Chapter 15

Problem Solving Sessions

15.1 Dynamic Models

Exercise 1.1 (6.1): Stability boundary for a second-order system.

Consider the second-order AR model

$$y_t + a_1 y_{t-1} + a_2 y_{t-2} = e_t$$

Derive and plot the area in the (a_1, a_2)-plane for which the model is asymptotically stable.

Solution:

The characteristic equation is

$$z^2 + a_1 z + a_2 = 0.$$

If z_1, z_2 denotes the roots of this equation, we have that

$$a_1 = -(z_1 + z_2),\ a_2 = z_1 z_2.$$

Consider the limiting case with one or both roots on the unit circle.

- One root in $z = 1$, the other one inside the interval $z \in]-1, 1[$.

$$a_1 = -1 - z_2,\ a_2 = z_2 \Rightarrow a_2 = -1 - a_2$$

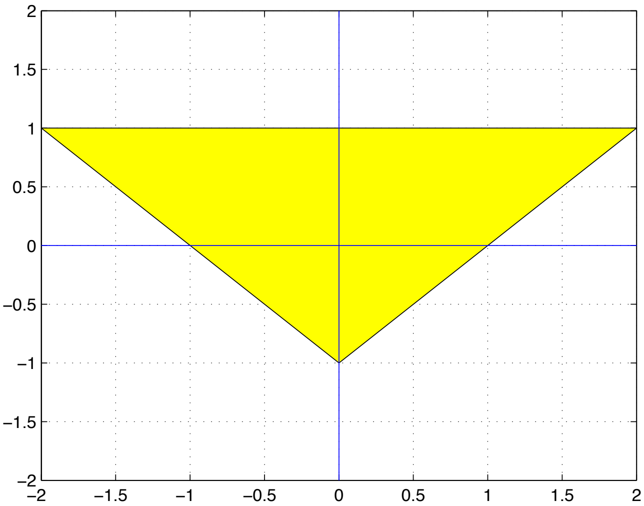
- One root in $z = -1$, the other one inside the interval $z \in]-1, 1[$.

$$a_1 = -1 - z_2,\ a_2 = -z_2 \Rightarrow a_2 = -1 + a_2$$

- Two complex conjugate roots $z_1, z_2 = \exp(\pm i\omega)$ with $\omega \in (0, \pi]$

$$a_1 = -2\cos\omega, a_2 = 1 \Rightarrow a_2 \in [-2, 2]$$

These cases define a closed contour that encloses the stability area as in Figure (15.1).



Exercise 1.2: Least Squares with Feedback

Consider the second-order AR model

$$y_t + ay_{t-1} = bu_{t-1} + e_t$$

where u_t is given by feedback as

$$u_t = -Ky_t.$$

Show that given realizations of this signal we cannot estimate a_0, b_0 separately, but we can estimate $a_0 + b_0k$. (Book p. 26)

Exercise 1.3

Determine the covariance function for an AR(1) process

$$y_t + ay_{t-1} = e_t$$

where e_t come from a white noise process with zero mean and unit variance. Determine the covariance function for an AR(2) process

$$y_t + ay_{t-1} + ay(t-2) = e_t$$

Determine the covariance function for an MA(1) process

$$y_t = e_t + be_{t-1}$$

Solution:
(a). The stochastic model is

$$Y_t + a_1 Y_{t-1} + a_2 Y_{t-2} = D_t$$

Then pre-multiplying both sides with $Y_t, Y_{t-1}, \dots, Y_{t-\tau}$ gives

$$\begin{cases} \mathbb{E}[Y_t Y_t] + a\mathbb{E}[Y_t Y_{t-1}] = \mathbb{E}[Y_t D_t] \\ \mathbb{E}[Y_{t-1} Y_t] + a\mathbb{E}[Y_{t-1} Y_{t-1}] = \mathbb{E}[Y_{t-1} D_t] \\ \mathbb{E}[Y_{t-2} Y_t] + a\mathbb{E}[Y_{t-2} Y_{t-1}] = \mathbb{E}[Y_{t-2} D_t] \\ \vdots \\ \mathbb{E}[Y_{t-\tau} Y_t] + a\mathbb{E}[Y_{t-\tau} Y_{t-1}] = \mathbb{E}[Y_{t-\tau} D_t], \end{cases}$$

and working out the expectations gives

$$\begin{cases} r_y(0) + ar_y(1) = 1 \\ r_y(1) + ar_y(0) = 0 \\ \vdots \\ r_y(\tau) + ar_y(\tau-1) = 0. \end{cases}$$

