Chapter 3

Flavors of Dynamic Systems and Models

"How can we mathematically represent and characterize a dynamical system? And what is a good representation for a certain task?"

This chapter is a survey of some important results in systems theory, and will serve to fix ideas, notation and language to be used in later chapters. Models come in many different flavors. We will mainly be interested in models relating input to output signals using formal equations. But keep in mind that this is but one choice. In some cases it is good practice to express models using predicate logic, software code, graphs and pictures, human language or even intuitions. At the end a model is only an internal representation of the actual system, and is as good proportionally to how well it serve its final purpose.

3.1 General Characterization of Dynamic Systems

In the context of this course, we are interested in dynamic systems, i.e. systems which have a nontrivial dependence on its past (future) signals. To make the ideas in this chapter easier to present, we will restrict ourself to systems accepting a univariate input signal $\{u_t\}_t \subset \mathbb{R}$, and outputting a univariate signal $\{y_t\}_t \subset \mathbb{R}$. Such systems are called SISO ('Single Input, Single Output'). It is not to hard then to figure out what MIMO, MISO, TITO stands for, is it?

Dynamic models come in different forms, but arguably the largest watershed is drawn between models represented in continuous-time, or in discrete-time. While in the physical sciences one often builds up models based on first-order principles expressed as continuous Ordinary Differential Equations (ODEs) or Partial Differential equations (PDEs), in the context of system identification one typically chooses discrete representations.

(Digital): The system's input-output behavior is recorded as numerical values, The keyword 'digital' suggests often a discrete nature either in time scale, or in the values of the signals.

(Parameters): The unknown quantities in the assumed model structure which is used to describe the system of interest. Those parameters will be estimated based on the collected input-output data of

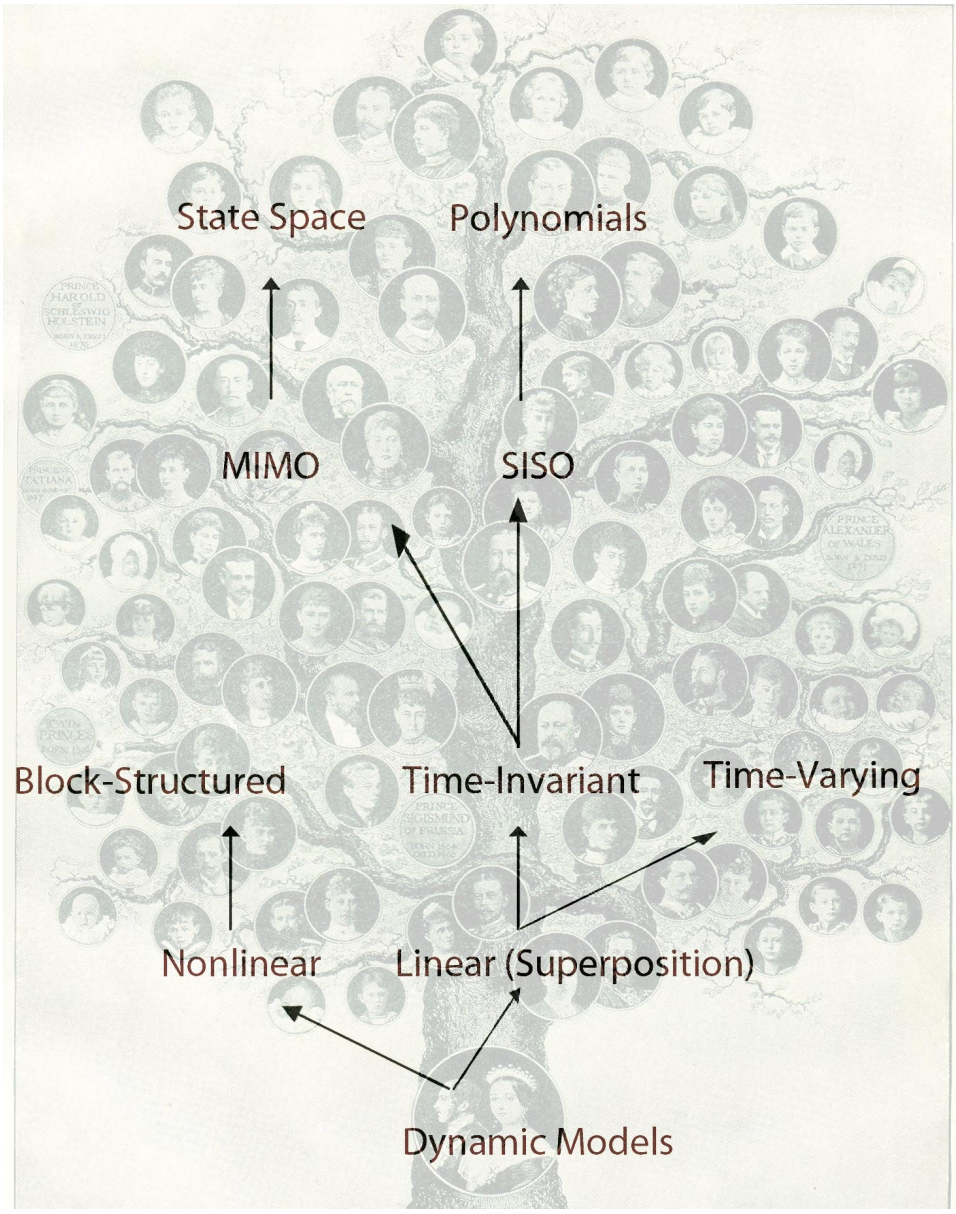


Figure 3.1: A Taxonomy of Systems

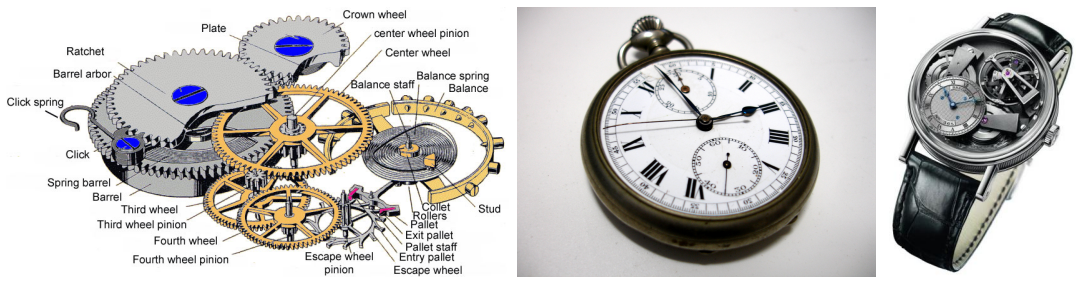
the systems. In general, the there are only a small, constant number of such parameters (as e.g. compared to n) to be estimated.

