The Kalman Predictor

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1 The Kalman Predictor Problem

Consider the state space system

$$\mathbf{x}(n+1) = F(n)\mathbf{x}(n) + \mathbf{v}(n) \quad (1.1)$$
$$\mathbf{y}(n) = H(n)\mathbf{x}(n) + \mathbf{e}(n) \quad (1.2)$$

where

$$\mathbf{x}(n) = \begin{pmatrix} \mathbf{x}_1(n) \\ \vdots \\ \mathbf{x}_N(n) \end{pmatrix} \qquad \mathbf{v}(n) = \begin{pmatrix} \mathbf{v}_1(n) \\ \vdots \\ \mathbf{v}_N(n) \end{pmatrix}$$

$$\mathbf{y}(n) = \begin{pmatrix} \mathbf{y}_1(n) \\ \vdots \\ \mathbf{y}_P(n) \end{pmatrix} \qquad \mathbf{e}(n) = \begin{pmatrix} \mathbf{e}_1(n) \\ \vdots \\ \mathbf{e}_P(n) \end{pmatrix}$$

$$\text{F(n)} = \begin{pmatrix} f_{11}(n) & \cdots & f_{1N}(n) \\ \vdots & \ddots & \vdots \\ f_{N1}(n) & \cdots & f_{NN}(n) \end{pmatrix}$$ $N \times N$ transition matrix

$$\text{H(n)}= \begin{pmatrix} h_{11}(n) & \cdots & h_{1N}(n) \\ \vdots & \ddots & \vdots \\ h_{P1}(n) & \cdots & h_{PN}(n) \end{pmatrix}$$ $P \times N$ measurement matrix

$\mathbf{v}(n)$ is a zero mean, white stochastic process with

$$E\left(\mathbf{v}(n)\right) = \mathbf{0}$$
$$E\left(\mathbf{v}(m)\mathbf{v}^H(n)\right) = \begin{cases} R_v & m = n \\ 0 & m \neq n \end{cases}$$

$\mathbf{e}(n)$ is a zero mean, white stochastic process with

$$E\left(\mathbf{e}(n)\right) = \mathbf{0}$$
$$E\left(\mathbf{e}(m)\mathbf{e}^H(n)\right) = \begin{cases} R_e & m = n \\ 0 & m \neq n \end{cases}$$

$\mathbf{v}(n)$ & $\mathbf{e}(n)$ are independent stochastic processes with

$$E\left(\mathbf{v}(m)\mathbf{e}^H(n)\right) = \mathbf{0} \;\; \forall \;\; m, n.$$

Let us begin by looking at the subspaces that the elements of the state space vector and the output vector lie in. It is easily seen that

1. The state space vector $\mathbf{x}(n)$ is a column vector of the state space N-tuple $\mathbf{X}(n) = \{\mathbf{x}_1(n),\ \mathbf{x}_2(n),\ \ldots\ \mathbf{x}_N(n)\}$.
2. The state space N-tuple $\mathbf{X}(n)$ lies in the subspace $\mathcal{V}_{n-1} = Sp\left\{\mathbf{V}(1),\ \mathbf{V}(2),\ \ldots\ ,\ \mathbf{V}(n-1)\right\}$ spanned by the N-tuples $\mathbf{V}(m) = \{\mathbf{v}_1(m),\ \ldots\ \mathbf{v}_N(m)\}$ for $m = 1,\ 2,\ \ldots\ , n-1$.

3. The output vector $\mathbf{y}(n)$ is a column vector of the output P-tuple $\mathbf{Y}(n) = \{\mathbf{y}_1(n),\ \mathbf{y}_2(n),\ \dots\ \mathbf{y}_P(n)\}$.

4. The output P-tuple $\mathbf{Y}(n)$ lies in the subspace $\mathcal{V}_{n-1} \bigoplus Sp\{\mathbf{E}(n)\}$, where $\mathbf{E}(n)$ is the N-tuple $\mathbf{E}(n) = \{\mathbf{e}_1(n),\ \mathbf{e}_2(n),\ \dots\ \mathbf{e}_P(n)\}$.

