System Identification, Lecture 8

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Overview Part II

1. Projects.
2. State Space Systems.
3. Subspace Identification.
4. Further Topics.
5. Identification of Nonlinear Models.
6. Wider View.

Projects

What do I expect from you:

1. I give you data + description - you give me good model.
2. Single out a SISO problem, make a model and assess why/whynot satisfactory.
3. Set a baseline - where do you want to improve on?
4. Make model of MIMO system.
5. Make plots of the results, and interpret results. What is good? What is not good?
6. Use for intended purpose.
7. What's next?

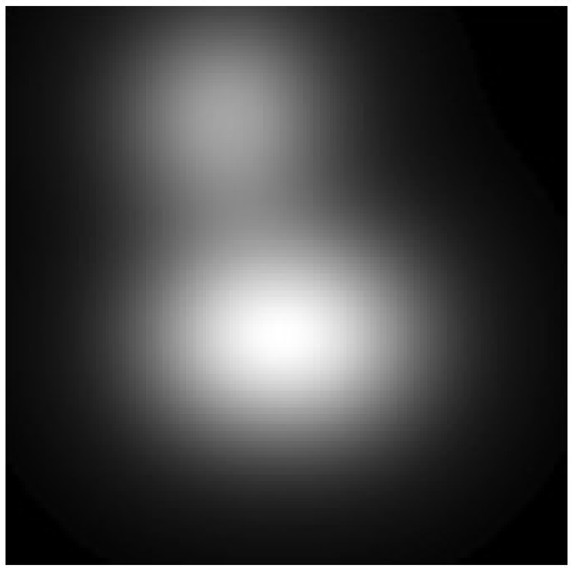
Projects

How I will evaluate Part II:

1. Groups 1-2 persons.
2. How you assess your result.
3. Report (June) (few graphs, working toward conclusion).
4. Presentation (June).

Projects

- Identification of a Multimedia stream



Challenges:

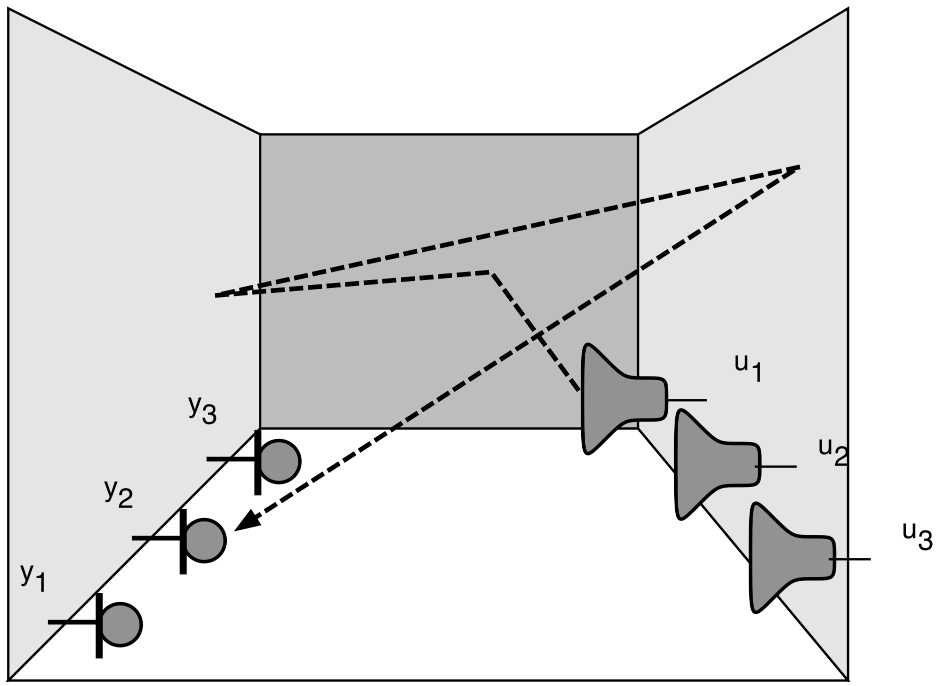
- Playing with indices.
- Simple system (shifting).
- Reconstruct example - (I give you the code if you ask for it).
- Other examples.