(White-box): Sometimes a model can be formalized in terms of physical laws, chemical relations, or other theoretical considerations for the studied system. Such a model is called white-box, as the intern model description directly appeals to the intern mechanism supposed to underly the system. As such, the system explains (in a sense) *why* it operates as is seen in input-output behavior.

(Black-box): A black-box model does not directly appeal to the actual mechanisms which are supposed to underly the studied system. A black-box model merely intends to make good predictions of the (future) system behavior: the internal description merely serves to relate input signals to output signals.

(Grey-box): Grey-box models are a mixture between black-box and white-box models. Some of the internal working of the system can be assumed to be dictated by laws, The complete input-output behavior is only known however up to some gaps. Those blanks can then be modeled using black-box techniques.

In the context of this course, mainly black-box models are considered. Section 5.3 will discuss how techniques can be extended to white- and grey-box models.



(a) (b) (c)

Figure 3.2: Metaphorical representation of a (a) white-box model where you see the internal working, (b) black-box model where you only see the output of the system, and (c) grey-box model where both are present to some degree.

The conversion of continuos time-models into an (approximate) discrete-time counterpart falls within the area of systems theory. This overall challenge is historically and research-wise related to numerical solvers used for finding solutions for sets of differential equations. Again, this book will adopt the convention to denote continuous signals using its 'argument form', e.g. $u(t)$. Its discrete counterpart are denoted as u_t where now t becomes an index taking integer values, i.e. $t = 1, 2, 3, \ldots$.

3.2 LTI Systems

The following definitions help us to narrow down the class of models studied here. The central notion is the idea of an impulse response, intuitively the timely effect observed at the output of a

system when injecting a pulse signal into the system. An example is given in Fig. (3.3).

Figure 3.3: An intuitive illustration of an impulse response: a sound beep ripples an acoustic signal through a room.

Definition 7 (Time-invariant System) *A system is called* time-invariant *if its response to a certain input signal does not depend on absolute time.*

Formally, the properties of the model (e.g. the orders, parameters or sampling time) do not depend on the precise index t. An example of a time-invariant model is $y_t = \theta + e_t$. If this model were time-varying it would be denoted as $y_t = \theta_t + e_t$.

Definition 8 (Causal System) *A system is called* causal *if an output response at a certain instant depends only on the inputs up to that time. That is, if a useful prediction can be made based on the past signals only.*

Definition 9 (Superposition Principle) *The superposition principle states that for any linear system the net response at a given place and time caused by a linear combination of different input signals equals the same linear combination of the output responses which would have been caused by individual application of each input signal.*

Theorem 1 (Continuous Impulse Response (IR) representation) *Given a causal LTI system $\mathcal{S}$, then its mapping from any continuous input signal $\{u(s)\}_{s \leq t}$ to a corresponding output $y(t)$, for any such t, can be represented using a fixed function $h : \mathbb{R}_+ \to \mathbb{R}$ as*

$$y(t) = \int_{\tau=0}^{\infty} h(\tau) u(t - \tau) d\tau. \tag{3.1}$$

The function h is called the continuous Impulse Response (IR) function.

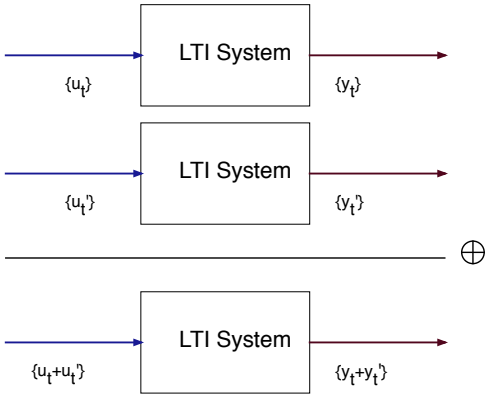


Figure 3.4: Illustration of the superposition principle as in Def. (9).

Theorem 2 (Discrete Impulse Response (IR) representation) *Given a causal LTI system $\mathcal{S}$, then its mapping from any input signal $\{\ldots, u_{t-1}, u_t\}$ to a corresponding output y_t can be represented using a fixed sequence $\{h_0, h_1, \ldots, h_d\}$ as*

$$y_t = \sum_{\tau=0}^{d} h_\tau u_{t-\tau}, \ \forall t = 1, \ldots. \tag{3.2}$$

with order d which is possibly infinite. The (infinite) vector $\mathbf{h} = (h_0, h_1, \ldots, h_d)^T \in \mathbb{R}^d$ is called the discrete Impulse Response (IR) vector.

This step from continuous representation (or 'model') to a discrete representation ('model') is intuitively seen as follows: As working assumption we take that we sample the (continuous) time as every other $\Delta > 0$ period, such that the sample after a time Δ correspond with the zeroth sample. Formally, this is written as the relation for every $t' = \ldots, -2, -1, 0, 1, 2, \ldots$ as

$$u_{t'} = u(t'\Delta + \tau), \ \forall 0 \leq \tau < \Delta, \tag{3.3}$$

and $u_0 = u(0)$. Note the use of the symbol t and t' for differentiating between continuous time and discrete index. In the rest van this text we will use t in both cases as its meaning is almost always clear from its context.

$$\begin{aligned} y(t'\Delta) &= \int_{\tau=0}^{\infty} h(\tau) u(t'\Delta - \tau) d\tau \\ &= \sum_{\tau'=1}^{\infty} \int_{\tau=(\tau'-1)\Delta}^{\tau'\Delta} h(\tau) u(t'\Delta - \tau) d\tau \\ &= \sum_{\tau'=1}^{\infty} \left(\int_{\tau=(\tau'-1)\Delta}^{\tau'\Delta} h(\tau) d\tau \right) u_{t'-\tau'} \\ &= \sum_{\tau'=1}^{\infty} h_{\tau'} u_{t'-\tau'}, \end{aligned} \tag{3.4}$$

where we define for any $\rho = 1, 2, 3, \ldots$ that

$$h_{\tau'} = \int_{\tau=(\tau'-1)\Delta}^{\tau'\Delta} h(\tau)d\tau. \tag{3.5}$$

Again we will use τ in general to denote both the displacement in the continuous case (i.e. τ), as the distant in the discrete case (i.e. τ').

Observe that no approximation need to be made if one assumes eq. (3.5), and that it is sufficient in the sense that $\{h_\tau\}_\tau$ fully specifies the (continuous) response to the input signal. Even if eq. (3.5) does not hold, $\{h_\tau\}_\tau$ might still give a good discrete to what is happening in continuous time, provided the signal $u(t)$ does not change too much during intersample intervals. The study of different sampling schemes and the influence on subsequent analysis steps goes to the heart of digital control theory and signal processing, and has its obvious relevance in the design of A/D convertors.

The following two examples are prototypical.

