System Identification, Lecture 9

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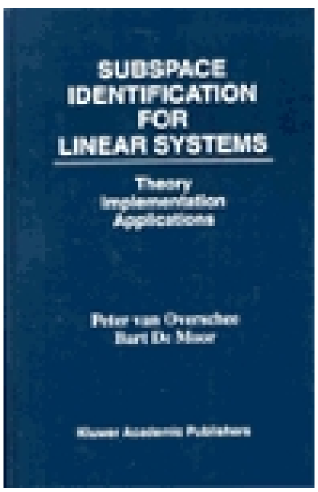
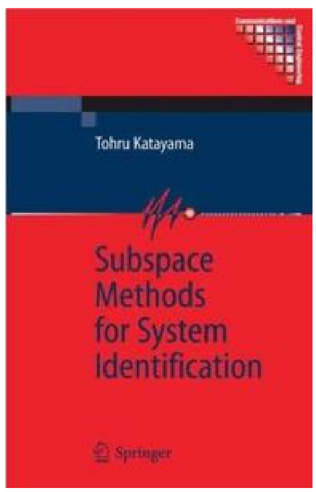
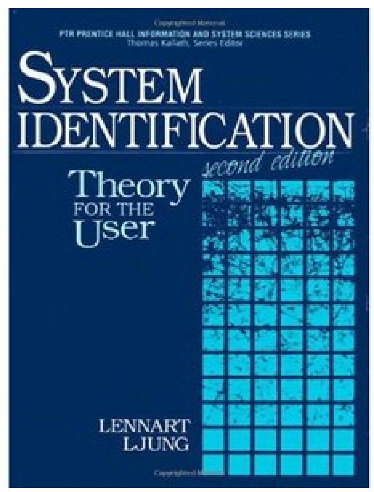
Overview Subspace Identification

1. Deterministic.
2. Stochastic.
3. Extensions.





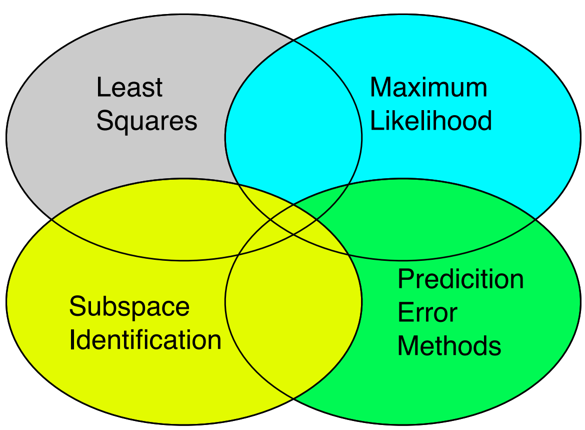


K. De Cock, B. De Moor, "Subspace Identification Methods", report, 2003.

Motivation

Why?

- MIMO.
- State space models.
- Inherent identifiability 'up to $\mathbf{T}$'.
- Numerical matching.
- Numerical Robust techniques (perturbations).
- Connection to systems theory.



State Space System

$$\begin{cases} \mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{B}\mathbf{u}_t \\ \mathbf{y}_t = \mathbf{C}\mathbf{x}_t + \mathbf{D}\mathbf{u}_t, \end{cases} \quad \forall t = -\infty, \dots, \infty.$$

with

- $\{\mathbf{x}_t\}_t \subset \mathbb{R}^n$ the state process.
- $\{\mathbf{u}_t\}_t \subset \mathbb{R}^p$ the input process.
- $\{\mathbf{y}_t\}_t \subset \mathbb{R}^q$ the output process.
- $\mathbf{A} \in \mathbb{R}^{n\times n}$ the system matrix.
- $\mathbf{B} \in \mathbb{R}^{n\times p}$ the input matrix.
- $\mathbf{C} \in \mathbb{R}^{q\times n}$ the output matrix.
- $\mathbf{D} \in \mathbb{R}^{q\times p}$ the feed-through matrix.

Problem Statement

Problem SI: Given multivariate timeseries $\{\mathbf{u}_t\}_{t=0}^N \subset \mathbb{R}^p$ and $\{\mathbf{y}\}_{t=0}^N \subset \mathbb{R}^q$, can you figure out the order n, matrices $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ and $\{\mathbf{x}_t\}_t \subset \mathbb{R}^n$?