- Identification of an industrial Petrochemical plant

Challenges:

– Real (IPCOS).
– Control and Kalman Filter.
– MIMO.

- Identification of an Acoustic Impulse Response



Challenges:

– High Orders but loads of data.
– Preprocessing (delay).
– Mixing structure.
– Demixing.
– Dirac.

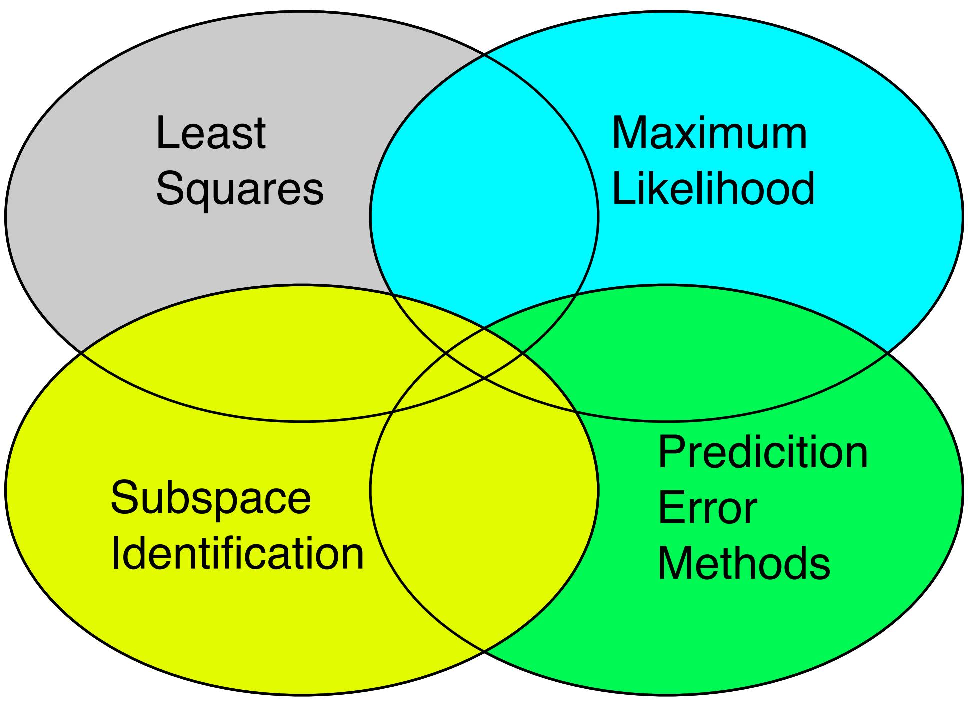
- Identification of Financial Stock Markets

Challenges:

- Stochastic System.
- Trends (inflation).
- Related Stocks.
- Model validation.
- Try to make money (in hindsight).

Summary - State Space Systems

- State Space Systems
- Observability - Controllability.
- Realization.
- Stochastic Systems



State Space System

$$\begin{cases} \mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{B}\mathbf{u}_t \\ \mathbf{y}_t = \mathbf{C}\mathbf{x}_t + \mathbf{D}\mathbf{u}_t, \end{cases} \quad \forall t = -\infty, \ldots, \infty.$$

with

- $\{\mathbf{x}_t\}_t \subset \mathbb{R}^n$ the state process.
- $\{\mathbf{u}_t\}_t \subset \mathbb{R}^p$ the input process.
- $\{\mathbf{y}_t\}_t \subset \mathbb{R}^q$ the output process.
- $\mathbf{A} \in \mathbb{R}^{n \times n}$ the system matrix.
- $\mathbf{B} \in \mathbb{R}^{n \times p}$ the input matrix.
- $\mathbf{C} \in \mathbb{R}^{q \times n}$ the output matrix.
- $\mathbf{D} \in \mathbb{R}^{q \times p}$ the feed-through matrix.

The 'state variable' has different interpretations.

