Exercise 1

Let $g : \Sigma \rightarrow \Sigma$ given by $g(\sigma) = \sigma[\alpha \mapsto \max(0, \sigma(\alpha))]$. Using the semantic definition prove that $C[\text{while } x < 0 \text{ do } x := x + 1] = g$, that is, prove that $g$ is the least fixed point of the operator $\Gamma$ corresponding to the while loop.

Exercise 2

Consider a very simple domain $U = \langle \{\star\}, id \rangle$ consisting of a single element $\star$.

(a) Draw Hasse diagrams of the following domains:

$$D_a = U \perp \quad D_c = (U + U) \perp \times (U + U) \perp$$

$$D_b = (U + U) \perp \quad D_d = [(U + U) \perp \rightarrow U \perp]$$

When applicable, use notation $[x]$ and $\text{int}_k(x)$ to name elements of these domains. Name elements of $D_d$ by $f_1, f_2, \ldots, f_n$.

(b) Consider functions $g, h : U \rightarrow U \perp$ given by

$$g(x) = [x] \quad h(x) = \perp$$

Functions $[g, h]^*$ and $[h, g]^*$ are elements of domain $D_d$. What names did you assign to them?

Exercise 3

Let $\text{Aexp}'$ be the set of extended arithmetic expressions in the language $\text{Imp}$:

$$a ::= n \mid x \mid a_0 + a_1 \mid a_0 - a_1 \mid a_0 \ast a_1 \mid b? a_0 : a_1$$

where $b \in \text{Bexp}$ and $a_0, a_1 \in \text{Aexp}'$.

Intended meaning of the $b? a_0 : a_1$ expression is as follows:

- if $b$ evaluates to true then the expression $b? a_0 : a_1$ has the value of $a_0$,
- if $b$ evaluates to false then the expression $b? a_0 : a_1$ has the value of $a_1$.

Now, write down the formal semantics for the $\text{?} -$ operator:

a) In the operational semantics (i.e., write inference rules for $(b? a_0 : a_1, \sigma) \rightarrow n$).

b) In the denotational semantics (i.e., extend the definition of $A[]$ to cover arithmetic expressions of the form $b? a_0 : a_1$).

Exercise 4

Prove that for any $\sigma \in \Sigma$, $a \in \text{Aexp}'$ and $n \in \mathbb{Z}$

$$\langle a, \sigma \rangle \rightarrow n \iff A[a] \sigma = n$$

(you can use the similar fact for Boolean expressions given in the lecture notes).

Exercise 5

Modify the ML implementation of the denotational semantics to include the language extension according to exercise 3b above. Submit a printout of the ML file with your modifications marked using a pen. Include a printout of an ML run where you demonstrate the changes by executing some simple example programs of your own invention.