Example 11 (A First Order Example) *At first an example is given of a continuous time first order linear system. Assume a system is modeled as a first order differential equation, or*

$$T\frac{dy(t)}{dt} + y(t) = Ku(t-\tau), \tag{3.6}$$

for a time delay $\tau \geq 0$, a gain $K > 0$ and time constant $T > 0$. Then the impulse response can be computed by equating the signal $u(t)$ to an impulse $\delta(t)$, and to solve for the corresponding $y(t)$. Similarly, one can compute the solution when the system is excited with the step signal $\{u(t) = 1(t \geq 0), t = -\infty, \ldots, \infty\}$. This solution is given in Fig. (3.5).

Conversely, one can determine the parameters τ, K, T by looking at the step response. Figure (3.5) demonstrates a graphical method for determining the parameters K, T, τ from the step response:

- *The gain K is given by the final value.*
- *By fitting the steepest tangent, T and τ can be obtained. The slope of this tangent is $\frac{K}{T}$*
- *This tangent crosses the time-axis at τ, the time delay.*

Example 12 (Step Response of a damped Oscillator) *Consider a second-order continuous system characterized by a differential equation, or*

$$\frac{d^2y(t)}{dt^2} + 2\xi\omega_0\frac{dy(t)}{dt} + \omega_0^2 y(t) = K\omega_0^2 u(t). \tag{3.7}$$

After some calculations the solution when of the differential equation when exited with a step input is found to be

$$y(t) = K\left(1 - \frac{e^{-\xi\omega_0 t}}{\sqrt{1-\xi^2}}\sin\left(\omega_0 t\sqrt{1-\xi^2} + \tau\right)\right), \tag{3.8}$$

where the time delay is $\tau = \arccos\xi$. Here the gain is parametrized by K, and the timescale is parametrized by ω_0. The parameter ξ regulates the damping of the oscillation of the system, hence the name. This step response is illustrated in Fig. (3.6) for different values of $\xi = 0.1, 0.2, 0.5, 0.7, 0.99$. In the example we fix the gain as $K = 1$, and the timescale as $\omega_0 = 1$.

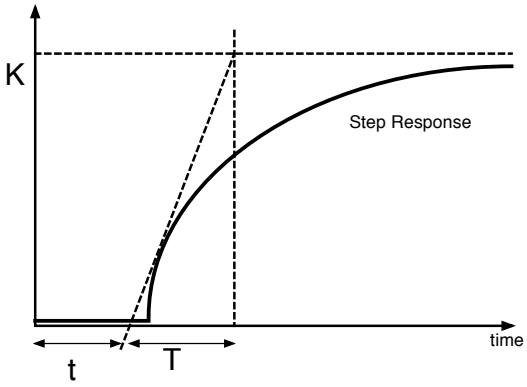


Figure 3.5: Example of a first-order system given in eq. (3.6)

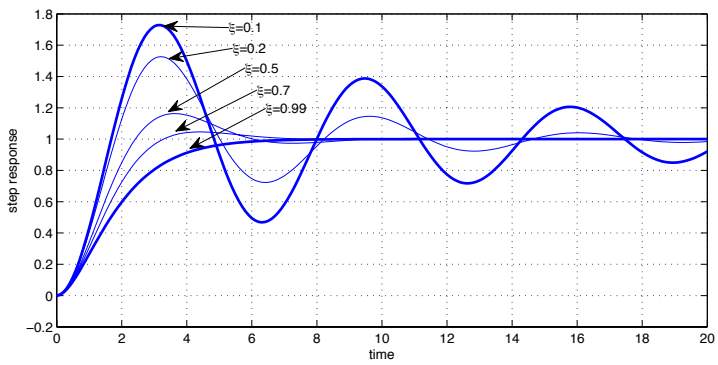


Figure 3.6: Example of a step response of a second-order system given in eq. (3.7) In this example $K = 1$ and $\omega_0 = 1$.

Chapter 13 discusses how one can extend this model class to account for nonlinear effects. That is, how identification techniques can be applied when the Linear superposition property is not valid any longer.

3.2.1 Transforms

In order to get insight into why, how and what a certain model is capable of, it turns out to be quite useful to express the dynamics using different languages. That is, the model dynamics are transformed to various descriptions. This amounts in general to the theory and practice of transforms.

Sinusoid Response and the Frequency Function

Let us elaborate on the simple case where the input of a system is a cosine function:

$$u_t = \cos(\omega t),\ \forall t = \ldots, -2, -1, 0, 1, 2, \ldots. \tag{3.9}$$

It will be convenient to rewrite this as

$$u_t = \Re e^{i\omega t},\ \forall t = -2, -1, 0, 1, 2, \ldots. \tag{3.10}$$

with $\Re\cdot$ denoting the 'real part'. This follows from Fermat's equality that $e^{i\omega_0 t} = \cos(i\omega_0 t) + i\sin(i\omega_0 t)$ as depicted in Fig. (3.7).

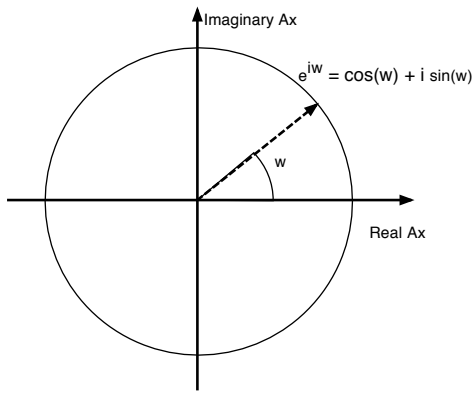


Figure 3.7: The Complex unit-circle.

Then given a (discrete) system $\{h_\tau\}_\tau$ we have for any $t = \ldots, -1, 0, 1, \ldots$ that

$$\begin{aligned} y_t &= \sum_{\tau=1}^{\infty} h_\tau \cos(\omega t - \tau) \\ &= \sum_{\tau=1}^{\infty} h_\tau \Re e^{i\omega(t-\tau)} \\ &= \Re \sum_{\tau=1}^{\infty} h_\tau e^{i\omega(t-\tau)} \\ &= \Re \left(e^{i\omega t} \sum_{\tau=1}^{\infty} h_\tau e^{-i\omega\tau} \right), \end{aligned} \tag{3.11}$$

the second equality follows as h_τ is real. Using the definition

$$H\left(e^{i\omega}\right) = \sum_{\tau=1}^{\infty} h_\tau e^{-i\omega\tau}, \tag{3.12}$$

one can write further

$$\begin{aligned} y_t &= \Re\left(e^{i\omega t} H\left(e^{i\omega}\right)\right) \\ &= \left|H\left(e^{i\omega}\right)\right| \cos\left(\omega t + \varphi\right), \end{aligned} \tag{3.13}$$

where

$$\varphi = \arg H\left(e^{i\omega}\right). \tag{3.14}$$

And the absolute value and the argument of a complex value $x + iy \in \mathbb{C}$ are defined as

$$|x + iy| = \sqrt{x^2 + y^2}, \tag{3.15}$$

and

$$\arg(x + iy) = \begin{cases} 2 \arctan \frac{y}{\sqrt{x^2+y^2}+x} & x > 0, y \neq 0 \\ \pi & x < 0, y = 0 \\ \infty & \text{else.} \end{cases} \tag{3.16}$$

This follows from the rules of complex calculus. This derivation is paramount into understanding how LTIs work. The engineering view is that an LTI H is a mapping from a sinusoid with frequency $-\pi \leq \omega \leq \pi$, to a corresponding one with the same frequency, but with amplitude magnified by $\left|H\left(e^{i\omega}\right)\right|$, and phase increased by $\arg H\left(e^{i\omega}\right)$. For that reason, the function $H\left(e^{i\omega}\right)$ is denoted as the *frequency function* or *transfer function* of the LTI H. The method of transforms - coming in the form of Laplace, z- or Fourier transforms is then all about the concept of $H\left(e^{i\omega}\right)$, as will be elaborated in the remainder of this section.

