Due: Tuesday 17 February 2004 1pm

**Exercise 1**

Prove or disprove each of the following.

a) \( C[\text{skip};c] = C[c] \).

b) \( C[\text{if } b \text{ then } (c_0;c) \text{ else } (c_1;c)] = C[(\text{if } b \text{ then } c_0 \text{ else } c_1);c] \).

c) \( C[\text{if } b \text{ then } (c;c_0) \text{ else } (c;c_1)] = C[(\text{if } b \text{ then } c) \text{ else } c_1] \).

d) If \( C[c_1] = C[c_2] \) then \( C[\text{while } b \text{ do } c_1] = C[\text{while } b \text{ do } c_2] \).

**Exercise 2**

Let \( g : \Sigma \rightarrow \Sigma \) given by \( g(\sigma) = \sigma[x \rightarrow \max(0,\sigma(x))] \). Using the semantic definition prove that \( C[\text{while } x < 0 \text{ do } x := x + 1] = g \), that is, prove that \( g \) is the least fixed point of the operator \( \Gamma \) corresponding to the while loop.

**Exercise 3**

Let \( O \) and \( T_\bot \) be the cpos described by the following Hasse diagrams:

\[
O = \begin{array}{c}
\top \\
\downarrow \\
\bot
\end{array} \quad T_\bot = \begin{array}{c}
\text{false} \\
\downarrow \\
\text{true}
\end{array}
\]

a) How many functions \( f : O \rightarrow T_\bot \) are there?
How many functions \( f : T_\bot \rightarrow O \) are there?

b) How many continuous functions \( f : O \rightarrow T_\bot \) are there?
How many continuous functions \( f : T_\bot \rightarrow O \) are there?

c) Which functions \( f : O \rightarrow T_\bot \) are not continuous?
Which functions \( f : T_\bot \rightarrow O \) are not continuous?

d) Draw the cpos \([O \rightarrow T_\bot]\) and \([T_\bot \rightarrow O]\) as Hasse diagrams, with functions ordered pointwise: \( f \sqsubseteq g \iff f(x) \sqsubseteq g(x) \) for all \( x \).

A function \( f : D \rightarrow E \) between cpos \((D, \sqsubseteq_D)\) and \((E, \sqsubseteq_E)\) is continuous iff

i) \( f \) is monotonic; and

ii) \( f \) “preserves lubs of chains”: for all chains \( d_0 \sqsubseteq_D d_1 \sqsubseteq_D d_2 \sqsubseteq_D \cdots \) in \( D \),

\[
f \left( \bigcup_{n \in \omega} d_n \right) = \bigcup_{n \in \omega} f(d_n).
\]

**Exercise 4**

Prove that the first clause is redundant, that is, prove the following:

If \( f \) preserves lubs of chains then \( f \) is monotonic.