Note that the only measurable signal in the state space system is the output signal $\mathbf{y}(n)$. Given the above state space system the Kalman predictor is the linear least squares estimator that estimates each random vector $\mathbf{x}_i(n)$ of the state space N-tuple $\mathbf{X}(n) = \{\mathbf{x}_1(n),\ \mathbf{x}_2(n),\ \dots\ \mathbf{x}_N(n)\}$ as the linear sum of the sequence of output P-tuples $\mathbf{Y}(1),\ \mathbf{Y}(2),\ \dots\ ,\ \mathbf{Y}(n-1)$,

$$\hat{\mathbf{x}}_i(n) = \sum_{m=1}^{n-1} \mathbf{Y}(m) A_{k,m}(n),$$

that minimizes the squared error

$$\|\mathbf{x}_i(n) - \hat{\mathbf{x}}_i(n)\|^2 = E\left((\mathbf{x}_i(n) - \hat{\mathbf{x}}_i(n))\,(\mathbf{x}_i(n) - \hat{\mathbf{x}}_i(n))^*\right).$$

This is a familiar minimization problem from the notes on The Geometric Tools of Hilbert Spaces with a familiar solution. The solution is that $\hat{\mathbf{x}}_i(n)$ is the projection of $\mathbf{x}_i(n)$ onto the subspace $\mathcal{Y}_{n-1} = Sp\{\mathbf{Y}(1),\ \mathbf{Y}(2),\ \dots\ ,\ \mathbf{Y}(n-1)\}$, which we denote by $\hat{\mathbf{x}}_i(n/\mathcal{Y}_{n-1})$. Note that the dimension of the subspace $\mathcal{Y}_{n-1}$ increases with n. However, as $\hat{\mathbf{x}}_i(n/\mathcal{Y}_{n-1})$ is estimated for each time instance n it is desirable to obtain a recursive evaluation of this projection.

2 The Kalman Predictor Solution

A recursive evaluation of $\hat{\mathbf{x}}_i(n/\mathcal{Y}_{n-1})$ is easily obtained through

1. An orthogonal decomposition of $\mathcal{Y}_n$ into $\mathcal{Y}_{n-1} \bigoplus \mathcal{E}_n^{\mathcal{Y}}$, where $\mathcal{E}_n^{\mathcal{Y}}$ is the space spanned by the projection error P-tuple from projecting the P-tuple $\mathbf{Y}(n)$ onto $\mathcal{Y}_{n-1}$.

2. The state space system equations.

The subspace sequence $\mathcal{E}_n^{\mathcal{Y}}$ is often called the innovation subspace sequence as it at each time instance n represents the new data information in the present subspace that is "orthogonal" to the data information in the past subspace $\mathcal{Y}_{n-1}$.
Let us first look at the orthogonal decomposition of $\mathcal{Y}_n$. It is easily seen that

$$\begin{aligned}
\mathcal{Y}_n &= Sp\{\mathbf{Y}(1),\ \dots\ ,\ \mathbf{Y}(n-1),\ \mathbf{Y}(n)\} \\
&= Sp\{\mathbf{Y}(1),\ \dots\ ,\ \mathbf{Y}(n-1)\} + Sp\{\mathbf{Y}(n)\} \\
&= Sp\{\mathbf{Y}(1),\ \dots\ ,\ \mathbf{Y}(n-1)\} \bigoplus \underbrace{Sp\left\{\mathbf{E}^Y(n)\right\}}_{\mathcal{E}_n^{\mathcal{Y}}}
\end{aligned}$$

where $\mathbf{E}^Y(n)$ is the the projection error P-tuple from projecting the P-tuple $\mathbf{Y}(n)$ onto $\mathcal{Y}_{n-1}$, $\mathbf{E}^Y(n) = \mathbf{Y}(n) - \hat{\mathbf{Y}}(n/\mathcal{Y}_{n-1}) = \{\mathbf{e}_1^y,\ \dots\ ,\ \mathbf{e}_P^y\}$ and $\mathbf{e}_i^y = \mathbf{y}_i(n) - \hat{\mathbf{y}}_i(n/\mathcal{Y}_{n-1})$. We know from Corollary 6-4 in the notes on The Geometric Tools of Hilbert Spaces that the projection of a vector onto a subspace that is decomposed into orthogonal subspaces is the sum of the projections of the vector onto the respective subspaces. We thus have that

$$\hat{\mathbf{x}}_i(n+1/\mathcal{Y}_n) = \hat{\mathbf{x}}_i(n+1/\mathcal{Y}_{n-1}) + \hat{\mathbf{x}}_i(n+1/\mathcal{E}_n^Y).$$

