Chapter 15

Problem Solving Sessions

15.1 Dynamic Models

Exercise 1.1 (6.1): Stability boundary for a second-order system.

Consider the second-order AR model

\[ y_t + a_1 y_{t-1} + a_2 y_{t-2} = \epsilon_t \]

Derive and plot the area in the \((a_1, a_2)\)-plane for which the model is asymptotically stable.

Solution:

The characteristic equation is

\[ z^2 + a_1 z + a_2 = 0. \]

If \(z_1, z_2\) denotes the roots of this equation, we have that

\[ a_1 = -(z_1 + z_2), \quad a_2 = z_1 z_2. \]

Consider the limiting case with one or both roots on the unit circle.

- One root in \(z = 1\), the other one inside the interval \(z \in ]1, 1[\).
  \[ a_1 = -1 - z_2, \quad a_2 = z_2 \Rightarrow a_2 = -1 - a_2 \]

- One root in \(z = -1\), the other one inside the interval \(z \in ]-1, 1[\).
  \[ a_1 = -1 - z_2, \quad a_2 = -z_2 \Rightarrow a_2 = -1 + a_2 \]

- Two complex conjugate roots \(z_1, z_2 = \exp(\pm i \omega)\) with \(\omega \in (0, \pi]\)
  \[ a_1 = -2 \cos \omega, a_2 = 1 \Rightarrow a_2 \in [-2, 2] \]

These cases define a closed contour that encloses the stability area as in Figure (15.1).
Exercise 1.2: Least Squares with Feedback

Consider the second-order AR model

\[ y_t + ay_{t-1} = bu_{t-1} + e_t \]

where \( u_t \) is given by feedback as

\[ u_t = -Ky_t. \]

Show that given realizations of this signal we cannot estimate \( a_0, b_0 \) separately, but we can estimate \( a_0 + b_0k \). (Book p. 26)
Exercise 1.3

Determine the covariance function for an AR(1) process

\[ y_t + ay_{t-1} = e_t \]

where \( e_t \) come from a white noise process with zero mean and unit variance. Determine the covariance function for an AR(2) process

\[ y_t + ay_{t-1} + ay(t-2) = e_t \]

Determine the covariance function for an MA(1) process

\[ y_t = e_t + be_{t-1} \]

Solution:

(a). The stochastic model is

\[ Y_t + a_1Y_{t-1} + a_2Y_{t-2} = D_t \]

Then pre-multiplying both sides with \( Y_t, Y_{t-1}, \ldots, Y_{t-\tau} \) gives

\[
\begin{align*}
E[Y_tY_1] + aE[Y_tY_{t-1}] &= E[Y_tD_t] \\
E[Y_{t-1}Y_1] + aE[Y_{t-1}Y_{t-1}] &= E[Y_{t-1}D_t] \\
E[Y_{t-2}Y_1] + aE[Y_{t-2}Y_{t-1}] &= E[Y_{t-2}D_t] \\
&\vdots \\
E[Y_{t-\tau}Y_1] + aE[Y_{t-\tau}Y_{t-1}] &= E[Y_{t-\tau}D_t],
\end{align*}
\]

and working out the expectations gives

\[
\begin{align*}
r_y(0) + ar_y(1) &= 1 \\
r_y(1) + ar_y(0) &= 0 \\
&\vdots \\
r_y(\tau) + ar_y(\tau-1) &= 0.
\end{align*}
\]

Hence we have that \( r_y(\tau) = (-a)^\tau r_y(0) \), and that \( r_y(0) + a(-ar_y(0)) = 1 \) or \( r_y(0) = \frac{1}{1-a^2} \).

(b). The stochastic model is

\[ Y_t + a_1Y_{t-1} + a_2Y_{t-2} = D_t \]

Then pre-multiplying both sides with \( Y_t, Y_{t-1}, \ldots, Y_{t-\tau} \) gives

\[
\begin{align*}
E[Y_tY_1] + a_1E[Y_tY_{t-1}] + a_2E[Y_tY_{t-2}] &= E[Y_tD_t] \\
E[Y_{t-1}Y_1] + a_1E[Y_{t-1}Y_{t-1}] + a_2E[Y_{t-1}Y_{t-2}] &= E[Y_{t-1}D_t] \\
E[Y_{t-2}Y_1] + a_1E[Y_{t-2}Y_{t-1}] + a_2E[Y_{t-2}Y_{t-2}] &= E[Y_{t-2}D_t] \\
&\vdots \\
E[Y_{t-\tau}Y_1] + a_1E[Y_{t-\tau}Y_{t-1}] + a_2E[Y_{t-\tau}Y_{t-2}] &= E[Y_{t-\tau}D_t],
\end{align*}
\]
and working out the expectations gives
\[
\begin{align*}
\begin{cases}
    r_y(0) + a_1 r_y(1) + a_2 r_y(2) = 1 \\
    r_y(1) + a_1 r_y(0) + a_2 r_y(1) = 0 \\
    r_y(2) + a_1 r_y(1) + a_2 r_y(0) = 0 \\
    \vdots \\
    r_y(\tau) + a_1 r_y(\tau - 1) + a_2 r_y(\tau - 2) = 0.
\end{cases}
\end{align*}
\]

The expressions of \( r_y(\tau) \) are then implied by this system and tend to zero when \( \tau \to 0 \).

