Uppsala University
Department of Information Technology
Division of Systems and Control
September 18, 2015

Automatic control III

Homework assignment 2
2015

Deadline (for this assignment):
Wednesday September 30, 23.59

All homework assignments are compulsory and form an important part of the examination.

This assignment is to be solved in groups, with up to 4 students. The solution has to be handed in in written form as a pdf file. All group members are responsible for the entire solutions. There will also be an oral examination (October 15), based on this assignment and Assignment 1. Each group member will be given a few individual questions related to the assignments.

Assignment 3 (not in this document) is to be solved individually.

Instructions on how to hand in the assignments are found on the course homepage. The solutions should be clearly presented, with well motivated reasoning and clearly stated answers. The solutions should be carefully written, with satisfactory equation typesetting, equation numbering, complete sentences, etc. However, no introduction, abstract, etc is needed, but just solutions to the problems.

Before handing in your solutions, make sure (for each exercise) that

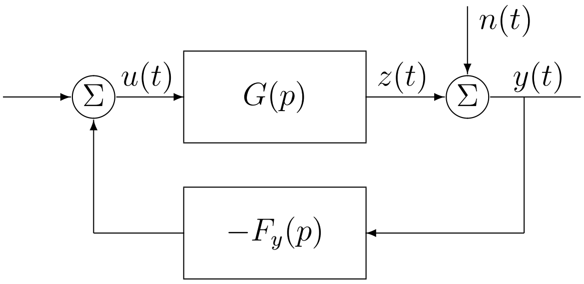
(I) You have answered all questions

(II) Your answer is reasonable

Solutions not fulfilling (I) and (II) will be rejected.

Problem I $\mathcal{H}_2$ control

This assignment deals with an inverted pendulum. Your task is to design a controller F_y, according to the picture below.



Here, u denotes the control input, z is the controlled variable, n is measurement noise, y is the measured (noisy) output, and F_y is the feedback controller. As a model for the system, the transfer function (from u to z)

$$G(s) = \frac{-(\beta s + 1)}{s^2 - 1}, \tag{1}$$

is used, which captures the fact that there is one stable and one unstable pole. Set $\beta = 0.1$. For designing the feedback controller, you will use continuous-time $\mathcal{H}_2$ methods.

(a) After a discussion with your colleagues about this particular application, you conclude that a good weighting of the transfer functions would be

$$W_S(s) = \frac{1}{s}, \quad W_T(s) = 1, \quad W_u(s) = 1. \tag{2}$$

Describe why one can expect this choice of weighting to lead to a controller with integral action.

(b) Write the transfer function (1) on controllable canonical form. Determine the extended model of the system. Check if all conditions related to the matrices M, D and innovation form are fulfilled. If these conditions are not fulfilled, how should the standard recipe for the determining the optimal $\mathcal{H}_2$ controller be modified?

(c) Determine the optimal $\mathcal{H}_2$ controller. What is the feedback transfer function $F_y(s)$ in this case? Does it have integral action?

(d) Plot the Bode diagrams of the weighted transfer functions $W_S(s)S(s)$, $W_T(s)T(s)$ and $W_u(s)G_{wu}(s)$. Do they look as you expected?

Problem II Equilibria and stability

In 1798 Reverend T.R. Malthus postulated[1] the following dynamical system to model growth of a human population:

$$\dot{x}(t) = rx(t) \tag{3}$$

where $r > 0, x(t) \in \mathbb{R}$ and $t \in \mathbb{R}_+$.

(a) Is the system (3) linear? What is the order of the system?

(b) Compute all equlibira of the system and determine their stability.

(c) Compute the solution $x(t)$ of system (3) starting at $x(0) = x_0 > 0$. What is its limit as $t \to \infty$?

Subsequent research in population dynamics postulated[2] the following system to model population growth in the presence of limited resources:

$$\dot{x}(t) = rx(t)\left(1 - \frac{x(t)}{K}\right) \tag{4}$$

where $r > 0, K > 0, x(t) \in \mathbb{R}$ and $t \in \mathbb{R}_+$.

(d) Is the system (4) linear? What is the order of the system?

(e) Compute all equlibira of the system and determine their stability using linearization.

