The proof of Lemma 3.10 on page 25 in the Course Notes part 1 has a step which is somewhat mysterious:

\[ \sigma[x\mapsto 1, y\mapsto (\sigma(x) - 1)!\sigma(x)\cdot y] = g(\sigma[x\mapsto \sigma(x) - 1, y\mapsto \sigma(x)\cdot y]) \]  

(1)

There is nothing deep about this, but we need to do a lot of simplifications. Recall the definition of \( g \) from the Course Notes:

\[
g(\sigma) = \begin{cases} 
\sigma[x\mapsto 1, y\mapsto \sigma(x)\cdot y] & \text{if } \sigma(x) > 1, \\
\sigma & \text{if } \sigma(x) \leq 1. 
\end{cases}
\]

In (1), the argument to \( g \) is \( \sigma[x\mapsto \sigma(x) - 1, y\mapsto \sigma(x)\cdot y] \) We need to substitute this into the definition of \( g \). One complication here is that the \( \sigma \) in the definition of \( g \) and the \( \sigma \) in (1) are different things! \( \sigma \) in the definition of \( g \) is the formal argument of a function definition. Since the names of formal arguments are arbitrary\(^1\), we can change it to something else, e.g. \( \sigma' \). We get:

\[
g(\sigma') = \begin{cases} 
\sigma'[x\mapsto 1, y\mapsto \sigma'(x)\cdot y] & \text{if } \sigma'(x) > 1, \\
\sigma' & \text{if } \sigma'(x) \leq 1. 
\end{cases}
\]

This will make the actual substitution of the argument expression \( \sigma[x\mapsto \sigma(x) - 1, y\mapsto \sigma(x)\cdot y] \) easier to follow. Substituting this in the definition of \( g \), we obtain

\[
g(\sigma[x\mapsto \sigma(x) - 1, y\mapsto \sigma(x)\cdot y]) =
\]

\[
\begin{cases} 
\sigma[x\mapsto \sigma(x) - 1, y\mapsto \sigma(x)\cdot y][x\mapsto 1, y\mapsto \sigma(x)\cdot y][x\mapsto \sigma(x) - 1, y\mapsto \sigma(x)\cdot y][x]\sigma[x\mapsto \sigma(x) - 1, y\mapsto \sigma(x)\cdot y][y] & \text{if } \sigma[x\mapsto \sigma(x) - 1, y\mapsto \sigma(x)\cdot y](x) > 1, \\
\sigma[x\mapsto \sigma(x) - 1, y\mapsto \sigma(x)\cdot y] & \text{if } \sigma[x\mapsto \sigma(x) - 1, y\mapsto \sigma(x)\cdot y](x) \leq 1.
\end{cases}
\]

(3)

We will start with the interesting (read “complicated”) case, namely when \( \sigma[x\mapsto \sigma(x) - 1, y\mapsto \sigma(x)\cdot y](x) > 1 \).

We’ll make the following general observations:

\[ \sigma[x \mapsto \cdots][x \mapsto \epsilon] = \sigma[x \mapsto \epsilon] \]  

(4)

\(^1\text{as long as we don’t get any collisions}\)
\[
\sigma[x\to\epsilon](x) = \epsilon
\]  
(5)

Why is this? In (4), the mapping of \(x\) by \(\sigma\) is changed \textit{twice}. Only the second (rightmost) change will be visible, the other has been “overwritten” or “shadowed”.

In (5), it does not matter what \(\sigma(x)\) is since the mapping of \(x\) by \(\sigma\) is changed to be \(\epsilon\).

The first case of (3) was

\[
\sigma[x\mapsto\sigma(x)-1,y\mapsto\sigma(x)\sigma(y)][x\mapsto\epsilon,y\mapsto\epsilon][x\mapsto\sigma(x)-1,y\mapsto\sigma(x)\sigma(y)][x\mapsto\sigma(x)-1,y\mapsto\sigma(x)\sigma(y)][x\mapsto\sigma(x)-1,y\mapsto\sigma(x)\sigma(y)][y] \]
(6)

Here we have a double change to both \(x\) and \(y\), so according to (4) we can throw away the first change, obtaining

\[
\sigma[x\mapsto\sigma(x)-1,y\mapsto\sigma(x)\sigma(y)][x\mapsto\epsilon][x\mapsto\sigma(x)-1,y\mapsto\sigma(x)\sigma(y)][y] \]
(7)

Here, the mapping of \(y\) is changed to

\[
\sigma[x\mapsto\sigma(x)-1,y\mapsto\sigma(x)\sigma(y)][x][y] \]
(8)

In this expression, as the mappings of both \(x\) and \(y\) have been explicitly changed, we can use (5) twice to read out the mappings directly from the change expression without involving \(\sigma\). We get:

\[
(\sigma(x) - 1)!\sigma(x)\sigma(y)
\]

Putting this back into (7), we get

\[
\sigma[x\mapsto\epsilon,y\mapsto\sigma(x)\sigma(y)][x\mapsto\sigma(x)-1][y\mapsto\sigma(x)\sigma(y)]
\]  
(9)

Since we obtained (9) by equalities from (2), the original equality (1) holds.

It now remains to do the second case of (3). The condition for that case was \(\sigma(x) - 1, y\mapsto\sigma(x)\sigma(y)](x) \leq 1\). Using (5) again, we see that this is the same as \(\sigma(x) - 1 \leq 1\), or (adding 1 to both sides) \(\sigma(x) \leq 2\). Since a premise of Lemma 3.10 was that \(\sigma(x) > 1\) it follows that in this case \(\sigma(x) = 2\), exactly.

In this case \(g(\sigma') = \sigma'\), so the application of \(g\) to the argument \(\sigma\mapsto\sigma(x)-1,y\mapsto\sigma(x)\sigma(y)\) will simply be the same argument. Make that application in the equality we want to show, (1), and we get:

2
\[\sigma[x+1,y\mapsto \sigma(x)-1]\sigma(x)\sigma(y)] = \sigma[x+\sigma(x)-1,y\mapsto \sigma(x)\sigma(y)]\]

Substituting 2 for \(\sigma(x)\) we get

\[\sigma[x+1,y\mapsto (2-1)\sigma(y)] = \sigma[x+2-1,y\mapsto 2\sigma(y)]\]

Now it is a piece of straightforward arithmetic to show that the two sides of this equality are actually the same.