History has provided us with many graphical tools to characterize LTIs, amongst which

(Bode) Represents the amplitudes $\left|H\left(e^{i\omega}\right)\right|$ and phases $\arg H\left(e^{i\omega}\right)$ as a function of the frequency $-\pi \leq \omega \leq \pi$.

(Nyquist) Represents for any $-\pi \leq \omega \leq \pi$ the complex numbers

$$\left(\left|H\left(e^{i\omega}\right)\right|, \arg H\left(e^{i\omega}\right)\right) \tag{3.17}$$

as curves in the 2D graph.

These concepts are often studied in the continuous-time case, but their basic properties carry over to the discrete-time case as well.

Now, those reasonings motivate us to decompose the given signals into contributions of sinusoids with various phases and amplitudes. Indeed if we know this decomposition it is straightforward to characterize the system from observed input- and output signals. Let us consider first the case where this decomposition is performed on a input-signal $\{u_t\}_{t=1}^n$ of finite length (!). Now define the function $\mathcal{U}_n : \mathbb{R} \to \mathbb{C}$ for any $-\pi \leq \omega \leq \pi$ as

$$\mathcal{U}_n(\omega) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} u_t e^{-i\omega t}. \tag{3.18}$$

The values obtained for $\omega = \frac{1}{n} 2\pi k$ for $k = 1, \ldots, n$ form the Discrete Fourier Transform of the finite sequence $\{u_t\}_{t=1}^n$. We can reconstruct the original sequence $\{u_t\}_{t=1}^n$ from $\{\mathcal{U}(\frac{i2\pi k}{n})\}_{k=1}^n$ as

$$u_t = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \mathcal{U}_n\left(\frac{2\pi k}{n}\right) e^{i2\pi k t}, \tag{3.19}$$

for any $t = 1, \ldots, n$. Indeed

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} \mathcal{U}_n\left(\frac{2\pi k}{n}\right) e^{i2\pi k t} = \frac{1}{n} \sum_{k=1}^{n} \sum_{s=1}^{n} u_s \exp\left(\frac{2i\pi k(t-s)}{n}\right) = \frac{1}{n} \sum_{s=1}^{n} u_s n \delta_{t-s} = u_t, \tag{3.20}$$

where

$$\delta_{k-s} = \frac{1}{n} \sum_{k=1}^{n} \exp\left(\frac{i2\pi r k}{n}\right) = \begin{cases} 1 & r = 0 \\ 0 & r \neq 0. \end{cases}. \tag{3.21}$$

This follows in turn as the different functions $\{\exp(i2\pi k/n)\}_{k=1}^n$ are carefully crafted as orthonormal functions.

As such, we have found a good decomposition of the signal $\{u_t\}$ in its frequency components $\{\mathcal{U}_n(\omega_k)\}_k$. It is instructive to consider the 'energy' of the different frequencies. This notion is captured by the *periodogram*, which is a function defined for any $-\pi \leq \omega \leq \pi$ as

$$|\mathcal{U}_n(\omega_k)|^2, \ \forall \omega_k. \tag{3.22}$$

Parseval's equality gives then that

$$\sum_{k=1}^{n} \left|\mathcal{U}_n\left(\frac{2\pi k}{n}\right)\right|^2 = \sum_{t=1}^{n} u_t^2. \tag{3.23}$$

3.3 Useful Representations

3.3.1 Disturbances

Again, we will focus on the LTI model

$$y_t = \sum_{\tau=0}^{\infty} h_\tau u_{t-\tau} = H(q^{-1})u_t. \tag{3.24}$$

A disturbance refers to the collection of effects such that the system at hand is not entirely described as in eq. (3.24). Herefor, the model of eq. (3.24) is extended as

$$y_t = H(q^{-1})u_t + d_t, \tag{3.25}$$

where $\{d_t\}_t \subset \mathbb{R}$ denote the disturbances. It is often realistic to have such terms as in practice one may be faced with data which is perturbed by

(Random Noise): The observations made during the experiment are often due to stochastic signals influencing the system during operation. Effects are roughly called stochastic if they differ from experiment to experiment. That is, if they result in different signals even when the experiment is repeated under identical circumstances. Noise terms may be due to inaccuracies in the measurement devices, or to external influences which are not directly measured, and which are beyond control of the designed experiment. Often, the stochastic framework and statistical models provide a reasonable description of such effects.

(Under-modeling): In case the system does not quite fit the model structure which is chosen, disturbance terms might reflect the 'residual' dynamics of the system. Those are often present as a model is a useful abstraction of the studied system, and a complete modeling is in many real-world cases beyond reach.

(Nonlinear Effects): A model satisfying the superposition principle is often accurate (convenient) enough for our purposes. However, most systems reflect some sort of deviation of this mathematical principle, in general denoted as a 'nonlinear effects'. In process industry for example, saturation effects often occur. But in case the system remains at more or less the same operation regime the system can be expected to behave linear. A disturbance term can absorb the occasional nonlinear effects nonetheless.

(Time Varying): In the same spirit as the nonlinear effects, any real system displays non time-invariant effects. But if one remains more or less in the same operation regime and avoids structural changes during the experiment, a disturbance term might get rid of minor time-varying effects. A particular time-varying effect is due to aging of the experimental setup, which can be avoided by collecting data in a relatively short timespan.