1. Representation of hidden 'state' of system (physical).

2. Summarization of what to remember from past.

3. Compact representation of information relevant to predict future.

4. Intersection of past and future.

5. Optimal estimate of the model parameters thus far (RLS).

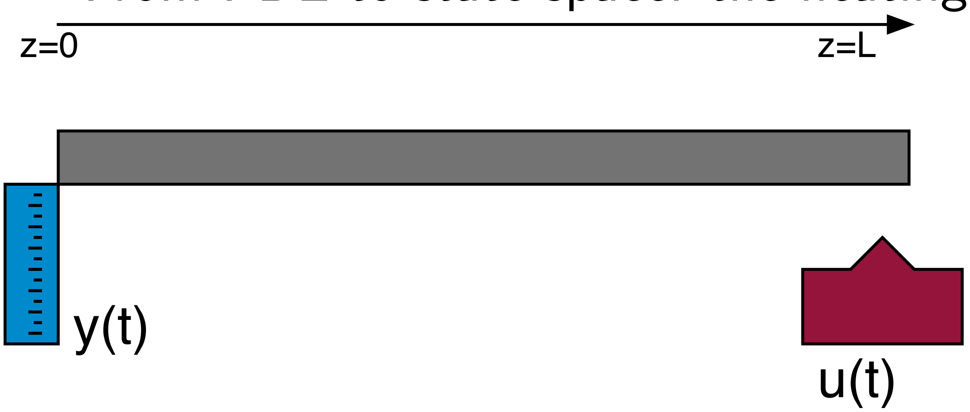
State Space System

Advantages over fractional polynomial models

- Closer to physical modeling.
- MIMO systems.
- Noise and Innovations.
- Canonical representation.
- Problems of identiafibility.

State Space System - ex. 1

From PDE to state-space: the heating-rod system:



Let $x(t,z)$ denote temperature at time t, and location z on the rod.

$$\frac{\partial x(t,z)}{\partial t} = \kappa \frac{\partial^2 x(t,z)}{\partial z^2}$$

The heating at the far end mens that

$$\left.\frac{\partial x(t,z)}{\partial z}\right|_{z=L} = K u(t),$$

The near-end is insulated such that

$$\left.\frac{\partial x(t,z)}{\partial z}\right|_{z=0} = 0.$$

The measurements are

$$y(t) = x(t, 0) + v(t), \forall t = 1, 2, \ldots$$

The unknown parameters are

$$\theta = \begin{bmatrix} \kappa \\ K \end{bmatrix}$$

This can be approximated as a system with n states

$$\mathbf{x}(t) = \Big(x(t, z_1), x(t, z_2), \ldots, x(t, z_n)\Big)^T \in \mathbb{R}^n$$

with $z_k = L(k-1)/(n-1)$.. Then we use the approximation that

$$\frac{\partial^2 x(t, z)}{\partial z^2} \approx \frac{x(t, z_{k+1}) - 2x(t, z_k) + x(t, z_{k-1})}{(L/(n-1))^2}$$

where $z_k = \operatorname{argmin}_{z_1,,\ldots,z_n} \|z - z_k\|$. Hence the continuous state-space approximation becomes

$$
\begin{cases}
\dot{\mathbf{x}}(t) = \left(\frac{n-1}{L}\right)^2 \kappa \begin{bmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ \vdots \\ K \end{bmatrix} u(t) \\
y(t) = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} \mathbf{x}(t) + v(t)
\end{cases}
$$

and a discrete Euler approximation

$$
\begin{cases}
\mathbf{x}_{t+1} - \mathbf{x}_t & = \Delta' \left(\frac{n-1}{L}\right)^2 \kappa \\
& \quad \begin{bmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \end{bmatrix} \mathbf{x}_t + \Delta' \begin{bmatrix} 0 \\ \vdots \\ K \end{bmatrix} \int_{\Delta'} u(t) \\
y_t & = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} \mathbf{x}_t + \int_{\Delta'} v(t)
\end{cases}
$$

State Space System - ex. 2

Models for the future size of the population (UN, WWF).

Leslie model: key ideas: discretize population in n aging groups and

- Let $\mathbf{x}_{t,i} \in \mathbb{R}^+$ denote the size of the ith aging group at time t.