The recursion for $\hat{\mathbf{x}}_i(n+1/\mathcal{Y}_n)$ is now obtained by expanding $\hat{\mathbf{x}}_i(n+1/\mathcal{Y}_{n-1})$ through the use of the state space recursion in Equation 1.1 and writing the projection $\hat{\mathbf{x}}_i(n+1/\mathcal{E}_n^Y)$

as the linear sum of the random vectors in the P-tuple $\mathbf{E}^Y(n)$. Using the state space recursion in Equation 1.1 $\hat{\mathbf{x}}_i(n+1/\mathcal{Y}_{n-1})$ can be written as

$$\begin{aligned}
\hat{\mathbf{x}}_i(n+1/\mathcal{Y}_{n-1}) &= \mathbf{P}_{\mathcal{Y}_{n-1}}\left(\mathbf{x}_i(n+1)\right) \\
&= \mathbf{P}_{\mathcal{Y}_{n-1}}\left(\sum_{j=1}^{N} f_{ij}(n)\mathbf{x}_j(n) + \mathbf{v}_j(n)\right) \\
&= \sum_{j=1}^{N} f_{ij}(n)\hat{\mathbf{x}}_j(n/\mathcal{Y}_{n-1}) + \underbrace{\hat{\mathbf{v}}_j(n/\mathcal{Y}_{n-1})}_{\mathbf{0}} \\
&= \sum_{j=1}^{N} f_{ij}(n)\hat{\mathbf{x}}_j(n/\mathcal{Y}_{n-1}).
\end{aligned}$$

The projection $\hat{\mathbf{x}}_i(n+1/\mathcal{E}_n^Y)$ is given by

$$\hat{\mathbf{x}}_i(n+1/\mathcal{E}_n^Y) = \sum_{j=1}^{P} k_{ij}(n)\mathbf{e}_j^y(n),$$

where the projection coefficient vector $k_i(n) = (k_{i1}(n), \cdots, k_{iP}(n))^T$ is given by

$$k_i(n) = \begin{pmatrix} k_{i1}(n) \\ \vdots \\ k_{iP}(n) \end{pmatrix} = \left\langle \mathbf{E}^Y(n), \mathbf{E}^Y(n) \right\rangle^{-1} \left\langle \mathbf{x}_i(n+1), \mathbf{E}^Y(n) \right\rangle.$$

Stacking the above results for $i = 1, \cdots, N$, into a vector equation we obtain

$$\begin{aligned}
\underbrace{\begin{pmatrix} \hat{\mathbf{x}}_1(n+1/\mathcal{Y}_n) \\ \vdots \\ \hat{\mathbf{x}}_N(n+1/\mathcal{Y}_n) \end{pmatrix}}_{\hat{\mathbf{x}}(n+1/\mathcal{Y}_n)} &= \underbrace{\begin{pmatrix} f_{11}(n) & \cdots & f_{1N}(n) \\ \vdots & \ddots & \vdots \\ f_{N1}(n) & \cdots & f_{NN}(n) \end{pmatrix}}_{F(n)} \underbrace{\begin{pmatrix} \hat{\mathbf{x}}_1(n/\mathcal{Y}_{n-1}) \\ \vdots \\ \hat{\mathbf{x}}_N(n/\mathcal{Y}_{n-1}) \end{pmatrix}}_{\hat{\mathbf{x}}(n/\mathcal{Y}_{n-1})} \\
&\quad + \underbrace{\begin{pmatrix} k_{11}(n) & \cdots & k_{1P}(n) \\ \vdots & \ddots & \vdots \\ k_{N1}(n) & \cdots & k_{NP}(n) \end{pmatrix}}_{K(n)} \underbrace{\begin{pmatrix} \mathbf{e}_1^y(n) \\ \vdots \\ \mathbf{e}_P^y(n) \end{pmatrix}}_{\mathbf{e}^y(n)},
\end{aligned}$$

which gives the state space vector estimate time recursion

$$\hat{\mathbf{x}}(n+1/\mathcal{Y}_n) = F(n)\hat{\mathbf{x}}(n/\mathcal{Y}_{n-1}) + K(n)\mathbf{e}^y(n),$$

where $K(n)$ is the Kalman gain given by

$$K(n) = \left\langle \mathbf{X}(n+1), \mathbf{E}^Y(n) \right\rangle^T \left\langle \mathbf{E}^Y(n), \mathbf{E}^Y(n) \right\rangle^{-T}.$$

To obtain a complete time recursive solution the above equation needs to be complemented with

1. An equation expressing the Kalman gain in other available variables, which will include the the covariance matrix of $\mathbf{e}^x(n+1) = \mathbf{x}(n+1) - \hat{\mathbf{x}}(n+1/\mathcal{Y}_n)$ denoted by $P(n+1)$.