(c). The MA(1) case goes along the same lines. The stochastic model is given as
\[
Y_t = D_t + c D_{t-1},
\]
then the Yule-walker equations are
\[
\begin{align*}
\begin{cases}
    \mathbb{E}[Y_t Y_t] &= \mathbb{E}[Y_t D_t] + c \mathbb{E}[Y_t D_{t-1}] = \mathbb{E}[(D_t + c D_{t-1}) D_t] + c \mathbb{E}[(D_t + c D_{t-1}) D_{t-1}] \\
    \mathbb{E}[Y_{t-1} Y_t] &= \mathbb{E}[Y_{t-1} D_t] + c \mathbb{E}[Y_{t-1} D_{t-1}] = \mathbb{E}[(D_{t-1} + c D_{t-2}) D_t] + c \mathbb{E}[(D_{t-1} + c D_{t-2}) D_{t-1}] \\
    \vdots \\
    \mathbb{E}[Y_{t-\tau} Y_t] &= \mathbb{E}[Y_{t-\tau} D_t] + c \mathbb{E}[Y_{t-\tau} D_{t-1}].
\end{cases}
\end{align*}
\]

and working out the terms gives
\[
\begin{align*}
\begin{cases}
    r_y(0) &= 1 + c^2 \\
    r_y(1) &= c \\
    \vdots \\
    r_y(\tau) &= 0.
\end{cases}
\end{align*}
\]

which gives a direct formula for the covariances. Note that the covariances equal zero for lags larger than the MA order.

**Exercise 1.4**

Given two systems
\[
H_1(z) = \frac{b}{z + a}
\]
and
\[
H_2(z) = \frac{b_0 z + b_1}{z^2 + a_1 z + a_2}
\]

(a) If those systems filters white noise \( \{e_t\} \) coming from a stochastic process \( \{D_t\} \), which is zero mean, and has unit variance. What is the variance of the filtered signal \( \{y_t\} \)?

(b) What happens to the output of the second system when you move the poles of \( H_2(z) \) towards the unit circle?

(c) Where to place the poles to get a 'low-pass' filter?

(d) Where to put the poles in order to have a resonance top at \( \omega = 1 \)?
(e) How does a resonant system appear on the different plots?

(f) What happens if $H_2(z)$ got a zero close to the unit circle?
Solution:
(a) A solution for computing the variance of the signal \( Y_t = H_1(z)u_t \) is to construct the Yule-Walker equations as in the correlation analysis. The model can be expressed in the time domain as a first order model

\[
Y_{t+1} + a Y_t = b D_t.
\]

By multiplication of both sides with \( Y_t \) and \( Y_{t+1} \), and taking expectations one gets

\[
\begin{align*}
E[Y_{t+1}Y_{t+1}] + aE[Y_{t+1}Y_t] &= bE[D_t \cdot Y_{t+1}], \\
E[Y_tY_{t+1}] + aE[Y_tY_t] &= bE[Y_tD_t].
\end{align*}
\]

working out the terms gives

\[
\begin{align*}
r_y(0) + ar_y(1) &= b^2, \\
r_y(1) &= ar_y(0) = 0.
\end{align*}
\]

And this implies that \( r_y(0) = 1 \).

The same can be worked out for the system \( H_2 \). Let

\[
Y_t = H_2(z)U_t
\]

where \( \phi_y(\omega) = \frac{1}{2\pi} \) for any frequency \( \omega \). Then

\[
\phi_y(\omega) = H_2(e^{i\omega})H_2(e^{-i\omega})\phi_u(\omega).
\]

Hence

\[
\phi_y(\omega) =
\]

(b) The system will display more oscillations (resonances), or equivalently, the sequence of covariances \( r_y(\tau) \) will decrease slower to zero when \( \tau \) increases.

(c) In order to get a low-pass filtering effect, the two (conjugate) poles should be placed close to the unit circle close to the point where \( \omega = 1 \) (right end).

(d) In order to make the system to have a resonance top, there should be one dominant frequency in the system. This frequency is then given as the \( \angle e^{i\omega} \approx 57^\circ \) as \( \omega = 1 \).

(e) see (b).

(f) The filter becomes high-pass.
Exercise 1.4

Given an input signal $V_t$ shaped by an ARMA filter,

$$A(q^{-1})X_t = C(q^{-1})V_t,$$

where $A$ and $C$ are monomials of appropriate order, and where $V_t$ white, zero mean and variance $\sigma_v^2$. Given noisy observations of this signal, or

$$Y_t = X_t + E_t$$

where $E_t$ follows a stochastic process with white, zero mean and variance $\sigma_e^2$ and uncorrelated to $D_t$. Rewrite this as an ARMA process, what would be the corresponding variance of the 'noise'? How would the spectrum of $Y_t$ look like?