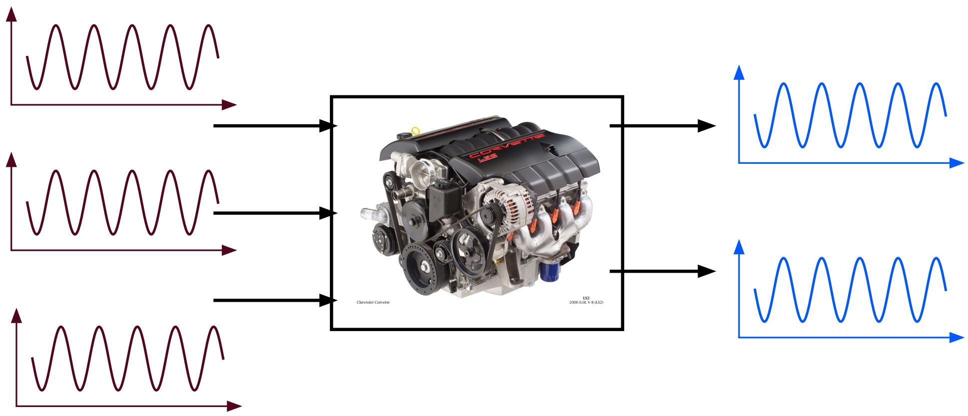
- Let $\mathbf{x}_{t+1,i+1} = s_i \mathbf{x}_{t,i}$ with $s_i \geq 0$ the 'survival' coefficient.
- Let $\mathbf{x}_{t+1,1} = s_0 \sum_{i=1}^n f_i \mathbf{x}_{t,i}$ with $f_i \geq 0$ the 'fertility' rate.

Hence, the dynamics of the population may be captured by the following discrete time model

$$
\begin{cases}
\mathbf{x}_{t+1} = \begin{bmatrix} s_0 f_1 & s_0 f_2 & \dots & s_0 f_n & \\ s_1 & 0 & & & \\ 0 & s_2 & 0 & & \\ & & \ddots & & \\ & & & s_{n-1} & 0 \end{bmatrix} \mathbf{x}_t + \mathbf{u}_t \\
y_t = \sum_{i=1}^n \mathbf{x}_{t,i}
\end{cases}
$$

Impulse Response to State Space System

What is now the relation of state-space machines, and the system theoretic tools seen in the previous Part?



Recall impulse response (SISO)

$$y_t = \sum_{\tau=0}^{\infty} h_\tau u_{t-\tau},$$

and MIMO

$$\mathbf{y}_t = \sum_{\tau=0}^{\infty} \mathbf{H}_\tau \mathbf{u}_{t-\tau},$$

where $\{\mathbf{H}_\tau\}_\tau \subset \mathbb{R}^{p\times q}$.

Recall: System identification studies method to build a model from observed input-output behaviors, i.e. $\{\mathbf{u}_t\}_t$ and $\{\mathbf{y}_t\}_t$.

Now it is a simple exercise to see which impulse response matrices $\{\mathbf{H}_\tau\}_\tau$ are implemented by a state-space model with matrices $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$:

$$\mathbf{H}_\tau = \begin{cases} \mathbf{D} & \tau = 0 \\ \mathbf{C}\mathbf{A}^{\tau-1}\mathbf{B} & \tau = 1, 2, \ldots \end{cases}, \ \forall \tau = 0, 1, 2, \ldots$$

Contrast with rational polynomials where typically

$$h_\tau \Leftrightarrow h(q^{-1}) = \frac{b_1 q^{-1} + b_2 q^{-2} + \ldots}{1 + a_1 q^{-1} + a_2 q^{-2} + \ldots}$$

Overlapping: consider FIR model

$$y_t = b_0 u_t + b_1 u_{t-1} + b_2 u_{t-2} + e_t$$

then equivalent state-space with states $\mathbf{x}_t = (u_t, u_{t-1}, u_{t-2})^T \in \mathbb{R}^3$ becomes

$$\begin{cases} \mathbf{x}_t &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{x}_{t-1} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_t \\ y_t &= \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} \mathbf{x}_t + e_t \end{cases}$$

and $\mathbf{x}_0 = (u_0, u_{-1}, u_{-2})^T$.