Hence we have that $r_y(\tau) = (-a)^\tau r_y(0)$. and that $r_y(0) + a(-ar_y(0)) = 1$ or $r_y(0) = \frac{1}{1-a^2}$.
(b). The stochastic model is

$$Y_t + a_1 Y_{t-1} + a_2 Y_{t-2} = D_t$$

Then pre-multiplying both sides with $Y_t, Y_{t-1}, \dots, Y_{t-\tau}$ gives

$$\begin{cases} \mathbb{E}[Y_t Y_t] + a_1\mathbb{E}[Y_t Y_{t-1}] + a_2\mathbb{E}[Y_t Y_{t-2}] = \mathbb{E}[Y_t D_t] \\ \mathbb{E}[Y_{t-1} Y_t] + a_1\mathbb{E}[Y_{t-1} Y_{t-1}] + a_2\mathbb{E}[Y_{t-1} Y_{t-2}] = \mathbb{E}[Y_{t-1} D_t] \\ \mathbb{E}[Y_{t-2} Y_t] + a_1\mathbb{E}[Y_{t-2} Y_{t-1}] + a_2\mathbb{E}[Y_{t-2} Y_{t-2}] = \mathbb{E}[Y_{t-2} D_t] \\ \vdots \\ \mathbb{E}[Y_{t-\tau} Y_t] + a_1\mathbb{E}[Y_{t-\tau} Y_{t-1}] + a_2\mathbb{E}[Y_{t-\tau} Y_{t-2}] = \mathbb{E}[Y_{t-\tau} D_t], \end{cases}$$

and working out the expectations gives

$$\begin{cases} r_y(0) + a_1 r_y(1) + a_2 r_y(2) = 1 \\ r_y(1) + a_1 r_y(0) + a_2 r_y(1) = 0 \\ r_y(2) + a_1 r_y(1) + a_2 r_y(0) = 0 \\ \vdots \\ r_y(\tau) + a_1 r_y(\tau-1) + a_2 r_y(\tau-2) = 0. \end{cases}$$

The expressions of $r_y(\tau)$ are then implied by this system, and tend to zero when $\tau \to 0$.

(c). The MA(1) case goes along the same lines. The stochastic model is given as

$$Y_t = D_t + cD_{t-1},$$

then the Yule-walker equations are

$$\begin{cases} \mathbb{E}[Y_t Y_t] = \mathbb{E}[Y_t D_t] + c\mathbb{E}[Y_t D_{t-1}] = \mathbb{E}[(D_t + cD_{t-1})D_t] + c\mathbb{E}[(D_t + cD_{t-1})D_{t-1}] \\ \mathbb{E}[Y_{t-1} Y_t] = \mathbb{E}[Y_{t-1} D_t] + c\mathbb{E}[Y_{t-1} D_{t-1}] = \mathbb{E}[(D_{t-1} + cD_{t-2})D_t] + c\mathbb{E}[(D_{t-1} + cD_{t-2})D_{t-1}] \\ \vdots \\ \mathbb{E}[Y_{t-\tau} Y_t] = \mathbb{E}[Y_{t-\tau} D_t] + c\mathbb{E}[Y_{t-\tau} D_{t-1}]. \end{cases}$$

and working out the terms gives

$$\begin{cases} r_y(0) = 1 + c^2 \\ r_y(1) = c \\ \vdots \\ r_y(\tau) = 0. \end{cases}$$

which gives a direct formula for the covariances. Note that the covariances equal zero for lags larger than the MA order.

