System Identification, Lecture 3

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Lecture 3

Models:

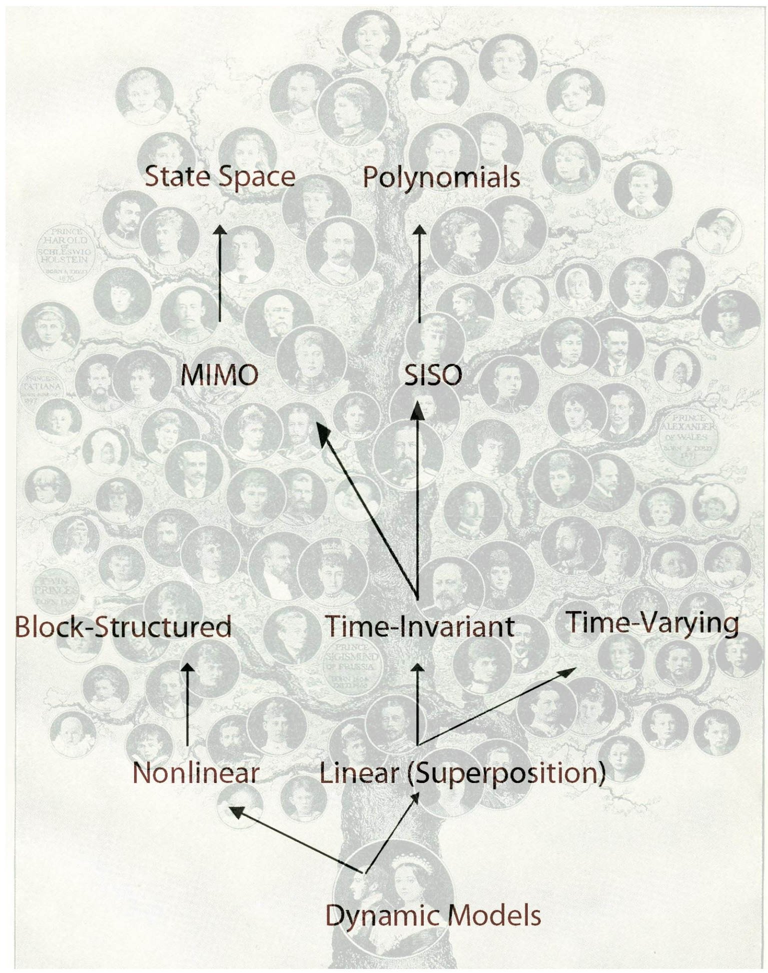
- A Taxonomy.
- Linear Time-Invariant Models.
- Polynomial Representations.
- Transforms.
- Identifiability.

Nonparametric Methods;

- Transient Analysis.
- Frequency Analysis.
- Correlation Analysis.
- Spectral Analysis.
- Other Input Signals.

Dynamic Models

A Taxonomy



- Dynamical Models
- White-, Black and Grey Box

Linear Time-Invariant Models

- Linear Superposition principle.
- Time-Invariant.
- Causal.
- Hence Impulse Response Representation for signals $\{u(t) : \forall t\}$ and $\{y(t) : \forall t\}$ as

$$y(t) = \int_{\tau=0}^{\infty} h(\tau) u(t-\tau) d\tau$$

- Discrete Impulse Response Model

$$y_t = \sum_{\tau=0}^{\infty} h_\tau u_{t-\tau}$$

- Conversion Continuous-Discrete $y(t\Delta + \tau) = y_t$ then

$$y(t\Delta) = \int_{\tau=0}^{\infty} h(\tau)u(t\Delta - \tau) = \sum_{s=1}^{\infty} \int_{\tau=(s-1)\Delta}^{s\Delta} h(s)u(t\Delta - \tau)ds$$