Controllability and Observability

A state-space model is said to be Controllable iff for any terminal state $\mathbf{x} \in \mathbb{R}^n$ one has that for all initial state $\mathbf{x}_0 \in \mathbb{R}^n$, there exists an input process $\{\mathbf{u}_t\}_t$ which steers the model from state $\mathbf{x}_0$ to $\mathbf{x}$.

A state-space model is said to be Reachable iff for any initial state $\mathbf{x}_0 \in \mathbb{R}^n$ one has that for all terminal states $\mathbf{x} \in \mathbb{R}^n$ there exists an input process $\{\mathbf{u}_t\}_t$ which steers the model from state $\mathbf{x}_0$ to $\mathbf{x}$.

The mathematical definition goes as follows: Define the reachability matrix $\mathcal{C} \in \mathbb{R}^{n \times np}$ as

$$\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \ldots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$$

The State space $(\mathbf{A}, \mathbf{B})$ is reachable (controllable) if

$$\operatorname{rank}(\mathcal{C}) = n.$$

Intuition: if the matrix $\mathcal{C}$ is full rank, the image of $\mathcal{C}$ equals $\mathbb{R}^n$, and the superposition principle states that any linear combination of states can be reached by a linear combination of inputs.

A state-space model is Observable iff any two different initial states $\mathbf{x}_0 \neq \mathbf{x}_0' \in \mathbb{R}^n$ lead to a different output $\{\mathbf{y}_s\}_{s\geq 0}$ of the state-space model in the future when the inputs are switched off henceforth (autonomous mode).

Define the Observability matrix $\mathcal{O} \in \mathbb{R}^{qn\times n}$ as

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}$$

Hence, a state-space model $(\mathbf{A}, \mathbf{C})$ is observable iff

$$\text{rank}(\mathcal{O}) = n$$

Intuition: if the (right) null space of $\mathcal{O}$ is empty, no two different $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^n$ lead to the $\mathcal{O}\mathbf{x} = \mathcal{O}\mathbf{x}'$.

Let

$$\mathbf{u}_- = (\mathbf{u}_0, \mathbf{u}_{-1}, \mathbf{u}_{-2}, \dots)^T$$

And

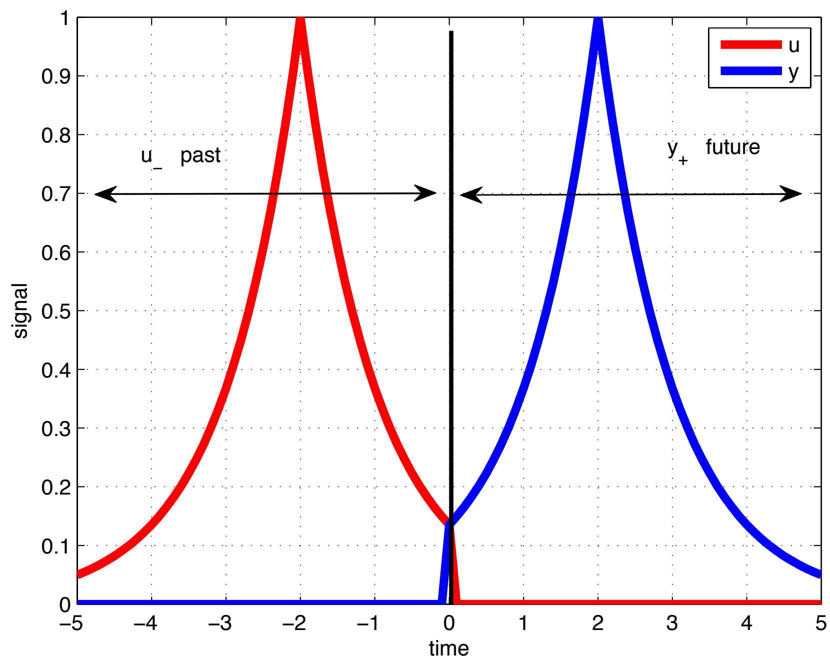
$$\mathbf{y}_+ = (\mathbf{y}_1, \mathbf{y}_2, \dots)^T$$

Then

$$\mathbf{x}_1 \propto \mathcal{C}\mathbf{u}_-$$

and

$$\mathbf{y}_+ \propto \mathcal{O}\mathbf{x}_1$$



Realization Theory

Problem: Given an impulse response sequence $\{\mathbf{H}_\tau\}_\tau$, can we recover $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$?