2. A time recursion for the covariance matrix $P(n+1)$, the so called Ricatti equation.

The two main components of the Kalman gain are $\left\langle \mathbf{X}(n), \mathbf{E}^Y(n) \right\rangle^T$ and $\left\langle \mathbf{E}^Y(n), \mathbf{E}^Y(n) \right\rangle^T$. Using the measurement equation of the state space system given in Equation 1.2 the inner product $\left\langle \mathbf{E}^Y(n), \mathbf{E}^Y(n) \right\rangle$ can be expressed as

$$
\begin{aligned}
&\left\langle \mathbf{E}^Y(n), \mathbf{E}^Y(n) \right\rangle \\
&= \left\langle \mathbf{Y}(n) - \hat{\mathbf{Y}}(n/\mathcal{Y}_{n-1}), \mathbf{Y}(n) - \hat{\mathbf{Y}}(n/\mathcal{Y}_{n-1}) \right\rangle \\
&= \left\langle \mathbf{X}(n)H^T(n) + \mathbf{E}(n) - \hat{\mathbf{X}}(n/\mathcal{Y}_{n-1})H^T(n), \mathbf{X}(n)H^T(n) + \mathbf{E}(n) - \hat{\mathbf{X}}(n/\mathcal{Y}_{n-1})H^T(n) \right\rangle \\
&= \left\langle \left(\mathbf{X}(n) - \hat{\mathbf{X}}(n/\mathcal{Y}_{n-1})\right) H^T(n) + \mathbf{E}(n), \left(\mathbf{X}(n) - \hat{\mathbf{X}}(n/\mathcal{Y}_{n-1})\right) H^T(n) + \mathbf{E}(n) \right\rangle \\
&= H(n) \underbrace{\left\langle \left(\mathbf{X}(n) - \hat{\mathbf{X}}(n/\mathcal{Y}_{n-1})\right), \left(\mathbf{X}(n) - \hat{\mathbf{X}}(n/\mathcal{Y}_{n-1})\right) \right\rangle}_{P^T(n)} H^T(n) + \underbrace{\langle \mathbf{E}(n), \mathbf{E}(n) \rangle}_{R_e^T} \\
&= H(n)P^T(n)H^T(n) + R_e^T.
\end{aligned}
$$

Similarly we obtain for the inner product $\left\langle \mathbf{X}(n+1), \mathbf{E}^Y(n) \right\rangle$ that

$$
\begin{aligned}
&\left\langle \mathbf{X}(n+1), \mathbf{E}^Y(n) \right\rangle \\
&= \left\langle \mathbf{X}(n)F^T(n) + \mathbf{V}(n), \mathbf{E}^Y(n) \right\rangle \\
&= \left\langle \mathbf{X}(n), \mathbf{E}^Y(n) \right\rangle F^T(n) + \underbrace{\left\langle \mathbf{V}(n), \mathbf{E}^Y(n) \right\rangle}_{\mathbf{0}} \\
&= \left\langle \mathbf{X}(n), \mathbf{Y}(n) - \hat{\mathbf{Y}}(n/\mathcal{Y}_{n-1}) \right\rangle F^T(n) \\
&= \left\langle \mathbf{X}(n), \mathbf{X}(n)H^T(n) + \mathbf{E}(n) - \hat{\mathbf{X}}(n/\mathcal{Y}_{n-1})H^T(n) \right\rangle F^T(n) \\
&= \left\langle \mathbf{X}(n), \left(\mathbf{X}(n) - \hat{\mathbf{X}}(n/\mathcal{Y}_{n-1})\right) H^T(n) + \mathbf{E}(n) \right\rangle F^T(n) \\
&= \left\langle \mathbf{X}(n), \mathbf{E}^X(n)H^T(n) + \mathbf{E}(n) \right\rangle F^T(n) \\
&= H(n) \left\langle \mathbf{X}(n), \mathbf{E}^X(n) \right\rangle F^T(n) + \underbrace{\langle \mathbf{X}(n), \mathbf{E}(n) \rangle}_{\mathbf{0}} F^T(n) \\
&= H(n) \underbrace{\left\langle \mathbf{E}^X(n), \mathbf{E}^X(n) \right\rangle}_{P^T(n)} F^T(n) \\
&= H(n)P^T(n)F^T(n)
\end{aligned}
$$

