Exam in Automatic Control III
Reglerteknik III 5hp

Date: December 12, 2011

Venue: Polacksbacken, skrivsal

Responsible teacher: Hans Norlander.

Aiding material: Textbooks (by Glad/Ljung), calculator, mathematical handbooks, copies of OH slides.

Preliminary grades: 13p for grade 4, 21p for grade 5. Maximum score is 30p.

Use separate sheets for each problem, i.e. no more than one problem per sheet. Write your exam code on every sheet.

Important: Your solutions should be well motivated unless else is stated in the problem formulation! Vague or lacking motivations may lead to a reduced number of points.

Your solutions can be given in Swedish or in English.

Good luck!

Problem 1 Consider the system

$$Y(s) = \frac{1}{s+1}\begin{bmatrix} 4.5\frac{s+3}{s+4} & 2 \\ 2 & 1 \end{bmatrix} U(s).$$

(a) Determine the relative gain array, RGA($G(s)$). Assume that the system should be controlled by decentralized control, with a cross-over frequency of approximately 2 rad/s. Is there any input-output pairing that should be preferred or avoided? **(3p)**

(b) Determine the poles and zeros of the system, including multiplicity. What is the order of a minimal realisation of the system? **(3p)**

(c) Based on your results from (b), discuss possible constraints on the bandwidth ω_B of the closed loop system considering that reasonable stability margins should still be achievable. **(2p)**

(d) Design an IMC controller, using the λ-tuning technique. Choose a reasonable value of λ. Also determine the resulting sensitivity function. **(3p)**

Problem 2 You have been asked to design a controller for a stable minimum phase system, and are given the following specifications (interpreted in control theory terminology):

- $|S(i\omega)| < 0.01$ for $\omega \leq 1$ rad/s
- $|T(i\omega)| < 0.01$ for $\omega \geq 80$ rad/s

(a) "Translate" these specifications to corresponding requirements for the loop gain $|G(i\omega)F_y(i\omega)|$. **(3p)**

(b) Use your expertise in control theory to judge whether or not these specifications are feasible for control design.
Hint: Bode's relation might be useful. **(3p)**

Problem 3 Robust loop shaping, according to Glover-McFarlane's approach, is used for the control of a DC motor. A preliminary proportional control is used yielding the loop gain

$$G_o(s) = \frac{24}{s(s+1)}.$$

The matrices X and Z are obtained as the solutions of the pertinent Riccati equations, and XZ has the eigenvalues

$$\lambda_1 = 0.1268 \quad \text{and} \quad \lambda_2 = 4.4357.$$

Then, for some $\alpha > 1$, a robustifying controller $F_y(s)$ is determined.

(a) Show that the system

$$G_1(s) = \frac{24}{s(s-1)}$$

belong to the class of systems that are *guaranteed* to be stabilized by the controller $F_y(s)$ above (for some $\alpha > 1$). **(4p)**

(b) Explain why the robust stability criterion

$$\| \Delta_G T \|_\infty < 1$$

never can be used to guarantee the stability of the closed loop system (for any controller $F_y(s)$) for the case where $G_o(s)$ above is the nominal model and $G_1(s)$ is a possible true system. **(3p)**

Problem 4 An inverted pendulum has the state space model

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \sin x_1 + u, \end{aligned}$$

where x_1 is the angular deviation from the vertical line (in erected position), and u is the input in form of an external torque about the hinged attachment.

(a) Use the Lyapunov function

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2)$$

as a tool for choosing the parameters α_i and β_i, $i = 1, 2$, in the control law

$$u = -(\alpha_1 x_1 + \alpha_2 x_2 + \beta_1 \sin x_1 + \beta_2 \sin x_2),$$

so that the equilibrium point $x = 0$ becomes asymptotically stable. **(3p)**

(b) When the inverted pendulum model is linearized around the equilibrium point $x = 0$, the linear model has the poles ± 1. The system is supposed to be stabilized by a linear sampling controller. Suggest a suitable sampling interval in order to achieve reasonable performance. **(3p)**

Solutions to the exam in Automatic Control III, 2011-12-12:

1. (a) Definition: $\text{RGA}(G(s)) = G(s). * (G^{-1}(s))^T$.