Realization: Given impulse response matrices $\{\mathbf{H}_\tau\}_\tau$, recover n and $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$.

A first (naive) approach:

(1) Estimate IR matrices $\{\hat{\mathbf{H}}_\tau\}_\tau$ by solving/approximating

$$\begin{bmatrix} \mathbf{y}_n^T \\ \mathbf{y}_{n+1}^T \\ \mathbf{y}_{n+2}^T \\ \vdots \\ \mathbf{y}_N^T \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1^T & \mathbf{u}_2^T & \dots & \mathbf{u}_n^T \\ \mathbf{u}_2^T & \mathbf{u}_3^T & & \mathbf{u}_{n+1}^T \\ \mathbf{u}_3^T & \mathbf{u}_4^T & & \mathbf{u}_{n+2}^T \\ \vdots & \vdots & & \vdots \\ \mathbf{u}_{N-n+1}^T & \mathbf{u}_{N-n+2}^T & & \mathbf{u}_N^T \end{bmatrix} \begin{bmatrix} \mathbf{H}_{n-1}^T \\ \mathbf{H}_{n-2}^T \\ \vdots \\ \mathbf{H}_0^T \end{bmatrix}$$

(2) Realization: transform $\{\hat{\mathbf{H}}_\tau\}_\tau$ into $\hat{n}$ and $(\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}, \hat{\mathbf{D}})$

But:

- Computational burdensome.
- Not robust.
- PE...
- Numerically ill-conditioned.
- Process Noise.
- State-Space structure.

That's why subspace ID:

- N4SID (*'enforce it'*) (Numerical algorithm for Subspace State-space System ID)
- MOESP (Multivariate Output Error State sPace)

The Deterministic Case

(From T. Katayama, 2005) Matrix matching

$$\begin{bmatrix} \mathbf{y}_t \\ \vdots \\ \mathbf{y}_{t+k-1} \end{bmatrix} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{k-1} \end{bmatrix} \mathbf{x}_t + \begin{bmatrix} \mathbf{D} & & & \\ \mathbf{CB} & \mathbf{D} & & \\ \vdots & & \ddots & \\ \mathbf{CA}^{k-2}\mathbf{B} & & \dots & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{u}_t \\ \vdots \\ \mathbf{u}_{t+k-1} \end{bmatrix}$$

In shorthand:

$$\mathbf{y}_k(t) = \mathcal{O}_k \mathbf{x}_t + \Psi_k \mathbf{u}_k(t)$$

This holds for any $t = 1, 2, \dots, N$, or

$$\begin{bmatrix} \mathbf{y}_k(0) & \mathbf{y}_k(1) & \dots & \mathbf{y}_k(i-1) \end{bmatrix} = \mathcal{O}_k \begin{bmatrix} \mathbf{x}_0 & \mathbf{x}_1 & \dots & \mathbf{x}_{i-1} \end{bmatrix} + \Psi_k \begin{bmatrix} \mathbf{u}_k(0) & \mathbf{u}_k(1) & \dots & \mathbf{u}_k(i-1) \end{bmatrix}$$

Or in even shorter hand

$$\mathbf{Y}_{k,0} = \mathcal{O}_k \mathbf{X}_0 + \Psi_k \mathbf{U}_{k,0}$$

Now the same trick for for data $k, \ldots, k+i-1$

$$\begin{cases} \mathbf{Y}_{k,s} = \begin{bmatrix} \mathbf{y}_k(s) & \mathbf{y}_k(1) & \ldots & \mathbf{y}_k(s+i-1) \end{bmatrix} \\ \mathbf{U}_{k,s} = \begin{bmatrix} \mathbf{u}_k(s) & \mathbf{u}_k(1) & \ldots & \mathbf{u}_k(s+i-1) \end{bmatrix} \\ \mathbf{X}_s = (\mathbf{x}_s, \ldots, \mathbf{x}_{s+i-1}) \end{cases}$$

Hence one has for all $s = 0, 1, \ldots, N-i$.

$$\mathbf{Y}_{k,s} = \mathcal{O}_k \mathbf{X}_s + \Psi_k \mathbf{U}_{k,s}.$$

We will use in our exposition

$$\begin{cases} \mathbf{Y}_{k,0} = \mathcal{O}_k \mathbf{X}_0 + \Psi_k \mathbf{U}_{k,0} \\ \mathbf{Y}_{k,k} = \mathcal{O}_k \mathbf{X}_k + \Psi_k \mathbf{U}_{k,k}. \end{cases}$$

