Uppsala University
Department of Information Technology
Division of Systems and Control
September 29, 2015

Automatic control III

Homework assignment 3
2015

Deadline (for this assignment):
Thursday October 15 (2015), 23:59

All homework assignments are compulsory and form an important part of the examination.

This assignment is to be solved **individually**. The solution has to be handed in in written form as a pdf file.

Instructions on how to hand in the assignments are found on the course homepage. The solutions should be clearly presented, with well motivated reasoning and clearly stated answers. The solutions should be carefully written, with satisfactory equation typesetting, equation numbering, complete sentences, etc. However, no introduction, abstract, etc is needed, but just solutions to the problems.

Before handing in your solutions, make sure (for each exercise) that

(I) You have answered all questions

(II) Your answer is reasonable

Solutions not fulfilling (I) and (II) will be rejected.

Problem I Analysis of nonlinear feedback systems

The Van der Pol oscillator[1] is a well-studied[2] and widely used[3] example of a second order system with a limit cycle. The system is governed by the differential equation

$$\frac{d^2y}{dt^2} - \mu(1 - y^2)\frac{dy}{dt} + y = 0, \tag{1}$$

where $\mu > 0$. Introduce the state variables $x_1 = y$ and $x_2 = \dot{y}$. The system (1) is then equivalent with the state space representation

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= \mu(1 - x_1^2)x_2 - x_1.\end{aligned}$$

(a) Perform a phase plane analysis of the Van der Pol oscillator, i.e. determine and characterize all stationary points.

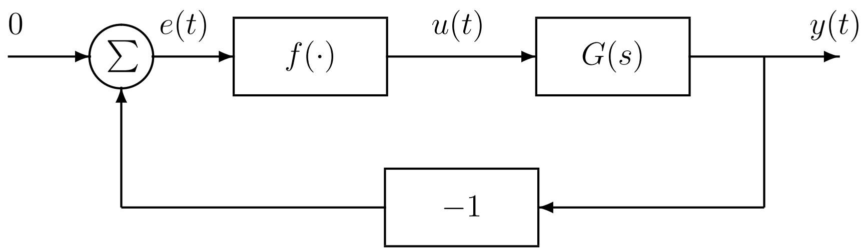
(b) Use Matlab to plot the phase portrait for $\mu = 0.1$, $\mu = 1.0$ and $\mu = 4.0$ respectively.

(c) Let $u = -y^3$. Show that the system

$$\frac{d^2y}{dt^2} - \mu\frac{dy}{dt} + y = \frac{\mu}{3} \cdot \frac{du}{dt} \tag{2}$$

is equivalent with (1) for this particular choice of u.

(d) The system (2), with $u = -y^3$, can be represented as the feedback system in the block-diagram below.



What is $G(s)$ and $f(\cdot)$ in this particular case?

[1] B. van der Pol, "A theory of the amplitude of free and forced triode vibrations", *Radio Review*, 1, 701-710, 754-762, 1920

[2] See, e.g., Parlitz, Ulrich, and Werner Lauterborn. "Period-doubling cascades and devil's staircases of the driven van der Pol oscillator." *Physical Review A* 36.3 (1987): 1428, and Rand, R. H., and P. J. Holmes. "Bifurcation of periodic motions in two weakly coupled van der Pol oscillators." *International Journal of Non-Linear Mechanics* 15.4 (1980): 387-399.

[3] Used, e.g., in seismology (Cartwright, Julyan HE, et al. "Dynamics of elastic excitable media." *International Journal of Bifurcation and Chaos* 9.11 (1999): 2197-2202) and biology (FitzHugh, Richard. "Impulses and physiological states in theoretical models of nerve membrane." Biophysical journal 1.6 (1961): 445-466, Nagumo, Jinichi, S. Arimoto, and S. Yoshizawa. "An active pulse transmission line simulating nerve axon." *Proceedings of the IRE* 50.10 (1962): 2061-2070.)

(e) Determine the sector and the circle in the circle criterion corresponding to this particular $f(\cdot)$. Also show that the circle criterion is not fulfilled in this case.

(f) Show that the describing function method indicates a (stable) limit cycle for the Van der Pol oscillator. Determine the amplitude C and the frequency ω indicated by the describing function for the cases $\mu = 0.1$, $\mu = 1.0$ and $\mu = 4.0$ respectively. Compare with the real values from simulations.

Problem II Feedback design for nonlinear systems

A DC motor

$$\Theta(s) = \frac{1}{s(s+1)}U(s)$$

is used as an actuator in a position servo. The input u is the voltage over the motor, and θ is the angle of the motor axis. A gear box is used to transform the rotational motion to linear motion. Due to an imperfection in the manufacture there is a backlash in the gear box.

Thus the linear position is $y = f(\theta)$, where f represents a backlash with $H = D = 0.02$, and its asociated describing function (for $C \geq 0.02$) is

$$\operatorname{Re} Y_f(C) = \frac{1}{\pi}\left[\frac{\pi}{2} + \arcsin\left(1 - \frac{0.04}{C}\right) + 2\left(1 - \frac{0.04}{C}\right)\sqrt{\frac{0.02}{C}\left(1 - \frac{0.02}{C}\right)}\right],$$

$$\operatorname{Im} Y_f(C) = -\frac{0.08}{\pi C}\left(1 - \frac{0.02}{C}\right).$$

(a) Assume that proportional control is used, i.e. $U(s) = K(R(s) - Y(s))$. How large values of the gain K can be used if a limit cycle is to be avoided according to the describing function method? Compare with simulations. If the results do not agree, try to explain why.

(b) Assume that the following requirements should be fulfilled:

- In the step response (from r to y) the rise time should be $T_r \leq 0.1$ seconds, and the overshoot should be $M \leq 20\%$,
- the controller $F(s)$ must have integral action,
- any latent oscillation in stationarity should have a frequency $\omega \leq 5$ rad/s and an amplitude $C \leq 0.025$ at θ.

Design a controller that meets the requirements (show this in simulations). Also analyse your obtained loop gain using the describing function method and compare these results with your simulations.

Problem III Optimal control

Design a feedback control $u(t)$ for the system

$$\dot{x}(t) = x(t) + u(t) + 1 \tag{3}$$

with $x(0) = 0$, minimizing the criterion

$$\int_0^T \left(x(t) + u^2(t)\right) dt \tag{4}$$

for a given value of T. Also plot your $u(t)$ and the evolution of $x(t)$ for this input. Using the plots, give an intuitive explanation why this is the solution to the given problem.