$$G^{-1}(s) = (s+1)\frac{1}{4.5\frac{s+3}{s+4} - 4}\begin{bmatrix} 1 & -2 \\ -2 & 4.5\frac{s+3}{s+4} \end{bmatrix} = \frac{(s+1)(s+4)}{0.5(s-5)}\begin{bmatrix} 1 & -2 \\ -2 & 4.5\frac{s+3}{s+4} \end{bmatrix}$$

Thus

$$\text{RGA}(G(s)) = \frac{2(s+4)}{s-5}\begin{bmatrix} 4.5\frac{s+3}{s+4} & -4 \\ -4 & 4.5\frac{s+3}{s+4} \end{bmatrix} = \begin{bmatrix} 9\frac{s+3}{s-5} & -8\frac{s+4}{s-5} \\ -8\frac{s+4}{s-5} & 9\frac{s+3}{s-5} \end{bmatrix}.$$

Then we get

$$\text{RGA}(G(0)) = \begin{bmatrix} -\frac{27}{5} & \frac{32}{5} \\ \frac{32}{5} & -\frac{27}{5} \end{bmatrix}.$$

One should avoid pairing asociated with negative elements in $\text{RGA}(G(0))$, so here the pairing $u_1 \leftrightarrow y_1$, $u_2 \leftrightarrow y_2$ should be avoided.

(b) The minors are $\frac{x}{s+1}$, $\frac{4.5(s+3)}{(s+1)(s+4)}$ and

$$\det G(s) = \frac{1}{(s+1)^2}\left(4.5\frac{s+3}{s+4} - 4\right) = \frac{0.5(s-5)}{(s+1)^2(s+4)}.$$

Theorem 3.5 ⇒ the pole polynomial is the least common denominator of all minors to $G(s)$, which in this case is $(s+1)^2(s+4)$. The system has one pole in -4 and a double pole in -1. A minimal realisation must then be of third order. Theorem 3.6 ⇒ the zero polynomial is the greatest common divisor of the numerators of the maximal minors of $G(s)$ (with the pole polynomial as denominator). Here the maximal minor is $\det G(s)$ (as for all square systems), and the zero polynomial is $s-5$. The system has one zero in $+5$.

(c) Non-minimum phase zeros (in the right half plane) limit the achievable bandwidth. A rule of thumb suggest $\omega_B < z/2$ (where z is a RHP zero), which here means that $\omega_B < 2.5$ rad/s.

(d) With IMC

$$F_y(s) = (I - Q(s)G(s))^{-1}Q(s), \quad T(s) = G(s)Q(s) \quad \text{and} \quad S(s) = I - G(s)Q(s).$$

With λ-tuning $Q(s) = \frac{1}{(\lambda s+1)^n}G^{-1}(s)$, but this is not directly applicable when $G(s)$ has non-minimum phase zeros. In this case

$$G^{-1}(s) = \frac{(s+1)(s+4)}{0.5(s-5)}\begin{bmatrix} 1 & -2 \\ -2 & 4.5\frac{s+3}{s+4} \end{bmatrix} = -\frac{2(s+1)(s+4)}{5(-s/5+1)}\begin{bmatrix} 1 & -2 \\ -2 & 4.5\frac{s+3}{s+4} \end{bmatrix}$$

(from (a)), with a zero in $+5$. Two standard solutions (according to Glad/Ljung):

2. (a) Ignore the factor $-s/5+1$ ⇒

$$Q(s) = \frac{-s/5+1}{(\lambda s+1)^2}G^{-1}(s) = -\frac{2(s+1)(s+4)}{5(\lambda s+1)^2}\begin{bmatrix} 1 & -2 \\ -2 & 4.5\frac{s+3}{s+4} \end{bmatrix}$$

$$\Rightarrow \quad F_y(s) = -\frac{2(s+1)(s+4)}{s(\lambda^2 s + 2\lambda + 0.2)}\begin{bmatrix} 1 & -2 \\ -2 & 4.5\frac{s+3}{s+4} \end{bmatrix},$$

$$S(s) = I - G(s)Q(s) = I - \frac{-s/5+1}{(\lambda s+1)^2}I = \frac{s(\lambda^2 s + 2\lambda + 0.2)}{(\lambda s+1)^2}I.$$