3.3.2 Polynomial Representations

Consider the SISO case. Then any LTI can be represented as

$$\begin{aligned} y_t &= \sum_{\tau=0}^{\infty} h_\tau u_{t-\tau} + \sum_{\tau=0}^{\infty} g_\tau e_{t-\tau} \\ &= (1 + h_1 q^{-1} + \cdots + h_\tau q^{-\tau} + \dots) u_t + (1 + g_1 q^{-1} + \cdots + g_\tau q^{-\tau} + \dots) e_t \\ &= H(q^{-1}) u_t + G(q^{-1}) e_t. \end{aligned} \tag{3.26}$$

with suitable polynomials $H(z)$ and $G(z)$. The former characterizes the dynamics of the (observed) input signals, the latter captures the dynamics of the disturbances to the system. Now there exists a wide spectrum of models which parameterize this general description in some convenient, appropriate way. A few are enumerated here:

FIR(m_b):

$$y_t = b_0 u_t + b_1 u_{t-1} + \cdots + b_{m_b} u_{t-m_b} + e_t = B(q^{-1}) u_t + e_t, \tag{3.27}$$

where

$$B(q^{-1}) = b_0 + b_1 q^{-1} + \cdots + b_{m_b} q^{-m_b}. \tag{3.28}$$

ARX(m_a, m_b): An Auto-Regressive model with eXogenous inputs is specified as

$$A(q^{-1}) y_t = B(q^{-1}) u_t + e_t, \tag{3.29}$$

where

$$A(q^{-1}) = 1 + a_1 q^{-1} + \cdots + a_{m_a} q^{-m_a}. \tag{3.30}$$

and $B(q^{-1})$ is given as in eq. (3.28). This implies that the model equals

$$y_t = \frac{B(q^{-1})}{A(q^{-1})} u_t + \frac{1}{A(q^{-1})} e_t, \tag{3.31}$$

and the noise influences the outcome in a nontrivial way. This is typical for situations where the disturbances come into the dynamical systems at earlier stages, i.e. the noise shares some important aspects of the dynamics with the influence of an input. Its appeal in practice comes however from a different phenomenon. This model fits straightforwardly a model description which is linear in the parameters.

$$y_t = \phi_t^T \theta, \tag{3.32}$$

where

$$\begin{cases} \phi_t = (-y_{t-1}, \dots, y_{t-m_a}, u_t, \dots, u_{t-m_b})^T \in \mathbb{R}^{m_a+m_b+1} \\ \theta = (a_1, \dots, a_{m_a}, b_0, b_1, \dots, b_{m_b})^T \in \mathbb{R}^{m_a+m_b+1}. \end{cases} \tag{3.33}$$

ARMAX(m_a, m_b, m_c): An Auto-Regressive model with eXogenous inputs and Moving Average model for the disturbances is given as

$$A(q^{-1}) y_t = B(q^{-1}) u_t + C(q^{-1}) e_t, \tag{3.34}$$

where

$$C(q^{-1}) = 1 + c_1 q^{-1} + \cdots + c_{m_c} q^{-m_c}. \tag{3.35}$$

and $A(q^{-1})$ is as in eq. (3.30) and $B(q^{-1})$ is given as in eq. (3.28). This model fits the description of eq. (3.26) as

$$y_t = \frac{B(q^{-1})}{A(q^{-1})}u_t + \frac{C(q^{-1})}{A(q^{-1})}e_t, \tag{3.36}$$

where the dynamics of the noise are parametrized more flexible than in the ARX(m_a, m_b) model.

OE(m_a, m_b): The Output Error model of order m_a, m_b is given as

$$y_t = \frac{B(q^{-1})}{A(q^{-1}}u_t + e_t, \tag{3.37}$$

where $A(q^{-1})$ is as in eq. (3.30) and $B(q^{-1})$ is given as in eq. (3.28). This model is often used in case the noise comes only in at the end-stages of the process to be modeled: it does not share many dynamics with the input.

n_a, m_b, m_c, m_d, m_f): The general fractional representation of a polynomial model is refered to as a Box-Jenkins (BJ) model structure of orders m_a, m_b, m_c, m_d, m_f defined as

$$A(q^{-1}y_t = \frac{B(q^{-1})}{F(q^{-1}}u_t + \frac{C(q^{-1})}{D(q^{-1}}e_t. \tag{3.38}$$

where

$$\begin{cases} D(q^{-1}) = 1 + d_1 q^{-1} + \cdots + d_{m_d} q^{-m_d} \\ F(q^{-1}) = 1 + f_1 q^{-1} + \cdots + f_{m_f} q^{-m_f}, \end{cases} \tag{3.39}$$

and $A(q^{-1}), B(q^{-1}), C(q^{-1})$ are as defined above. It should be stressed that its not often useful to use this model structure in its general form. On the contrary, it is good practice to reduce it by setting one or more of the polynomials to unity.

3.3.3 Models for Timeseries

W enow study some common model structured useful for characterizing timeseries $\{y_t\}$. The input signals to such systems are not observed, and are commonly assumed to be noise signals. In general, we consider the model

$$y_t = \sum_{\tau=0}^{\infty} g_\tau e_{t-\tau} = (1 + g_1 q^{-1} + \cdots + g_\tau q^{-\tau} + \dots)e_t = G(q^{-1})e_t. \tag{3.40}$$

MA(m): A Moving Average model of order m:

$$y_t = e_t + c_1 e_{t-1} + \cdots + c_m e_{t-m} = C(q^{-1})e_t. \tag{3.41}$$

Such an all-zero model is useful to model signals with power spectra which have sharp valleys toward zero.

AR(m): An Auto-Regressive model of order m:

$$y_t + a_1 y_{t-1} + \cdots + a_m y_{t-m} = A(q^{-1})y_t = e_t. \tag{3.42}$$

Equivalently, one writes

$$y_t = A^{-1}(q^{-1})e_t. \tag{3.43}$$

Such an all-pole model is useful to model signals with power spectra which have sharp upward peaks.

ARMA(m_a, m_c): An Auto-Regressive Moving Average model of orders m_a and m_c:

$$A(q^{-1})y_t = C(q^{-1})e_t, \tag{3.44}$$

where

$$\begin{cases} A(q^{-1}) = 1 + a_1 q^{-1} + \cdots + a_{m_a} q^{-m_a} \\ C(q^{-1}) = 1 + c_1 q^{-1} + \cdots + c_{m_c} q^{-m_c}. \end{cases} \tag{3.45}$$

Equivalently, one has

$$y_t = \frac{C(q^{-1})}{A(q^{-1})} e_t, \tag{3.46}$$

and this model as such uses a fractional noise model.

ARIMA(d, m_a, m_c): An Auto-Regressive Integrated Moving Average model of orders m_a and m_c:

$$(1 - q^{-1})^d A(q^{-1})y_t = C(q^{-1})e_t, \tag{3.47}$$

with $A(q^{-1})$ and $C(q^{-1})$ the polynomials as defined in (3.45). In case $d = 1$, this is equivalent to the model

$$A(q^{-1})(y_t - y_{t-1}) = C(q^{-1})e_t, \tag{3.48}$$

hence explaining the naming convention.

3.3.4 Stability and Minimal Phase

A system $\mathcal{S}$ is called Bounded-Input, Bounded Output (BIBO) stable if any input signal of bounded input can only imply an output which is bounded.