Exercise 1.4

Given two systems

$$H_1(z) = \frac{b}{z+a}$$

and

$$H_2(z) = \frac{b_0 z + b_1}{z^2 + a_1 z + a_2}$$

(a) If those systems filters white noise $\{e_t\}$ coming from a stochastic process $\{D_t\}_t$ which is zero mean, and has unit variance. What is the variance of the filtered signal $\{y_t\}$?

(b) What happens to the output of the second system when you move the poles of $H_2(z)$ towards the unit circle?

(c) Where to place the poles to get a 'low-pass' filter?

(d) Where to put the poles in order to have a resonance top at $\omega = 1$?

(e) How does a resonant system appear on the different plots?

(f) What happens if $H_2(z)$ got a zero close to the unit circle?

Solution:

(a). A solution for computing the variance of the signal $Y_t = H_1(z)u_t$ is to construct the Yule-Walker equations as in the correlation analysis. The model can be expressed in the time domain as a first order model

$$Y_{t+1} + aY_t = bD_t.$$

By multiplication of both sides with Y_t and Y_{t+1}, and taking expectations one gets

$$\begin{cases} \mathbb{E}[Y_{t+1}Y_{t+1}] + a\mathbb{E}[Y_{t+1}Y_t] = b\mathbb{E}[D_tY_{t+1}] \\ \mathbb{E}[Y_tY_{t+1}] + a\mathbb{E}[Y_tY_t] = b\mathbb{E}[Y_tD_t]. \end{cases}$$

working out the terms gives

$$\begin{cases} r_y(0) + ar_y(1) = b^2 \\ r_y(1) = ar_y(0) = 0. \end{cases}$$

And this implies that $r_y(0) = 1$.

The same can be worked out for the system H_2. Let

$$Y_t = H_2(z)U_t$$

where $\phi_u(\omega) = \frac{1}{2\pi}$ for any frequency ω. Then

$$\phi_y(\omega) = H_2(e^{i\omega})H_2(e^{-i\omega})\phi_u(\omega).$$

Hence

$$\phi_y(\omega) =$$

(b). The system will display more oscillations (resonances), or equivalently, the sequence of covariances $r_y(\tau)$ will decrease slower to zero when τ increases.

(c). In order to get a low-pass filtering effect, the two (conjugate) poles should be placed close to the unit circle close to the point where $\omega = 1$ (right end).

(d). In order to make the system to have a resonance top, there should be one dominant frequency in the system. This frequency is then given as the $\angle e^{i\omega} \approx 57°$ as $\omega = 1$.

(e). see (b).

(f). The filter becomes high-pass.

Exercise 1.4

Given an input signal V_t shaped by an ARMA filter,

$$A(q^{-1})X_t = C(q^{-1})V_t,$$

where A and C are monomials of appropriate order, and where V_t white, zero mean and variance σ_v^2. Given noisy observations of this signal, or

$$Y_t = X_t + E_t$$

where E_t follows a stochastic process with white, zero mean and variance σ_e^2 and uncorrelated to D_t. Rewrite this as a ARMA process, what would be the corresponding variance of the 'noise'? How would the spectrum of Y_t look like?

Solution:

Rewrite the system as

$$Y_t = E_t + \frac{C(q^{-1})}{A(q^{-1})}V_t$$

and hence the spectrum of the output becomes

$$\phi_y(\omega) = \phi_e(\omega) + \frac{C(e^{i\omega})}{A(e^{i\omega})}\phi_v(\omega)$$

Let us rewrite this system as an ARMA system based on a possibly different noise source $\{G_t\}_t$ with variance σ_g^2, that is we impose the form

$$Y_t = \frac{D(q^{-1})}{A(q^{-1})}G_t$$

with monomial D. Hence $\phi_y(\omega) = \frac{D(e^{i\omega})}{A(e^{i\omega})}\phi_g(\omega)$. Then equation both models gives that for any ω one has that

$$A(e^{i\omega})\frac{\sigma_e^2}{2\pi} + C(e^{i\omega})\frac{\sigma_e^2}{2\pi} = A(e^{i\omega})\phi_y(\omega) = D(e^{i\omega})\frac{\sigma_g^2}{2\pi}$$

Since A, C, D are monomials (i.e. $A(1) = C(1) = D(1) = 1$), calculation then gives that $\sigma_g^2 = \sigma_e^2 + \sigma_v^2$.