and switching indices and application of Euler's method gives

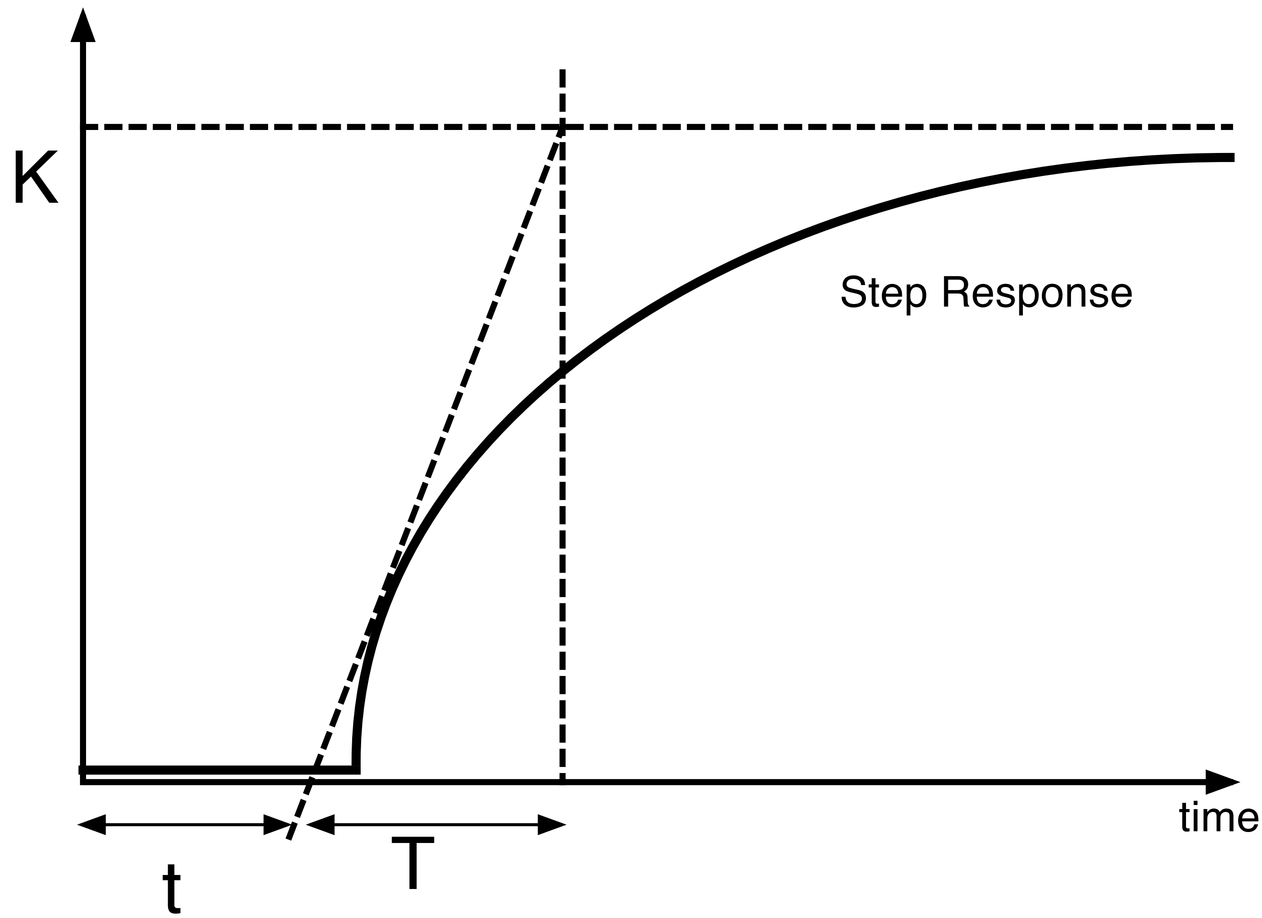
$$= \sum_{s=1}^{\infty} \left(\int_{\tau=(s-1)\Delta}^{s\Delta} h(s)ds \right) u_{t-s}$$

or $h_\tau = \int_{\tau'=(\tau-1)\Delta}^{\tau'\Delta} h(\tau)ds$. Works if $u_t \approx u(t\Delta + \tau)$.