Inserting the above two inner product equations into the equation for the Kalman gain we obtain

$$K(n) = F(n)P(n)H^T(n) \left(H(n)P(n)H^T(n) + R_e\right)^{-1}.$$

The final missing component needed to obtain a complete recursive solution is the time recursion for $P(n+1) = E\left(\mathbf{e}^x(n+1)(\mathbf{e}^x(n+1))^H\right)$, which is obtained from the following time recursion for $\mathbf{e}^x(n+1) = \mathbf{x}(n+1) - \hat{\mathbf{x}}(n+1/\mathcal{Y}_n)$.

$$
\begin{aligned}
\mathbf{e}^x(n+1) &= \mathbf{x}(n+1) - \hat{\mathbf{x}}(n+1/\mathcal{Y}_n) \\
&= F(n)\mathbf{x}(n) + \mathbf{v}(n) - (F(n)\hat{\mathbf{x}}(n/\mathcal{Y}_{n-1}) + K(n)\mathbf{e}^y(n)) \\
&= F(n) \underbrace{(\mathbf{x}(n) - \hat{\mathbf{x}}(n/\mathcal{Y}_{n-1}))}_{\mathbf{e}^x(n)} - K(n)\mathbf{e}^y(n) + \mathbf{v}(n)
\end{aligned}
$$

$$
\begin{aligned}
&= F(n)\mathbf{e}^x(n) - K(n)\left(\mathbf{y}(n) - \hat{\mathbf{y}}(n/\mathcal{Y}_{n-1})\right) + \mathbf{v}(n)\\
&= F(n)\mathbf{e}^x(n) - K(n)\left(H(n)\mathbf{x}(n) + \mathbf{e}(n) - H(n)\hat{\mathbf{x}}(n/\mathcal{Y}_{n-1})\right) + \mathbf{v}(n)\\
&= F(n)\mathbf{e}^x(n) - K(n)H(n)\underbrace{\left(\mathbf{x}(n) - \hat{\mathbf{x}}(n/\mathcal{Y}_{n-1})\right)}_{\mathbf{e}^x(n)} + \mathbf{v}(n) - K(n)\mathbf{e}(n)\\
&= \left(F(n) - K(n)H(n)\right)\mathbf{e}^x(n) + \mathbf{v}(n) - K(n)\mathbf{e}(n).
\end{aligned}
$$

Now as $\mathbf{e}^x(n)$, $\mathbf{v}(n)$ and $\mathbf{e}(n)$ are independent from one another the time recursion for $P(n+1) = E\left(\mathbf{e}^x(n+1)(\mathbf{e}^x(n+1))^H\right)$ becomes

$$
P(n+1) = \left(F(n) - K(n)H(n)\right)P(n)\left(F(n) - K(n)H(n)\right)^T + R_v + K(n)R_e(n)K^T(n).
$$

Summarizing the Kalman predictor equations we obtain the recursive algorithm below.

$$
\begin{aligned}
\mathbf{e}^y(n) &= \mathbf{y}(n) - H(n)\hat{\mathbf{x}}(n/\mathcal{Y}_{n-1})\\
K(n) &= F(n)P(n)H^T(n)\left(H(n)P(n)H^T(n) + R_e\right)^{-1}\\
\hat{\mathbf{x}}(n+1/\mathcal{Y}_n) &= F(n)\hat{\mathbf{x}}(n/\mathcal{Y}_{n-1}) + K(n)\mathbf{e}^y(n)\\
P(n+1) &= \left(F(n) - K(n)H(n)\right)P(n)\left(F(n) - K(n)H(n)\right)^T + R_v + K(n)R_e(n)K^T(n).
\end{aligned}
$$

For furter reading on the geometric development of the Kalman predictor and filter see for example `http://www.tele.ucl.ac.be/EDU/INMA2731/cours/Kalman.pdf`.