Solution:

Rewrite the system as

$$Y_t = E_t + \frac{C(q^{-1})}{A(q^{-1})}V_t$$

and hence the spectrum of the output becomes

$$\phi_y(\omega) = \phi_c(\omega) + \frac{C(e^{i\omega})}{A(e^{i\omega})} \phi_e(\omega)$$

Let us rewrite this system as an ARMA system based on a possibly different noise source $\{G_t\}_t$ with variance $\sigma_g^2$, that is we impose the form

$$Y_t = \frac{D(q^{-1})}{A(q^{-1})}G_t$$

with monomial $D$. Hence $\phi_y(\omega) = \frac{D(e^{i\omega})}{A(e^{i\omega})} \phi_g(\omega)$. Then equation both models gives that for any $\omega$ one has that

$$A(e^{i\omega})\frac{\sigma_e^2}{2\pi} + C(e^{i\omega})\frac{\sigma_v^2}{2\pi} = A(e^{i\omega})\phi_y(\omega) = D(e^{i\omega})\frac{\sigma_g^2}{2\pi}$$

Since $A, C, D$ are monomials (i.e. $A(1) = C(1) = D(1) = 1$), calculation then gives that $\sigma_g^2 = \sigma_v^2 + \sigma_e^2$. 

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Exercise 1.5 (3.1): Determine the time constant $T$ from a step response.

A first order system $Y(s) = G(s)U(s)$ with

$$G(s) = \frac{K}{1 + sT}e^{-s\tau}$$

or in time domain as a differential equation

$$T\frac{dy}{dt} + y(t) = Ku(t - \tau)$$

derive a formula of the step response of an input $u_t = I(t > 0)$.

Solution: The system is

$$T\frac{dy}{dt} + y(t) = Ku(t - \tau)$$

derive a formula of the step response of an input $u_t$.

The tangent at $t = \tau$ is given as

$$y'(t) = \frac{K}{T}(t - \tau)$$

The tangent reaches the steady state value $K$ at time $t = \tau + T$.

Exercise 1.6 (3.10): Step response as a special case of spectral analysis.

Let $(y_t)_t$ be the step response of an LTI $H(q^{-1})$ to an input $u_t = aI(t \geq 0)$. Assume $y_t = 0$ for $t < 0$ and $y_t \approx c$ for $t > N$. Justify the following rough estimate of $H$

$$\hat{h}_k = \frac{y_k - y_{k-1}}{a}, \forall k = 0, \ldots, N$$

and show that it is approximatively equal to the estimate provided by the spectral analysis.

Solution:

From

$$y_t = \sum_{k=0}^{t} h_k u_{t-k} = a \sum_{k=0}^{t} h_k$$

and since $y_t$ remains constant for values $t > N$ it follows that

$$h_t = \frac{y_t - y_{t-1}}{a}$$

for $t = 0, 1, 2, \ldots, n$, and since $h_t \approx 0$ for large $n$. Thus the following is a possible estimate of the transfer function:

$$\hat{H}(e^{i\omega}) = \sum_{k=0}^{n} h_k \exp(-i\omega k)$$

$$= \frac{1}{a} \sum_{k=0}^{n} (y_k - y_{k-1}) \exp(-i\omega k)$$

$$\approx \frac{1}{a} \sum_{k=0}^{n} y_k \exp(-i\omega k) - \frac{1}{a} \sum_{k=0}^{n} y_k \exp(-i\omega k) \exp(-i\omega) = \frac{1}{a} Y_n(\omega)(1 - \exp(-i\omega)).$$
Now

\[ U(\omega) = \sum_{k=0}^{\infty} u_k \exp(-i\omega k) \sum_{k=0}^{\infty} \exp(-i\omega k) = \frac{a}{1 - \exp(-i\omega)} \]
Exercise 1.7 (4.5): Ill-conditioning of the normal equations in case of a polynomial trend model.

Given model
\[ y_t = a_0 + a_1 t + \cdots + a_r t^r + e_t \]

Show that the condition number of the associated matrix \( \Phi^T \Phi \) is ill-conditioned:
\[ \text{cond}(\Phi^T \Phi) \geq O\left(\frac{N^{2r}}{(2r + 1)}\right) \]

for large \( n \), and where \( r > 1 \) is the polynomial order. Hint. Use the relations for a symmetric matrix \( A \):
- \( \lambda_{\text{max}}(A) \geq \max_i A_{ii} \)
- \( \lambda_{\text{min}}(A) \leq \min_i A_{ii} \)

Solution:
Since for large values of \( n \) one has
\[ \sum_{t=1}^{n} t^k = O\left(\frac{n^{k+1}}{k+1}\right) \]
for all \( k = 1, 2, \ldots \), it follows that
\[ \text{cond}(\phi^\phi) = \frac{\lambda_{\text{max}}(\phi^\phi)}{\lambda_{\text{min}}(\phi^\phi)} \geq \frac{\max_i [\phi^T \phi]_{ii}}{\min_i [\phi^T \phi]_{ii}} = O\left(\frac{n^{2r+1}}{2r + 1}\right) / O(n) = O\left(\frac{n^{2r}}{2r + 1}\right), \]
which is very large even for moderate values of \( n \) and \( r \).