Which we will denote as the matrix input-output relations of 'past' and 'future'.

or

$$\begin{cases} \mathbf{U}_{k,0} = \begin{bmatrix} \mathbf{u}_0 & \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_{i-1} \\ \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \dots & \mathbf{u}_i \\ \vdots & & & & \vdots \\ \mathbf{u}_{k-1} & \mathbf{u}_k & & \dots & \mathbf{u}_{k+i-2} \end{bmatrix} \in \mathbb{R}^{kp\times i} \end{cases}$$

$$\begin{cases} \mathbf{Y}_{k,0} = \begin{bmatrix} \mathbf{y}_0 & \mathbf{y}_1 & \mathbf{y}_2 & \dots & \mathbf{y}_{i-1} \\ \mathbf{y}_1 & \mathbf{y}_2 & \mathbf{y}_3 & \dots & \mathbf{y}_i \\ \vdots & & & & \vdots \\ \mathbf{y}_{k-1} & \mathbf{y}_k & & \dots & \mathbf{y}_{k+i-2} \end{bmatrix} \in \mathbb{R}^{kq\times i} \end{cases}$$

$$\begin{cases} \mathbf{U}_{k,k} = \begin{bmatrix} \mathbf{u}_k & \mathbf{u}_{k+1} & \mathbf{y}_{k+2} & \dots & \mathbf{u}_{k+i-1} \\ \mathbf{u}_{k+1} & \mathbf{u}_{k+2} & \mathbf{y}_{k+3} & \dots & \mathbf{u}_{k+i} \\ \vdots & & & & \vdots \\ \mathbf{u}_{2k-1} & \mathbf{u}_k & & \dots & \mathbf{u}_{k+i-2} \end{bmatrix} \in \mathbb{R}^{kq\times i} \end{cases}$$

$$\begin{cases} \mathbf{Y}_{k,k} = \begin{bmatrix} \mathbf{y}_k & \mathbf{y}_{k+1} & \mathbf{y}_{k+2} & \dots & \mathbf{y}_{k+i-1} \\ \mathbf{y}_{k+1} & \mathbf{y}_2 & \mathbf{y}_3 & \dots & \mathbf{y}_{k+i} \\ \vdots & & & & \vdots \\ \mathbf{y}_{2k-1} & \mathbf{y}_{2k} & & \dots & \mathbf{y}_{2k+i-1} \end{bmatrix} \in \mathbb{R}^{kq\times i} \end{cases}$$

Let

$$\mathbf{W}_- = \begin{bmatrix} \mathbf{U}_{k,0} \\ \mathbf{Y}_{k,0} \end{bmatrix} \qquad \mathbf{W}_+ = \begin{bmatrix} \mathbf{U}_{k,k} \\ \mathbf{Y}_{k,k} \end{bmatrix}$$

Now we study the relation of $\mathbf{W}_-, \mathbf{W}_+$ and $\mathbf{H}$. From above, we have that

$$\mathbf{W}_- = \begin{bmatrix} \mathbf{U}_{k,0} \\ \mathbf{Y}_{k,0} \end{bmatrix} = \begin{bmatrix} I_{kp} & 0 \\ \psi_k & \mathcal{O}_k \end{bmatrix} \begin{bmatrix} \mathbf{U}_{k,0} \\ \mathbf{X}_0 \end{bmatrix}$$

Or

$$\mathbf{W}_- = \begin{bmatrix} \mathbf{U}_{k,0} \\ \mathbf{Y}_{k,0} \end{bmatrix} = \begin{bmatrix} I_{kp} & 0 \\ \psi_k & \mathcal{O}_k \mathcal{C}_k \end{bmatrix} \begin{bmatrix} \mathbf{U}_{k,0} \\ \mathbf{U}_{k,0} \end{bmatrix}$$

Relation - ex.1.