2. (b) Replace the factor $-s/5+1$ with $s/5+1 \Rightarrow$

$$Q(s) = \frac{-s/5+1}{(\lambda s+1)(s/5+1)} G^{-1}(s) = -\frac{2(s+1)(s+4)}{5(\lambda s+1)(s/5+1)} \begin{bmatrix} 1 & -2 \\ -2 & 4.5\frac{s+3}{s+4} \end{bmatrix}$$

$$\Rightarrow \quad F_y(s) = -\frac{2(s+1)(s+4)}{s(\lambda s+5\lambda+2)} \begin{bmatrix} 1 & -2 \\ -2 & 4.5\frac{s+3}{s+4} \end{bmatrix},$$

$$S(s) = I - G(s)Q(s) = I - \frac{-s/5+1}{(\lambda s+1)(s/5+1)} I = \frac{s(\lambda s+5\lambda+2)}{(\lambda s+1)(s+5)} I.$$

The bandwidth is $\omega_B \approx 1/\lambda$, so a suitable choice here is e.g. $\lambda = 0.5$.

2. (a) For small ϵ the approximate relations

$$|S(i\omega)| < \epsilon \quad \Leftrightarrow \quad |G(i\omega)F_y(i\omega)| > 1/\epsilon, \quad |T(i\omega)| < \epsilon \quad \Leftrightarrow \quad |G(i\omega)F_y(i\omega)| < \epsilon$$

hold. Here this means that the loop gain should be

- $|G(i\omega)F_y(i\omega)| > 1/0.01 = 100$ for $\omega \leq 1$ rad/s
- $|G(i\omega)F_y(i\omega)| < 0.01$ for $\omega \geq 80$ rad/s.

(b) The cross-over frequency ω_c must lie in the interval $[1, 80]$ rad/s. Bode's relation states that for a minimum phase system

$$\arg G(i\omega) = \int_{-\infty}^{\infty} \frac{d}{dx} f(x) \psi(x - \log\omega) dx,$$

where $\psi(x) = \log\frac{e^x+1}{|e^x-1|}$, $f(x) = \log|G(i\omega)|$ and $x = \log x$. Thus $\frac{d}{dx}f(x)$ is the slope of the amplitude curve in the Bode plot. In intervals where there are only small variations in the slope, a good approximation is $\arg G(i\omega) \approx \frac{\pi}{2}\frac{d}{dx}f(x)$. For a stable closed loop system $\arg G(i\omega_c) > -\pi$ is required (the Nyquist criterion), and for reasonable performance the phase margin should be sufficiently large. Thus the slope around ω_c should be larger than -2. Approximate $f(x)$ with a straight line such that the specifications in (a) are fulfilled, i.e.

$$f(x) = kx + m, \quad f(x_1) = y_1, \quad f(x_2) = y_2,$$

where $x_1 = \log 1$, $x_2 = \log 80$, $y_1 = \log 100$ and $y_2 = \log 0.01$. The slope is then

$$k = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\log 0.01 - \log 100}{\log 80 - \log 0} = \frac{\log 0.0001}{\log 80} \approx -2.1.$$

Thus, it will be hard to find a stabilizing controller giving reasonable performance and stability margins.

3. (a) Let $G_o(s) = M^{-1}(s)N(s)$ be a nominal model, where $M(s)$ and $N(s)$ are stable rational transfer functions given by

$$I = M^T(-s)M(s) + N^T(-s)N(s).$$

Then all transfer functions given by

$$G_p(s) = (M(s) + \Delta_M(s))^{-1}(N(s) + \Delta_N(s)),$$

where $\Delta_M(s)$ and $\Delta_N(s)$ are any stable transfer functions for which the condition $\| \begin{bmatrix}\Delta_M(s) & \Delta_N(s)\end{bmatrix} \|_\infty < 1/\gamma$ is fulfilled, are stabilized by the Glover-McFarlane controller $F_y(s)$ based on the nominal model. For this present SISO system we can write

$$G(s) = \frac{N(s)}{M(s)} = \frac{\frac{24}{s^2+\alpha s+\beta}}{\frac{s(s+1)}{s^2+\alpha s+\beta}},$$

where $\alpha, \beta > 0$ ($M(s)$ and $N(s)$ must be stable). With $\Delta_N(s) = 0$ and $\Delta_M(s) = -\frac{2s}{s^2+\alpha s+\beta}$ we get

$$\frac{N(s) + \Delta_N(s)}{M(s) + \Delta_M(s)} = \frac{\frac{24}{s^2+\alpha s+\beta}}{\frac{s(s+1)-2s}{s^2+\alpha s+\beta}} = \frac{24}{s(s-1)} = G_1(s)$$