Def. Minimal Realization. A state-space model $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is a minimal realization of order n iff the corresponding $\mathcal{C}$ and $\mathcal{O}$ are full rank, that is iff the model is reachable (observable) and controllable.

Thm. (Kalman) If $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ and $(\mathbf{A}', \mathbf{B}', \mathbf{C}', \mathbf{D}')$ are two minimal realizations of the same impulse response $\{\mathbf{H}_\tau\}$, then they are linearly related by a nonsingular matrix $\mathbf{T} \in \mathbb{R}^{n\times n}$ such that

$$\begin{cases} \mathbf{A}' = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} \\ \mathbf{B}' = \mathbf{T}^{-1}\mathbf{B} \\ \mathbf{C}' = \mathbf{C}\mathbf{T} \\ \mathbf{D}' = \mathbf{D} \end{cases}$$

Intuition: a linear transformation of the states does not alter input-output behavior; that is, the corresponding $\{\mathbf{H}_\tau\}_\tau$ is the same. The thm states that those are the only transformations for which this is valid.

Hence, it is only possible to reconstruct a minimal realization of a state-space model $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ from $\{\mathbf{H}_\tau\}_\tau$ *up to a linear transformation of the states*.

In case we only observe sequences $\{\mathbf{u}_t\}_{t\geq 1}$ and $\{\mathbf{y}_t\}_{t\geq 1}$, we have to account for the transient effects and need to estimate $\mathbf{x}_0 \in \mathbb{R}^n$ as well. This is in many situations crucial. The above thm. is extended to include $\mathbf{x}_0$ as well.

Now the celebrated Kalman-Ho realization algorithm goes as follows:

- Hankel-matrix

$$\mathbf{H}^n = \begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_2 & \mathbf{H}_3 & \dots & \mathbf{H}_n \\ \mathbf{H}_2 & \mathbf{H}_3 & \mathbf{H}_4 & & \\ & & \ddots & & \\ \mathbf{H}_n & & & & \mathbf{H}_{2n+1} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{CB} & \mathbf{CAB} & \mathbf{CA}^2\mathbf{B} & \dots & \mathbf{CA}^{n-1}\mathbf{B} \\ \mathbf{CAB} & \mathbf{CA}^2\mathbf{B} & & & \\ & & \ddots & & \\ \mathbf{CA}^{n-1}\mathbf{B} & & & & \mathbf{CA}^{2n-1}\mathbf{B} \end{bmatrix} = \mathcal{O}\mathcal{C}$$

- The state space is identifiable up to a non-singular matrix $\mathbf{T} \in \mathbb{R}^{n \times n}$ such that

$$\mathbf{H}^n = \mathcal{O}\mathcal{C} = \mathcal{O}\mathbf{T}\mathbf{T}^{-1}\mathcal{C}$$

.

- Then take the SVD of $\mathbf{H}^n$, such that

$$\mathbf{H}^n = \mathbf{U}\Sigma\mathbf{V}^T$$

with $\mathbf{U} \in \mathbb{R}^{pn \times n}, \mathbf{V} \in \mathbb{R}^{n \times nq}$ and $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^{n \times n}$.

- Hence a minimal realization is given as

$$\begin{cases} \mathcal{O}' = \mathbf{U}\sqrt{\Sigma} \\ \mathcal{C}' = \sqrt{\Sigma}\mathbf{V} \end{cases}$$

- From $\mathcal{O}', \mathcal{C}'$ it is not too difficult to extract $(\mathbf{A}, \mathbf{B}, \mathbf{C})$