- Ex. 1: a first order model

$$T\frac{dy(t)}{dt} + y(t) = Ku(t-\tau),$$

solving by Euler when $\{u(t) = 1(t \geq 0)\}$:

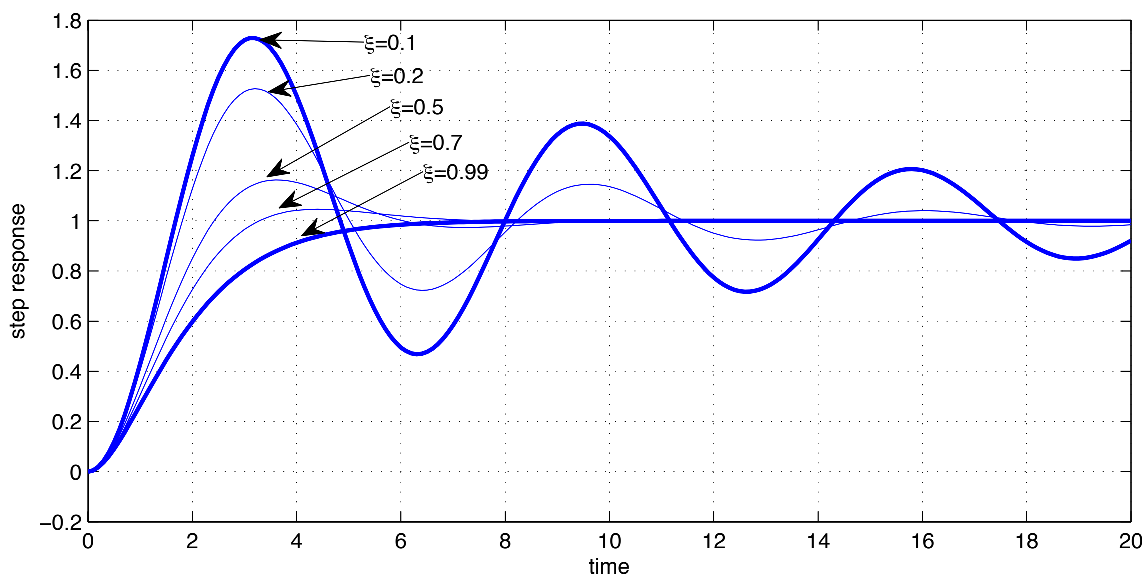


- Ex. 2: A second order system

$$\frac{d^2y(t)}{dt^2} + 2\xi\omega_0\frac{dy(t)}{dt} + \omega_0^2 y(t) = K\omega_0^2 u(t).$$

solving by Euler when $\{u(t) = \delta_t\}$, then solving by Euler when $\{u(t) = \delta(t)\}$, then (with $\tau = \arccos \xi$)

$$y(t) = K\left(1 - \frac{e^{-\xi\omega_0 t}}{\sqrt{1-\xi^2}}\sin\left(\omega_0 t\sqrt{1-\xi^2} + \tau\right)\right)$$



Transforms

Given the cos signal

$$y_t = \cos(\omega t),\ \forall t = \ldots, -1, 0, 1, \ldots$$

Then this can be written as

$$y_t = \Re \exp(i\omega t)$$

when applied to a LTI H we have

$$y_t = \sum_{\tau=0}^{\infty} h_\tau \cos(\omega t - \tau) = \Re \left(\sum_{\tau=0}^{\infty} h_\tau \exp(i\omega(t-\tau)) \right)$$

and

$$= \Re \left(\exp(i\omega t) \sum_{\tau=0}^{\infty} h_\tau \exp(-i\omega\tau) \right)$$
$$\rightarrow |H(\exp(i\omega))| \cos\left(\omega t + \arg H(\exp(i\omega))\right)$$

where for all $s \in \mathbb{C}$ one has

$$H(s) = \sum_{\tau=0}^{\infty} h_\tau e^{-s\tau}$$

Polynomial Representations

In polynomial representation

$$y_t = (h_0 + h_1 q^{-1} + h_2 q^{-2} + \dots)u_t = H(q^{-1})u_t.$$

Typically, also disturbances $\{d_t\}_t$ in model

$$y_t = H(q^{-1})u_t + d_t = H(q^{-1})u_t + G(q^{-1})e_t,$$

with $\{e_t\}_t$ driving noise.

- Stochastic, unpredictable random noise.
- Under-modelling.
- Nonlinearities.
- Time-Varying Aspects.