Definition 10 (Minimum Phase) *A polynomial $A(z) = 1 + a_1 z + \cdots + a_m z^m$ with real-valued coefficients $\{a_1, \ldots, a_m\}$ is called minimum phase if it has all its zeros $\{z \in \mathbb{C}\}$ strictly inside the unit circle. This naming convention is as for any other polynomial with real-valued coefficients and such that $|B(e^{\mathbf{i}\omega})| = |A(e^{\mathbf{i}\omega})|$ for all ω has larger or equal phase lag $-\arg B(e^{\mathbf{i}\omega})$. That implies in turn that a minimal phase system has zero delay, and is causally invertible. That is the inverse system is BIBO stable.*

This nomenclature is used as polynomial with minimal phase does imply the smallest phase-lag of all polynomials sharing the same characteristics in terms of the magnitudes of the frequency responses. A model is in minimal phase if its zeros lie strictly inside the unit circle.

3.4 Simulation and Prediction

A model is often used in two different regimes in order to forecast what is going to happen next.

(Simulation): The most basic use of model is to simulate the system's response to various input scenarios. This simply means that an input sequence $\{u_t^*\}_{t=1}^n$ is chosen by the user, and and is applied to the model $H(q^{-1})$ in order to obtain the undisturbed output

$$y_t^* = H(q^{-1})u_t^*, \ \forall t = 1, \ldots, n. \tag{3.49}$$

This is the output produced by the model when there are no external disturbances which need to be taken into account.

(Prediction): Given the input-signals and the past output signals recorded before instant t, as well as the model, what will the outcome be at instance t? That is, in this case we have some knowledge about the disturbances which acted in the past on the system, and hence for the disturbance terms in the model.

Let us now see how the latter can be formalized. We shall start by discussing how a future value of v_t can be predicted in case it is described as

$$v_t = H(q^{-1})e_t = \sum_{\tau=0}^{\infty} h_\tau e_{t-\tau}. \tag{3.50}$$

For this equation to be meaningful we assume that H is stable, that is

$$\sum_{\tau=0}^{\infty} |h_\tau| < \infty. \tag{3.51}$$

A crucial property of eq. (3.50) which we will impose is that it should be invertible, that is, if v_s is known for all $s \leq t$, then we should be able to compute e_t as

$$e_t = \tilde{H}(q^{-1})v_t = \sum_{\tau=0}^{\infty} \tilde{h}_\tau v_{t-\tau} \tag{3.52}$$

with

$$\sum_{\tau=0}^{\infty} |\tilde{h}_\tau| < \infty. \tag{3.53}$$

The filters H and $\tilde{H}$ are related as follows. Conider the polynomial in $z \in \mathbb{C}$ defined as

$$H(z) = \sum_{\tau=0}^{\infty} h_\tau z^{-\tau}, \tag{3.54}$$

and assume that the inverse function $\frac{1}{H(z)}$ is analytic in $|z| \geq 1$, or

$$\frac{1}{H(z)} = \sum_{\tau=0}^{\infty} \bar{h}_\tau z^{-\tau}. \tag{3.55}$$

Define then the filter $H^{-1}(q^{-1})$ as

$$H^{-1}(q^{-1}) = \sum_{\tau=0}^{\infty} \bar{h}_\tau q^{-\tau}. \tag{3.56}$$

Then $H^{-1}(q^{-1}) = \tilde{H}(q^{-1})$. The proof of this result needs quite a few subtle reasonings. However, this result is quite powerful as it indicates that the properties of the filter $H(q^{-1})$ are similar to those of the function $H(z)$. All that is needed is that the function $\frac{1}{H(z)}$ be analytic in $|z| \geq 1$. That is, it has no poles on or outside the unit circle. We could also phrase the condition as $H(z)$ must have zeros on or outside the unit circle.

Example 13 (Moving Average) *Suppose that one has for all $t = -\infty, \ldots, \infty$ that*

$$v_t = e_t + ce_{t-1}, \tag{3.57}$$

That is

$$H(q^{-1}) = 1 + cq^{-1}, \tag{3.58}$$

that is the process $\{v_t\}_t$ is a MA(1) process of order 1. Then

$$H(z) = 1 + cz^{-1} = \frac{z+c}{z}, \tag{3.59}$$

$$H^{-1}(z) = \frac{1}{1+cz^{-1}} = \sum_{\tau=0}^{\infty} (-c)^\tau z^{-\tau}, \tag{3.60}$$

where we use the geometric series expansion. Then e_t can be computed from $\{v_t\}_{s \leq t}$ as

$$e_t = \sum_{\tau=0}^{\infty} (-c)^\tau v_{t-\tau}. \tag{3.61}$$

Suppose now that we have observed only $\{v_t\}_{s<t}$, and that we want to predict the value of v_t based on these observations. We have then that, since $H(z)$ is assumed to be monic, that

$$v_t = \sum_{\tau=0}^{\infty} h_\tau e_{t-\tau} = e_t + \sum_{\tau=1}^{\infty} h_\tau e_{t-\tau}. \tag{3.62}$$

Now, knowledge of v_s implies knowledge of e_s for all $s < t$ as $v_t = H(q^{-1})e_t$ or

$$e_t = H^{-1}(q^{-1})v_t, \tag{3.63}$$

and thus a prediction of v_t is made as

$$\hat{v}_{t|t-1} = (H(q^{-1}) - 1)H^{-1}(q^{-1})v_t, = (1 - H^{-1}(q^{-1})v_t. \tag{3.64}$$

This is in a sense the best one can do in case $\{e_t\}_t$ contains no information information which contains information to predict next values. That is, the information of e_t is not predictable based on $\{e_s\}_{s<t}$, nor of linear combinations of those (as e.g. $\{v_t\}_{s<t}$). Such terms are denoted as *innovations*.

3.5 Identifiability Issues

System identification is concerned with finding appropriate models from experimental input-output behavior of the system. Conversely, an essential limitation of tools of system identification is that they cannot be used to recover properties of the system which are not reflected by its input-output behavior. The formal way to characterize this notion is by using the following definition

Definition 11 (Globally Identifiability at θ^* of a Model Structure) *Consider a class of models parameterized by a vector θ, or $\mathcal{H} = \{H(z,\theta), G(z,\theta) : \forall\theta\}$. Identification then tries to recover θ using observations of $\{(u_t, y_t)\}_t$. where $y_t = H(z,\theta)u_t + G(z,\theta)v_t$ for all t. Then the model class $\mathcal{H}$ is globally identifiable at θ^* if and only if*

$$\theta = \theta^* \Leftrightarrow H(z,\theta) = H(z,\theta^*), G(z,\theta) = G(z,\theta^*),\ \forall z \in \mathbb{C}. \tag{3.65}$$

This property comes also in a global flavor.

Definition 12 (Globally Identifiability of a Model Structure) *Consider a class of models parameterized by a vector θ, or $\mathcal{H} = \{H(z,\theta), G(z,\theta) : \forall\theta\}$. The class $\mathcal{H}$ is globally identifiable if and only it is globally identifiable at any possible value θ.*

Or in other words, no two different sets of parameter $\theta \neq \theta'$ can express models having the same LTI. Both notions are used as asymptotically, we are only interested at global identifiability at the actual parameter θ_0, a property which is obviously implied by global identifiability. The following counterexample is informative.

