encircle or enters the circle.
**Goddard’s rocket problem:** How should the thrust of a vertically ascending rocket be optimized to reach as high altitudes as possible (taking into account atmospheric drag and the gravitational field)?

Robert Goddard (on March 16, 1926), holds the launching frame of his most notable invention — the **first liquid-fueled rocket**.

Goddard’s diary entry the day after (March 17, 1926) the successful launch:

“The first flight with a rocket using liquid propellants was made yesterday at Aunt Effie’s farm in Auburn... Even though the release was pulled, the rocket did not rise at first, but the flame came out, and there was a steady roar. After a number of seconds it rose, slowly until it cleared the frame, and then at express train speed, curving over to the left, and striking the ice and snow, still going at a rapid rate.”

To read more about Goddard and his endeavours, see http://en.wikipedia.org/wiki/Robert_H._Goddard

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1. Newton’s force law gives us

\[ \dot{v} = \frac{1}{m} (u - D(v, h)) - g, \]

where \( m \) denotes the mass of the rocket, \( v \) denotes the velocity, \( u \) denotes the engine thrust, \( h \) denotes altitude and \( g \) denotes gravity. Furthermore, \( D(v, h) \) denotes the air drag.

2. The rocket ascends vertically: \( \dot{h} = v \)

3. The mass of the rocket is reduced as more fuel is consumed. Assume that the fuel consumption is proportional to the thrust

\[ \dot{m} = -\gamma m \]

4. The thrust is limited

\[ 0 \leq u \leq u_{\text{max}}. \]

5. We start from the ground. Fully fueled the rocket mass is \( m_0 \) and empty the rocket mass is \( m_1 \),

\[ v(0) = 0, \quad h(0) = 0, \quad m(0) = m_0, \quad m(t_f) \geq m_1, \]

where \( t_f \) is the final time.

6. Resulting **optimization** problem:

\[ \max h(t_f) \]

subject to the above restrictions 1 – 5.
The maximum principle – special case (I/III) 9(15)

Study the following special case:

\[ \min \phi(x(t_f)) \]
\[ \dot{x}(t) = f(x(t), u(t)), \]
\[ u(t) \in U, \quad 0 \leq t \leq t_f, \]
\[ x(0) = x_0. \]

**Step 1.** Assume that we have a control signal \( u^*(t) \) with a corresponding state \( x^*(t) \) that fulfills the constraints.

Let us now test if \( u^*(t) \) and \( x^*(t) \) are also optimal (i.e. minimizes \( \phi(x(t_f)) \)) by investigating what happens if we perturb the control signal somewhat.

The maximum principle – special case (II/III) 10(15)

We can derive the following variational equation

\[ \dot{\eta}(t) = f_x(x^*(t), u^*(t)) \eta(t) \]

**Step 2 continued.** The change in the cost function is given by (first order approximation)

\[ \epsilon \phi_x(x^*(t_f)) \eta(t_f) \]

The maximum principle – special case (III/III) 11(15)

**Step 3 continued.** The cost function change can also be written

\[ \lambda^T(t_1) \eta(t_1) = \lambda^T(t_1) (f(x^*(t_1), \bar{u}) - f(x^*(t_1), u^*(t_1)) ] \]

where \( \lambda \) is given by

\[ \dot{\lambda}(t) = -f_x(x^*(t), u^*(t))^T \lambda(t), \]
\[ \lambda(t_f) = \phi_x(x^*(t_f))^T. \]

This means that optimality requires

\[ \lambda^T(t_1) (f(x^*(t_1), \bar{u}) - f(x^*(t_1), u^*(t_1))) \geq 0 \]

for all choices of \( \bar{u} \in U \) and for all \( t_1 \).

The maximum principle 12(15)

**Theorem:** Assume that the optimization problem

\[ \min_{u(t)} \lambda^T(t_f) f(x^*(t), u) = \lambda^T(t) f(x^*(t), u^*(t)), \quad 0 \leq t \leq t_f, \]

where \( \lambda(t) \) fulfills

\[ \dot{\lambda}(t) = -f_x(x^*(t), u^*(t))^T \lambda(t), \quad \lambda(t_f) = \phi_x(x^*(t_f))^T. \]
## Short history of optimal control

Roots in calculus of variations (Bernoulli, Euler, Lagrange, Weierstrass, ...)

- Optimal control emerged in the 1950s during the space race
- Dynamic programming (Richard Bellman in the US)
- Maximum principle (Lev Pontryagin in the former Soviet union)
- Linear quadratic control (Rudolph Kalman)

## Motion control – industrial robots

The are many many industrial application, let me just show you one.

Generation of suitable reference trajectories is a very common application of optimal control.

Computing optimal trajectories (position, velocity, ...) for the tool. This reference trajectory is then handed to the robot control system that controls the various parts of the robot such that the trajectory is followed as well as possible.

http://www.youtube.com/watch?v=SOESSCXGhFo

## A few concepts to summarize lecture 9

**Optimal control:** Optimal control deals with the problem of finding a control law for a given system such that a certain optimality criterion is achieved.

**Maximum principle:** The maximum principle is used in optimal control to find the best possible control for taking a dynamical system from one state to another. A necessary condition for an optimum.

**Variational equation:** A variational equation describes how perturbations evolve along a trajectory.