(f) Show (analytically) that

$$x(t) = \frac{Kx_0e^{rt}}{K + x_0(e^{rt} - 1)} \tag{5}$$

is the solution of system (4) starting at initial condition $x(0) = x_0 > 0$. Compute the limit of the solution as $t \to \infty$ and relate to your anwer to **(e)**. Comment on the relation of this solution and the one computed in **(c)** as $K \to \infty$.

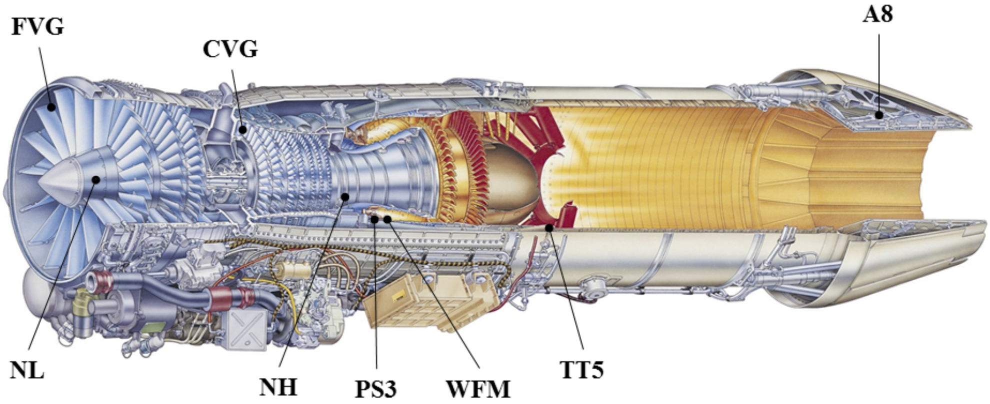
(g) Simulate the system (4) for some relevant and interesting values of the parameters, without using the closed-form solution (5). If using MATLAB, a useful command could be **ode45**.

[1]T.R. Malthus, "An Essay on the Principle of Population" *Printed for J. Johnson, in St. Paul's Church-Yard, London*, London, UK, 1798.

[2]Hutchinson, G. Evelyn. "Circular causal systems in ecology." *Annals of the New York Academy of Sciences* 50.4 (1948): 221-246.

Problem III Controller design for a jet engine

The jet engine RM12 is used in the fighter aircraft JAS 39 Gripen version A/B/C/D, and is schematically sketched in the following picture:



A sketch of the RM12 jet engine.

One proposed[3] design method for the feedback control of the engine is the $\mathcal{H}_\infty$ method. Of course, a jet engine is a nonlinear system, but for a certain working point, the dynamics of the system can be approximated with a linear model. Writing the linear model on state space form, it has 30 states. With some model reduction (see, e.g., section 3.6 in the course literature Glad-Ljung if you are interested in the principles), the model can be reduced to 9 states, 4 inputs and 5 outputs. The inputs are

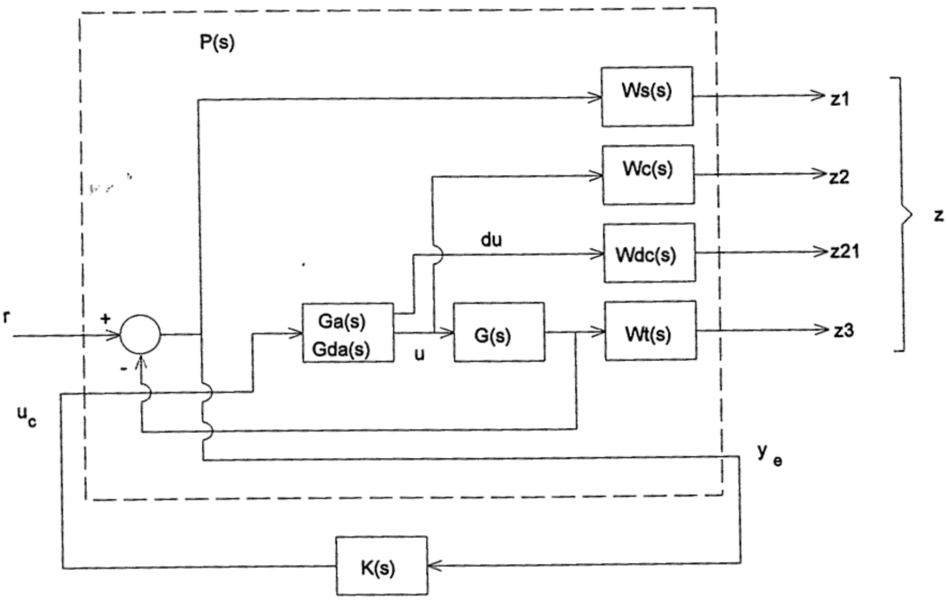
- u_{WFM} fuel flow to the main combustion chamber
- u_{FVG} the angle of the inlet fan vanes
- u_{CVG} the angle of the inlet compressor vanes
- u_{A8} the area of the outlet nozzle.