1. Input-Output Models relating input $\{u_t\}_t$ to $\{y_t\}_t$

(FIR): $y_t = B(q^{-1})u_t + e_t$
(ARX): $A(q^{-1})y_t = B(q^{-1})u_t + e_t$
RMAX): $A(q^{-1})y_t = B(q^{-1})u_t + C(q^{-1})e_t$
(OE): $y_t = \frac{B(q^{-1})}{A(q^{-1})}u_t + e_t$
(BJ): $A(q^{-1})y_t = \frac{B(q^{-1})}{F(q^{-1})}u_t + \frac{C(q^{-1})}{D(q^{-1})}e_t$

2. Time-series Models explaining $\{y_t\}_t$

(MA): $y_t = C(q^{-1})e_t$
(AR): $A(q^{-1})y_t = e_t$
ARMA): $A(q^{-1})y_t = C(q^{-1})e_t$
RIMA): $(1 - q^{-1})^d A(q^{-1})y_t = C(q^{-1})e_t$

3. BIBO Stable.

4. Minimum Phase.

5. Simulation and Prediction.

Identifiability

- Globally Identifiable at θ^* iff for all θ

$$\begin{cases} H(z,\theta) = H(z,\theta^*) \\ G(z,\theta) = G(z,\theta^*) \end{cases} \quad , \forall z \Leftrightarrow \theta = \theta^*$$

- Globally Identifiable.
- Sequence of two LTIs
- State Space Systems.

Nonparametric Methods

Transient Analysis

Impulse:

- Inject input

$$u_t = \begin{cases} K & t = 0 \\ 0 & \text{else} \end{cases}$$

- Outcome (without noise)

$$y_t = K \begin{cases} h_t & t \geq 0 \\ 0 & \text{else} \end{cases}$$

Step Response:

- Inject input

$$u_t = \begin{cases} K & t \geq 0 \\ 0 & \text{else} \end{cases}$$

- Outcome (without noise)

$$y_t = K \begin{cases} \sum_{\tau=0}^{t} h_\tau & t \geq 0 \\ 0 & \text{else} \end{cases}$$

- Or

$$y_t - y_{t-1} = K \begin{cases} h_t & t \geq 0 \\ 0 & \text{else} \end{cases}$$

Frequency Analysis

- Sinusoid:

$$u_t = a\cos(\omega t)$$

then output is

$$y_t = a|H(e^{i\omega})|\cos\left(\omega t + \arg H(e^{i\omega})\right)$$

- Sum of Sinusoids:

$$u_t = \sum_{k=1}^{K} a_k \cos(\omega_k t + b_k)$$

then output is

$$y_t = \sum_{k=1}^{K} a_k |H(e^{i\omega_k})| \cos\left(\omega_k t + b_k + \arg H(e^{i\omega_k})\right)$$

- Decompose input using Fourier

$$U(e^{i\omega}) = \int_t u_t e^{-i\omega t}$$

- Then transform

$$Y(s) = H(s)U(s) + E(s)$$

for $s = e^{i\omega}$

Correlation Analysis

- Specify input as 'random' sequence characterized by mean

$$m_u = \mathbb{E}[u_t] = \lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} u_t$$

  and covariance

$$r_u(\tau) = \mathbb{E}[u_t u_{t-\tau}] = \lim_{n\to\infty} \frac{1}{n} \sum_{t=\tau+1}^{n} u_t u_{t-\tau}$$

- Then outcome

$$y_t = \sum_{\tau=0}^{\infty} h_\tau u_{t-\tau}$$

- Multiplication of both sides by $u_{t-\tau}$ and taking expectation

$$r_{uy}(\tau) = \sum_{s=1}^{\infty} h_s r_u(\tau)$$

- Rearrange as a system of linear equations (Wiener-Hopf) or

$$\begin{bmatrix} ruy(0) \\ ruy(1) \\ \vdots \\ ruy(\tau) \\ \vdots \end{bmatrix} = \begin{bmatrix} ru(0) & ru(1) & ru(2) & \dots & ru(\tau) & \dots \\ ru(1) & ru(0) & & & ru(\tau-1) & \dots \\ ru(2) & & & & & \\ \vdots & & & \ddots & & \\ ru(\tau) & ru(\tau-1) & & & ru(0) & \dots \\ \vdots & & & & \vdots & \ddots \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_\tau \\ \vdots \end{bmatrix}$$

- For timeseries, if

$$y_t = \sum_{\tau=0}^{\infty} h_\tau e_{t-\tau}$$

with $\sum_{\tau=0}^{\infty} |g_\tau| < \infty$, then stationary.