We have $\lambda_m = 4.4357$ as the greatest eigenvalue of XZ, and thus

$$\gamma = \alpha\sqrt{1+\lambda_m} > \sqrt{1+\lambda_m} \approx 2.3315.$$

We must find α and β and then show that $|\Delta_M(i\omega)| < 1/\gamma \approx 0.4289$ for all ω (since $\| \begin{bmatrix}\Delta_M & \Delta_N\end{bmatrix} \|_\infty = \| \Delta_M \|_\infty$, as $\Delta_N = 0$). First find α and β:

$$\begin{gathered}
1 = M(-s)M(s) + N(-s)N(s) \quad \Leftrightarrow \\
(s^2 - \alpha s + \beta)(s^2 + \beta s + \alpha) = (s^2 - s)(s^2 + s) + 24^2 \\
\Leftrightarrow \quad s^4 + (2\beta - \alpha^2)s + \beta^2 = s^4 - s^2 + 24^2
\end{gathered}$$

Equating coefficients for equal powers of s gives

$$\begin{cases} 2\beta - \alpha^2 & = -1 \\ \beta^2 & = 24^2 \end{cases} \quad \Rightarrow \quad \begin{cases} \alpha & = 7 \\ \beta & = 24 \end{cases}$$

We get

$$|\Delta_M(i\omega)|^2 = \frac{(2\omega)^2}{(24-\omega^2)^2 + (7\omega)^2} = \frac{4\omega^2}{\omega^4 + \omega^2 + 24^2} = \frac{4x}{x^2 + x + 24^2} = f(x)$$

with $x = \omega^2$. To find maximum, solve $0 = \frac{df}{dx}$:

$$\begin{aligned}
0 = \frac{df}{dx} = \frac{4(x^2 + x + 24^2) - 4x(2x+1)}{(x^2 + x + 24^2)^2} = \frac{4(-x^2 + 24^2)}{(x^2 + x + 24^2)^2} \quad \Rightarrow \\
x = 24 \quad \Rightarrow \quad |\Delta_M(i\omega)| \leq \sqrt{f(24)} = \sqrt{\frac{4 \cdot 24}{24^2 + 24 + 24^2}} \approx 0.286 < \frac{1}{\gamma},
\end{aligned}$$

which proves the statement.

(b) With $G_1(s) = G_o(1 + \Delta_G(s))$ we have

$$\Delta_G(s) = \frac{G_1(s) - G_o(s)}{G_o(s)} = \frac{\frac{24}{s(s-1)} - \frac{24}{s(s+1)}}{\frac{24}{s(s+1)}} = \frac{2}{s-1}.$$

Since $\Delta_G(s)$ is unstable the small gain theorem is not applicable.

4. (a) We get

$$\begin{aligned}\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2 &= x_1x_2 + x_2\sin x_1 + x_2u \\ &= x_1x_2 + x_2\sin x_1 - \alpha_1x_1x_2 - \alpha_2x_2^2 - \beta_1x_2\sin x_1 - \beta_2x_2\sin x_2,\end{aligned}$$

so with $\alpha_1 = 1$, $\beta_1 = 1$, $\alpha_2 > 0$ and $\beta_2 = 0$ we have $\dot{V} = -\alpha_2x_2^2 \leq 0$. Since $\dot{x}_2 = \sin x_1 \neq 0$, except for $x_1 = 0$, in the interval $|x_1| < \pi$, no solution can stay where $\dot{V} = 0$ outside $x = 0$, and according to Theorem 12.4 $x = 0$ is an asymptotically stable equilibrium point.

(b) According to the rule of thumb the bandwidth should be chosen $\omega_B > 2p$ where p is an unstable pole. Thus we should have $\omega_B > 2$ rad/s. Furthermore, the sampling frequency should be chosen as at least $\omega_s \geq 10\omega_B$. So $\omega_s \geq 20$ rad/s $\Rightarrow h \leq \frac{2\pi}{\omega_s} \approx 0.31$ seconds.