Let

$$u = (0, 0, 0, 1, 0, 0, 0, \dots)^T$$

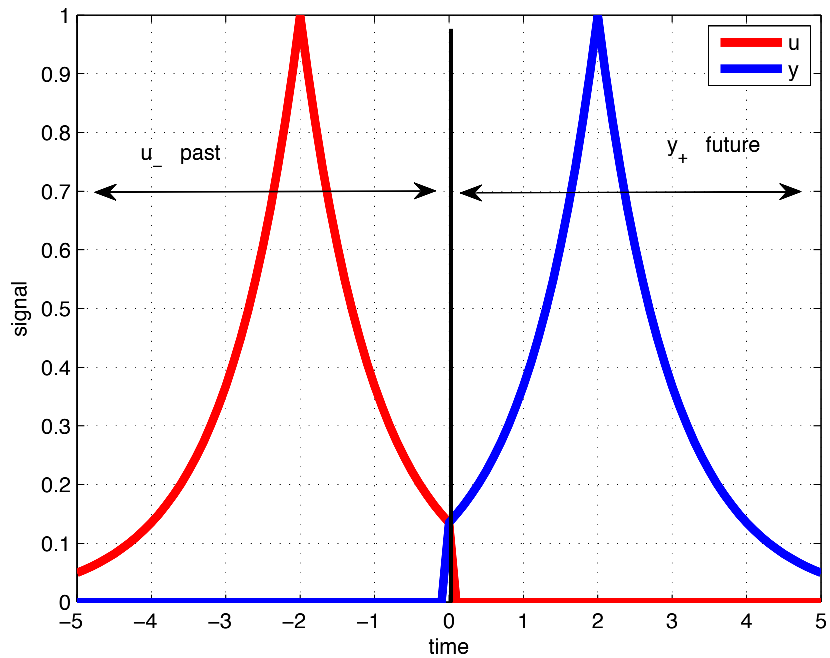
Apply this input signal to a noiseless LTI, and suppose the outcome is

$$y = (0, 0, 0, g_1, g_1, g_3, g_4, g_5, \dots)^T$$

Let $k = 4, i = 8$, then we get

$$\mathbf{W}_- = \left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & g_0 & g_1 & g_2 & g_3 & g_4 \\ 0 & 0 & g_0 & g_1 & g_2 & g_3 & g_4 & g_5 \\ 0 & g_0 & g_1 & g_2 & g_3 & g_4 & g_5 & g_6 \\ g_0 & g_1 & g_2 & g_3 & g_4 & g_5 & g_6 & g_7 \end{array}\right]$$

This datamatrix ressembles $\begin{bmatrix} I_4 & 0 \\ \psi_4 & \mathcal{O}_4\mathcal{C}_4 \end{bmatrix}$ (up to permutation).



This is a general structure, using a LQ (QR)-decomposition one can bring any $\mathbf{W}_-$ into this structure, from which we have the matrix $\mathbf{H}_k$, and can apply realization. This approach is taken in MOESP

1. Using LQ to recover matrix $\mathcal{O}_k\mathcal{C}_k$

2. Use realization to recover $\mathbf{A}, \mathbf{B}$, and then $\mathbf{B}, \mathbf{D}$.

3. Then use Kalman filter to obtain corresponding state sequence.

Relation - N4SID

A different road:

- Recover the order and the state *subspace* by relating $\mathbf{W}_-$ to $\mathbf{W}_+$,
- Then recover $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ by LS.

How does that work?

Thm. $\mathrm{span}(\mathbf{W}_-) \cap \mathrm{span}(\mathbf{W}_+) = \mathrm{span}(\mathbf{X}_k)$, or

$$\mathbf{Y}_{k,0} = \mathcal{O}_k \mathbf{X}_0 + \Psi \mathbf{U}_{k,0}$$

So find the subspace by oblique projection (SVD).

$$\Pi_{\mathbf{U}}^{+} = I - \mathbf{U}^T (\mathbf{U}\mathbf{U}^T)^{-1} \mathbf{U}$$

Then $\mathbf{Y}_{k,0} \Pi_{\mathbf{U}}^{+} = \mathcal{O}_k \mathbf{X}_0 \Pi_{\mathbf{U}}^{+}$.

Stochastic Realization

Problem: Given $\mathbb{E}[Y_t Y_{t-\tau}^T] = \Lambda(\tau)$ for $\tau = 0, 1, 2, \ldots$, find a realization $(\mathbf{A}, \mathbf{B})$ such that the outcome $\{Y_t\}$ of the system

$$\begin{cases} X'_{t-1} = \mathbf{A} X'_t + \mathbf{K} D_t \\ Y_t = \mathbf{C} X'_t + D_t \end{cases}$$

when driven by white noise $\{D_t\}$ taking values in $\mathbb{R}^n$ has properties $\{\Lambda(\tau)\}_\tau$. Richer in history: Parzen, Akaike,Kalman, Faurre, De moor/Van Overschee, but Messier in results