An Example

Given an input

$$u = (1, 0, 0, 0, \dots)^T$$

and output signal

$$y = (0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \dots)^T$$

with system

$$y_t = y_{t-1} + y_{t-2},\ y_0 = 0, y_1 = u_1$$

or SS with $\mathbf{x}_0 = (0, 0)^T$ as

$$\begin{cases} \mathbf{x}_{t+1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{x}_t + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_t \\ y_t = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}_t \end{cases}$$

or transfer function

$$G(z) = \frac{z}{z^2 - z - 1}$$

Now realization

$$\mathbf{H}_5 = \begin{bmatrix} 1 & 1 & 2 & 3 & 5 \\ 1 & 2 & 3 & 5 & 8 \\ 2 & 3 & 5 & 8 & 13 \\ 3 & 5 & 8 & 13 & 21 \\ 5 & 8 & 13 & 21 & 34 \end{bmatrix}$$

Then SVD of $\mathbf{H}_5$ gives $\sigma_1 = 54.5601$ and $\sigma_2 = 0.4399$, and a minimal realization is

$$\begin{cases} \mathbf{x}'_{t+1} = \begin{bmatrix} 1.6179 & 0.0185 \\ 0.0185 & -0.6179 \end{bmatrix} \mathbf{x}'_t + \begin{bmatrix} 0.8550 \\ -0.5187 \end{bmatrix} u_t \\ y_t = \begin{bmatrix} 0.8550 & -0.5187 \end{bmatrix} \mathbf{x}_t \end{cases}$$

Stochastic Systems

Stochastic disturbances (no inputs)

$$\begin{cases} X_{t+1} & = \mathbf{A} X_t + W_t \\ Y_t & = \mathbf{C} X_t + V_t \end{cases}$$

with

- $\{X_t\}_t$ the stochastic state process taking values in $\mathbb{R}^n$.
- $\{Y_t\}_t$ the stochastic output process, taking values in $\mathbb{R}^p$.
- $\mathbf{A} \in \mathbb{R}^{n \times n}$ the (deterministic) system matrix.
- $\mathbf{C} \in \mathbb{R}^{p \times n}$ the (deterministic) output matrix.
- $\{W_t\}_t$ the stochastic process disturbances taking values in $\mathbb{R}^n$.

- $\{V_t\}_t$ the stochastic measurement disturbances taking values in $\mathbb{R}^p$.

The stochastic vectors follow a probability law assumed to follow

- $\mathbb{E}[W_t] = 0_n$, and $\mathbb{E}[W_t W_s^T] = \delta_{s,t}\mathbf{Q} \in \mathbb{R}^{n\times n}$.

- $\mathbb{E}[V_t] = 0_p$, and $\mathbb{E}[V_t V_s^T] = \delta_{s,t}\mathbf{R} \in \mathbb{R}^{p\times p}$.

- $\mathbb{E}[W_t V_t^T] = \delta_{s,t}\mathbf{S} \in \mathbb{R}^{n\times p}$.

- W_t, V_t assumed independent of $\ldots, X_t$.

Main questions:

- Covariance matrix states $\mathbb{E}[X_t X_t^T] = \Pi$:

$$\Pi = \mathbf{A}\Pi\mathbf{A}^T + \mathbf{Q}$$

  - Lyapunov, stable.

- Covariance matrix outputs $\mathbb{E}[Y_t Y_t^T]$.

This model can equivalently be described in its innovation form

$$\begin{cases} X'_{t+1} & = \mathbf{A}X'_t + \mathbf{K}D_t \\ Y_t & = \mathbf{C}X'_t + D_t \end{cases}$$

with $\mathbf{K} \in \mathbb{R}^{n\times p}$ the Kalman gain, such that $\mathbf{P}, \mathbf{K}$ solves

$$\begin{cases} \mathbf{P} = \mathbf{APA} + (\mathbf{G} - \mathbf{APC}^T)(\Lambda_0 - \mathbf{CPC}^T)^{-1}(\mathbf{G} - \mathbf{APC}^T)^T \\ \mathbf{K} = (\mathbf{G} - \mathbf{APC}^T)(\Lambda_0 - \mathbf{CPC}^T) \end{cases}$$

and

- $\mathbb{E}[D_t {D'_t}^T] = (\Lambda_0 - \mathbf{CPC}^T)$
- $\mathbf{P} = \mathbb{E}[X'_t {X'_t}^T]$

Conclusions

- State-space systems for MIMO - distributed parameter systems.
- Relation impulse response - state-space models.
- Controllability - Observability
- Kalman - Ho
- Stochastic Systems

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