- For Time-series with AR(M) model

$$\sum_{\tau=0}^{M} a_\tau y_{t-\tau} = e_t$$

where

$$\mathbb{E}[e_t] = 0, \mathbb{E}[e_t e_{t-\tau}] = \sigma^2 \delta_\tau$$

multiply with y_{t-s} and take expectations

$$\mathbb{E}[e_t y_{t-s}] = \sum_{\tau=0}^{M} a_\tau \mathbb{E}[y_{t-s} y_{t-\tau}]$$

Then (Yule-Walker)

$$\begin{bmatrix} \sigma^2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} ry(0) & ry(1) & ry(2) & \dots & ry(M) \\ ry(1) & ry(0) & & & ry(M-1) \\ ry(2) & & & & \\ \vdots & & & \ddots & \\ ry(M) & ry(M-1) & & & ry(0) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_M \end{bmatrix}$$

Spectral Analysis

- Rather than working with covariance r, work with Spectral density
$$\phi_u(\omega) = \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} r_u(\tau) e^{-i\omega\tau}$$

- Then
$$\phi_{uy}(\omega) = H(e^{i\omega})\phi_u(\omega)$$
where $H(e^{i\omega}) = \sum_{\tau=0}^{\infty} h_\tau e^{-i\omega\tau}$.

- Hence for all $\omega =]-\pi, \pi[$ one has
$$H(e^{i\omega}) = \frac{\phi_{uy}(\omega)}{\phi_u(\omega)}$$

- But estimate based on
$$\hat{r}_u(\tau) = \frac{1}{n} \sum_{t=\tau+1}^{n} u_t u_{t-\tau}$$

not so accurate, better to take windowed Fourier transform

$$\hat{\phi}_u = \frac{1}{2\pi} \sum_{t=-n}^{n} \hat{r}_u(\tau) w(\tau) e^{-i\omega\tau}$$

– Rectangular.
– Bartlett.
– Tukey.

Spectral Factorization

- Assume ϕ_y can be written in terms of a rational polynomial of order m:

$$\phi_y(s) = \frac{C(s)}{D(s)} = \frac{c_{-m}s^{-m} + \cdots + c_m s^m}{d_{-m}s^{-m} + \cdots + d_m s^m}$$

Then

$$\phi_y(s) = \frac{B(s)}{A(s)} \frac{B^*(s^{-*})}{A^*(s^{-*})}$$

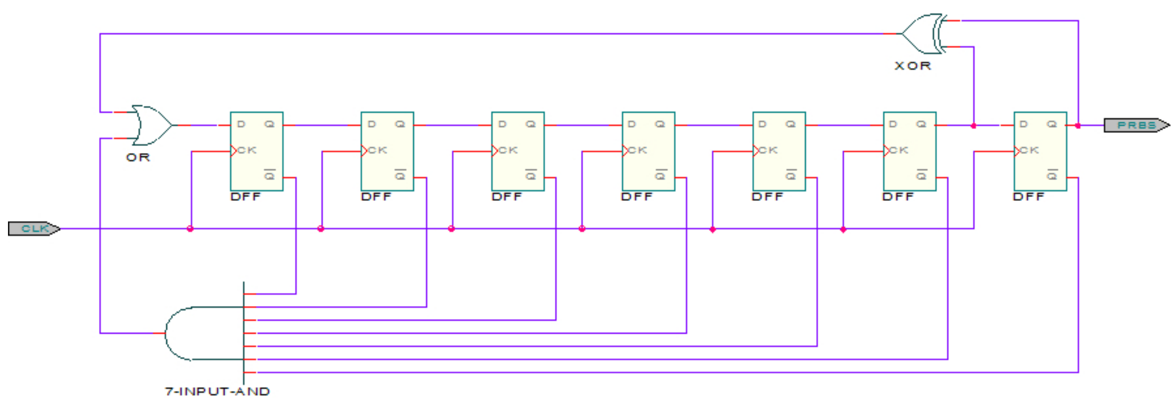
where

$$\begin{cases} A(s) = a_0 + a_1 s + \cdots + a_m s^m \\ B(s) = b_0 + b_1 s + \cdots + b_m s^m \end{cases}$$

R(m):

Other Input Sequences

- ARMA.
- PRBS.
- Sum of Sinusoids.



Conclusions

- LTI Models
- Polynomial Representations vs. Discrete z-transform
- Initial Analysis by injecting certain input signals and recording output.