Example 14 (Sequence of LTIs) *Consider two LTI models given as*

$$\begin{cases} y_t = a_1 \frac{1}{1-a_2 q^{-1}} u_t \\ y_t = b_1(1 - b_2 q^{-1}) u_t, \end{cases} \tag{3.66}$$

where $a_2 \neq b_2$. Then samples $\{(u_t, y_t)\}_t$ originating from joining the two systems sequentially would obey the relation

$$y_t = (b_1 a_1)\frac{1 - b_2 q^{-1}}{1 - a_2 q^{-1}} u_t, \tag{3.67}$$

and the outcome is identifiable up to the term $(a_1 b_1)$. That is, merely looking at the input-output behavior, one cannot recover which part of the overall gain is to the first subsystem, and which is due to the second one.

In case $a_2 = b_2$, the overall model behaves as

$$y_t = (b_1 a_1) u_t. \tag{3.68}$$

Conversely, if one does not know the orders of the subsystems, and whether they do share common factors, then an input behavior $\{(u_t, y_t)\}_t$ described as eq. (3.67) could as well be caused by the systems

$$\begin{cases} y_t = a_1 \frac{1}{(1-a_2 q^{-1})(1-c_1 q^{-1})(1-c_2 q^{-1})} u_t \\ y_t = b_1(1 - b_2 q^{-1})(1 - c_1 q^{-1})(1 - c_2 q^{-1}) u_t, \end{cases} \tag{3.69}$$

for all $c_1, c_2 \in \mathbb{C}$ where $|c_1| < 1$ and $|c_2| < 1$, as far as we know.

Hence a sequence of two LTIs is only identifiable up to the gains of the subsystem and under the assumption that the models do not share canceling factors.

The following example gives a flash-forward of the difficulties we will discuss in the context of state-space models

Example 15 (Identifiability of State Space Systems) *Consider a sequence of data* $(u_1, y_1), (u_2, y_2), \ldots, (u_t,$ *which obey the difference equations for all* $t = 1, 2, \ldots, t, \ldots$

$$\begin{cases} \mathbf{x}_t = \mathbf{A}\mathbf{x}_{t-1} + \mathbf{b}u_t \\ y_t = \mathbf{c}^T\mathbf{x}_t \end{cases} \tag{3.70}$$

with $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_t, \ldots$ *which are elements of* $\mathbb{R}^d$ *a sequence of (unknown)* state *vectors Then the input-output behavior is determined up to a linear transformation of the system matrix* $\mathbf{A} \in \mathbb{R}^{d\times d}$*. That is, let* $\mathbf{G} \in \mathbb{R}^{d\times d}$ *be a matrix which is full rank such that its inverse* $\mathbf{G}^{-1}$ *exists, then the sequence* $(u_1, y_1), (u_2, y_2), \ldots, (u_t, y_t), \ldots$ *obeys as well the equations*

$$\begin{cases} \tilde{\mathbf{x}}_t = \mathbf{G}\mathbf{A}\mathbf{G}^{-1}\tilde{\mathbf{x}}_{t-1} + \mathbf{G}\mathbf{b}u_t \\ y_t = \mathbf{c}^T\mathbf{G}^{-1}\tilde{\mathbf{x}}_t \end{cases} \tag{3.71}$$

where now $\tilde{\mathbf{x}}_t = \mathbf{G}\mathbf{x}_t$*. We say that a state-space system is only identifiable up to a (full rank) linear transformation.*

3.5.1 Persistency of Exitation

As seen in the previous chapter, in order for the LS estimate to give unique solutions, we need to have that the associated sample covariance matrix is full rank. The notion of Persistency of Excitation (PE) is an interpretation of this condition when the LS estimate is applied for estimating the parameters of a dynamical model based on input- and output behavior. Let us first illustrate this with the following example

Example 16 (PE for a FIR(d) Model) *Let us consider signals* $\{u_t\}_t$ *and* $\{y_t\}$ *of length* n*, and suppose their relation can adequately be captured using the following model structure*

$$y_t = \sum_{\tau=1}^{d} h_{0,\tau}u_{t-\tau} + e_t. \tag{3.72}$$

where $\theta_0 = (h_{0,1}, \ldots, h_{0,d})^T \in \mathbb{R}^d$ *are unknown. Stacking all* $n - d$ *such equalities yields the linear system*

$$\begin{bmatrix} u_1 & \ldots & u_d \\ u_2 & & u_{d+1} \\ & \vdots & \\ u_{n-d+1} & \ldots & u_n \end{bmatrix} \begin{bmatrix} h_{0,1} \\ \vdots \\ h_{0,d} \end{bmatrix} = \begin{bmatrix} y_{d+1} \\ \vdots \\ y_n \end{bmatrix} + \begin{bmatrix} e_{d+1} \\ \vdots \\ e_n \end{bmatrix}, \tag{3.73}$$

or shortly $\Phi\theta_0 = \mathbf{y} + \mathbf{e}$*, using appropriate definitions of the matrix* Φ *and the vectors* $\mathbf{y}, \mathbf{e}$*. Then the LS estimate of those parameters is given as* θ_n *which solves the system*

$$\begin{bmatrix} \hat{r}_{uu}(0) & \ldots & \hat{r}_{uu}(d-1) \\ & \ddots & \\ \hat{r}_{uu}(d-1) & \ldots & \hat{r}_{uu}(0) \end{bmatrix} \theta_n = \hat{\mathbf{R}}_d\theta_n = \begin{bmatrix} \hat{r}_{uy}(1) \\ \ldots \\ \hat{r}_{uy}(d) \end{bmatrix} = \hat{\mathbf{r}}_d, \tag{3.74}$$

where $\hat{r}_{uu}(\tau) = \frac{1}{n}\sum_{t=1}^{n-\tau} u_t u_{t+\tau}$ *and* $\hat{r}_{uy}(\tau) = \frac{1}{n}\sum_{t=1}^{n-\tau} u_t y_{t+\tau}$*, and where* $\hat{\mathbf{R}}_d \in \mathbb{R}^{d\times d}$ *and* $\hat{\mathbf{r}}_d \in \mathbb{R}^d$ *are defined appropriately. Then, this set of equations has a unique solution if and only if* $\hat{\mathbf{R}}_d$ *is of full rank, e.g. invertible. This requires in turn that the input signals are sufficiently rich: for example if* $d > 1$ *and* $u_t = 1$ *for all* $t = 1, \ldots, n$*, this condition is obviously not satisfied.*

This intuition leads to the general definition of PE:

Definition 13 (PE) *A signal* $\{u_t\}_t$ *of infinite length is called Persistently Exciting (PE) of order* d *in case the following two conditions are satisfied.*

(i): For any $\tau = 0, \ldots, d-1$*, the limit*

$$r_{uu}(\tau) = \lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n-\tau} u_t u_{t+\tau}, \tag{3.75}$$

exists.