The really interesting control variable for a jet engine is, of course, the thrust. The thrust can however not be measured in-flight, instead a number of different parameters are measured:

- u_{NL} the rotational speed of the low pressure axis
- y_{EPR} engine pressure ratio; the ratio between the pressure in the afterburner and the intake
- y_{NH} the rotational speed of the low pressure axis
- y_{TT5} the turbine downstream temperature
- y_{PS3} the compressor downstream static pressure.

In addition to the model of the engine, each actuator (e.g., the actuator for u_{WFM}, a fuel flow, is a valve) is modelled as a first order system. Each input has its own actuator, and the models of the actuators are gathered in the transfer function block G_A in the figure below. There are also limits for the rate of change in the inputs (e.g., the fan vanes may not be rotated too fast), hence a weighting function W_{dc} for the derivatives of the control signals is added. That is, the extended system with the weighting functions looks as follows (cf. figure 10.1 in the course literature):

[3] Härefors, M. "A Study in Jet Engine Control Control Structure Selection and Multivariable Design." *Dissertation, Chalmers University of Technology*, 1999.



Block diagram of the extended system with weighting functions W_s, W_c, W_{dc}, W_t, the actuator model G_a and G_{da} (describing the derivatives), the engine model G and the controller K.

To describe the present limitations and the desired properties, the weighting functions have to be of the following orders, respectively: W_s of order 5, W_c of order 2, W_{dc} of order 0, and W_t of order 7. (Remember they are multivariable, hence the orders may be rather high even for 'simple' requirements.)

(a) Of which order can you expect the controller to be, when using the $\mathcal{H}_\infty$ design method? What had been the order, if no model reduction is used?

The answer in **(a)** suggests that another approach might be worth examining. Hence, the next step is to look at a decentralized controller, using RGA analysis, for some control signals.

(b) Now, the coupling between the inputs $[u_{WFM} \;\; u_{A8}]^T$ and the outputs $[y_{NL} \;\; y_{TT41}]^T$ (where u_{TT41} is the turbine inlet temperature) is analyzed. The steady-state RGA matrix is given by

$$\begin{pmatrix} 0.212 & 0.788 \\ 0.788 & 0.212 \end{pmatrix} \tag{6}$$

and the RGA matrix for the crossover-region at 10 rad/s is

$$\begin{pmatrix} -0.078 - 0.008i & 1.088 + 0.008i \\ 1.088 + 0.008i & -0.078 - 0.008i \end{pmatrix}. \tag{7}$$

Does this pair appear to be suitable for a decentralized controller? If so, which pairing should be preferred? If not, could (static) decoupling be used for this problem?

(b) Next, the coupling between the inputs $[u_{WFM} \;\; u_{A8}]^T$ and the outputs $[y_{NL} \;\; y_{FPR}]^T$ is analyzed (where y_{FPR} is the ratio between the pressure before and after the fan). The steady-state RGA matrix is given by

$$\begin{pmatrix} -2.779 & 3.779 \\ 3.779 & -2.779 \end{pmatrix} \tag{8}$$

and the RGA matrix for the crossover-region at 10 rad/s is

$$\begin{pmatrix} 0.801 + 0.296i & 0.199 - 0.296i \\ 0.199 - 0.296i & 0.801 + 0.296i \end{pmatrix}. \tag{9}$$

Does this pair appear to be suitable for a decentralized controller? If so, which pairing should be preferred? If not, could (static) decoupling be used for this problem?