Build up the data matrices $\mathbf{Y}_{k,0}$ and $\mathbf{Y}_{k,k}$, and use those to reconstruct the internal states. One common way to do that is using Canonical Correlation Analysis, solving

$$\max_{\mathbf{a},\mathbf{b}} \frac{\mathbf{a}^T \mathbf{Y}_{k,0} \mathbf{Y}_{k,k}^T \mathbf{b}}{\sqrt{\mathbf{a}^T \mathbf{Y}_{k,0} \mathbf{Y}_{k,0}^T \mathbf{a}} \sqrt{\mathbf{b}^T \mathbf{Y}_{k,k} \mathbf{Y}_{k,k}^T \mathbf{b}}}$$

- Solutions given by generalized eigenvalue problem.

- Detection of n by number of significant eigenvalues of $\Sigma_{--}^{-1/2}\Sigma_{-+}\Sigma_{++}^{-1/2}$ where

$$\begin{cases} \Sigma_{--} = \frac{1}{N}\mathbf{Y}_{k,0}\mathbf{Y}_{k,0}^T \\ \Sigma_{-+} = \frac{1}{N}\mathbf{Y}_{k,0}\mathbf{Y}_{k,k}^T \\ \Sigma_{--} = \frac{1}{N}\mathbf{Y}_{k,k}\mathbf{Y}_{k,k}^T \end{cases}$$

- Basis given by corresponding eigenvectors.

- Again, compute matrices $\mathcal{O}_k$ and $\mathcal{C}_k$, and realize a $(\mathbf{A}, \mathbf{C})$.

Extensions

- Innovation representation.
- Reduced Realization.
- Weighting matrices.
- Positive Real and Stable.
- Relation to Kalman filter.
- SVD and LQ are robust and efficient techniques.

Combined Stochastic - Deterministic

System

$$\begin{cases} X_{t+1} = \mathbf{A} X_t + \mathbf{B} \mathbf{u}_t + V_t \\ Y_t = \mathbf{C} X_t + \mathbf{D} \mathbf{u}_t + W_t, \end{cases} \quad \forall t = -\infty, \ldots, \infty.$$

with

- $\{\mathbf{x}_t\}_t \subset \mathbb{R}^n$ the state process.
- $\{\mathbf{u}_t\}_t \subset \mathbb{R}^p$ the input process.
- $\{V_t\}_t \subset \mathbb{R}^n$ the process noise with covariance $\mathbf{R}$.
- $\{\mathbf{y}_t\}_t \subset \mathbb{R}^q$ the output process.
- $\{V_t\}_t \subset \mathbb{R}^n$ the measurement noise with covariance $\mathbf{Q}$.
- $\mathbf{A} \in \mathbb{R}^{n \times n}$ the system matrix.
- $\mathbf{B} \in \mathbb{R}^{n \times p}$ the input matrix.
- $\mathbf{C} \in \mathbb{R}^{q \times n}$ the output matrix.
- $\mathbf{D} \in \mathbb{R}^{q \times p}$ the feed-through matrix.

Problem: Given $\{\mathbf{u}_t\}_t \subset \mathbb{R}^p$ and $\{\mathbf{y}_t\}_t \subset \mathbb{R}^q$, find $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{P}, \mathbf{Q})$ and $\{\mathbf{x}_t\}_t$.

Basic equation

$$\mathbf{Y}_{k,0} = \mathcal{O}_k \mathbf{X}_0 + \Psi \mathbf{U}_{k,0} + \mathbf{V}$$

- Razor away $\mathbf{U}$ by oblique projection.
- Razor away $\mathbf{V}$ using appropriate instruments.

Algorithm:

- Build data matrices.
- Estimate $\mathcal{O}_k$, or $\{\mathbf{x}_t\}_t$.
- Recover $(\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}, \hat{\mathbf{D}})$.
- Estimate $\mathbf{P}, \mathbf{Q}$ from sample covariance of residuals.

Extensions:

- Feedback - rank conditions.
- Bilinear (States $\times$ Inputs) and nonlinear systems (Hammerstein).
- Recursive.
- Selection of the order.
- Statistical analysis.
- Finite data.

Conclusions

To remember

- Problem.
- Subspace as extended realization.
- SVD and LQ.
- Stochastic.
- Combined Deterministic - Stochastic.
- Optimality?