Exercise 1.5 (3.1): Determine the time constant T from a step response.

A first order system $Y(s) = G(s)U(s)$ with

$$G(s) = \frac{K}{1+sT}e^{-s\tau}$$

or in time domain as a differential equation

$$T\frac{dy(t)}{dt} + y(t) = Ku(t-\tau)$$

derive a formula of the step response of an input $u_t = I(t > 0)$.

Solution: The system is $T\frac{dy(t)}{dt} + y(t) = Ku(t-\tau)$. The step response is therefor

$$y(t) = \begin{cases} 0 & t < \tau \\ K\left(1 - \exp(-(t-\tau)/T)\right) \end{cases}$$

The tangent at $t = \tau$ is given as

$$y'(t) = \frac{K}{T}(t-\tau)$$

The tangent reaches the steady state value K at time $t = \tau + T$.

Exercise 1.6 (3.10): Step response as a special case of spectral analysis.

Let $(y_t)_t$ be the step response of an LTI $H(q^{-1})$ to an input $u_t = aI(t \geq 0)$. Assume $y_t = 0$ for $t < 0$ and $y_t \approx c$ for $t > N$. Justify the following rough estimate of H

$$\hat{h}_k = \frac{y_k - y_{k-1}}{a}, \ \forall k = 0, \ldots, N$$

and show that it is approximatively equal to the estimate provided by the spectral analysis.

Solution:

From

$$y_t = \sum_{k=0}^{t} h_k u_{t-k} = a\sum_{k=0}^{t} h_k$$

and since y_t remains constant for values $t > N$ it follows that

$$h_t = \frac{y_t - y_{t-1}}{a}$$

for $t = 0, 1, 2, \ldots, n$, and since $h_t \approx 0$ for large n. Thus the following is a possible estimate of the transfer function:

$$\begin{aligned} \hat{H}(e^{i\omega}) &= \sum_{k=0}^{n} h_k \exp(-i\omega k) \\ &= \frac{1}{a}\sum_{k=0}^{n}(y_k - y_{k-1})\exp(-i\omega k) \\ &\approx \frac{1}{a}\sum_{k=0}^{n} y_k \exp(-i\omega k) - \frac{1}{a}\sum_{k=0}^{n} y_k \exp(-i\omega k)\exp(-i\omega) = \frac{1}{a}Y_n(\omega)(1-\exp(-i\omega)). \end{aligned}$$

Now

$$U(\omega) = \sum_{k=0}^{\infty} u_k \exp -i\omega k a \sum_{k=0}^{\infty} \exp -i\omega k = \frac{a}{1 - \exp -i\omega}.$$

Exercise 1.7 (4.5): Ill-conditioning of the normal equations in case of a polynomial trend model.

Given model

$$y_t = a_0 + a_1 t + \cdots + a_r t^r + e_t$$

Show that the condition number of the associated matrix $\Phi^T \Phi$ is ill-conditioned:

$$\text{cond}(\Phi^T \Phi) \geq O(N^{2r}/(2r+1))$$

for large n, and where $r > 1$ is the polynomial order. Hint. Use the relations for a symmetric matrix A:

- $\lambda_{\max}(A) \geq \max_i A_{ii}$
- $\lambda_{\min}(A) \leq \min_i A_{ii}$

Solution:
Since for large values of n one has

$$\sum_{t=1}^{n} t^k = O\left(\frac{n^{k+1}}{k+1}\right)$$

for all $k = 1, 2, \ldots$, it follows that

$$\text{cond}(\phi^{\phi}) = \frac{\lambda_{\max}(\phi^{\phi})}{\lambda_{\min}(\phi^{\phi}))} \geq \frac{\max_i[\phi^T \phi]_{ii}}{\min_i[\phi^T \phi]_{ii}} = O\left(\frac{n^{2r+1}}{2r+1}\right) / O(n) = O\left(\frac{n^{2r}}{2r+1}\right),$$

which is very large even for moderate values of n and r.