(ii): The matrix

$$\mathbf{R}_d = \begin{bmatrix} r_{uu}(0) & \ldots & r_{uu}(d-1) \\ & \ddots & \\ r_{uu}(d-1) & \ldots & r_{uu}(0) \end{bmatrix} \succ 0, \tag{3.76}$$

i.e., this matrix is positive definite, and hence is of full rank.

Example 17 (White Noise) *Let* $\{U_t\}_t$ *be a sequence of zero mean white noise with variance* $\sigma^2 > 0$*, then* $r_u(\tau) = \mathbb{E}[U_t U_{t-\tau}] = \sigma^2\delta_\tau$*, and the matrix* $\mathbf{R}_d = \sigma^2 I_d$*, and is thus of full rank for any* d*. That means that a sequence of white noise* $\{U_t\}_t$ *is PE of any order.*

Example 18 (Step) *Let* $\{u_t\}$ *be a step function where the step is made at* $t = 1$*, i.e.* $u_t = I(t \geq 1)$*. Hence for finite* n *the matrix* $\hat{\mathbf{R}}_d$ *is full rank for* $d \leq 2$*, but when considering the limit for* $n \to \infty$*, the initial difference at step* $t = 0, 1$ *is averaged out, and the matrix* $\mathbf{R}_d$ *is only full rank for* $d = 1$*. Hence the step function is PE of order 1.*

Example 19 (Finite Impulse) *Let* $\{u_t\}$ *be a finite impulse function, such that* $u_t = 1$ *only if* $t = 0$*, zero otherwise. Then this signal is not PE of any order as* $\mathbf{R}_0 = 0$*. Note that the situation is entirely different when* $u_1 = a_n$ *with* $a_n \to \infty$ *when* $n \to \infty$*, as then the excitation in the initial steps is not necessarily averaged out.*

The notion of PE is not restricted to the use of FIR models, but one has that a model of order d is identifiable in case the input signal $\{u_t\}_t$ is PE of order d when using the IV or PEM method as will be introduced in later chapters. However, if the model can be written as a FIR model of order d, PE of order d is already sufficient for identifiability (uniqueness) of the solution.

Also observe that the concept of PE is useful for noisy systems. If the system is noiseless, one can obtain identifiability using less restrictive conditions on the input signal. Specifically, one does not have to consider the limiting behavior. For example, if the given signals $\{u_t\}_t$ and $\{y_t\}_t$ obey a

FIR system without noise, i.e. $y_t = \sum_{\tau=1}^{d} h_{0,\tau} u_{t-d}$, the parameters can be recovered *exactly* from the system

$$\begin{bmatrix} u_1 & \dots & u_d \\ u_2 & & u_{d+1} \\ \vdots & & \vdots \\ u_d & \dots & u_{2d-1} \end{bmatrix} \begin{bmatrix} h_{0,1} \\ \vdots \\ h_{0,d} \end{bmatrix} = \Phi_d \theta_0 = \begin{bmatrix} y_{d+1} \\ \vdots \\ y_{2d} \end{bmatrix} \tag{3.77}$$

in case the matrix $\Phi_d \in \mathbb{R}^{d \times d}$ were full rank.

Let us now exemplify PE conditions derived for more complex ways used to generate signals. The proofs as in the SI book, albeit simple, are not reproduced here.

Example 20 (PE for d distinct frequencies) *Let $\{U_t\}$ be a (multivariate) ergodic stochastic process. Assume the its spectral density (matrix) is positive (definite) in at least d distinct frequencies, then $\{U_t\}$ is PE of at least order d.*

Example 21 (frequency of a PE signal) *Let $\{u_t\}_t$ be a signal which is PE of at least order d, then its spectral density is nonzero in at least d different frequencies.*

Example 22 (PE for filtered signals) *Let $\{u_t\}$ be a signal which is PE of order d. Let $H(q^{-1})$ be an asymptotically stable, linear filter with k zeros on the unit circle, then the filtered signal $\{y_t = H(q^{-1})u_t\}$ is PE of order m with $d - k \leq m \leq d$.*

Example 23 (PE and zero filtering) *Let $\{U_t\}_t$ be a stationary stochastic process which is PE of order at least d. Define*

$$Z_t = \sum_{\tau=1}^{d} h_\tau U_{t-\tau}. \tag{3.78}$$

Then the condition that $\mathbb{E}[Z_t Z_t] = 0$ implies (if and only if) that $h_1 = \cdots = h_d = 0$.

3.5.2 Input Signals

It is hence clear that a successful SI experiment relies on a good input signal. We already mentioned that a stochastic white noise sequence has good properties w.r.t. PE. Again, we use capital letters to denote random quantities (i.e. depending on some sort of sampling mechanism), and lower case letters indicate Some other examples are given here.

Example 24 (A PRBS) *A* Pseudo Random Binary Sequence *(PRBS) is a signal that shifts between two levels (typically $\pm a$) in a deterministic fashion. Typically, such signals are realized by using a circuit with shift registers such that the outcome 'looks similar to white stochastic noise'. However, the essential difference is that when computing a PRBS again on a different occasion, the signal will be exactly the same. The signal is necessarily periodic, that is, it repeats itself after a given period. In most practical cases however, the period would be chosen such that it exceeds the number of samples, such that no artifacts come up in the analysis due to such property. When applying a PRBS, the user has to design the two levels, the period as well as the clock period. The clock period is the minimal time the signal varies its level. Typically, the clock period is taken equal to one sampling interval.*

Example 25 (A PRBS of period m) *Let $\{u_t\}_t$ be a signal which is generated as a pseudo-random binary sequence (PRBS) with period m, and magnitude a, that is one has*

1. *The signal is deterministic (e.g. not depending on any sampling mechanism).*
2. *The magnitudes of the signal are a, such that $u_t = \pm a$.*
3. *The autocovariances are almost zero for $\tau = 1, \ldots, m-1$, specifically one has that $\hat{\mathbf{R}}_m^{-1}$ equals*

$$\hat{\mathbf{R}}_m^{-1} = \begin{bmatrix} a^2 & \frac{-a^2}{m} & \ldots & \frac{-a^2}{m} \\ \frac{-a^2}{m} & a^2 & \ldots & \frac{-a^2}{m} \\ & & \ddots & \\ \frac{-a^2}{m} & \frac{-a^2}{m} & \ldots & a^2 \end{bmatrix} \tag{3.79}$$

4. *The signal repeats itself exactly after m time instances.*

Then this signal is PE of order exactly m.

Example 26 (An ARMA PRBS Process) *Let $\{e_t\}$ be a PRBS. Then this process filtered by an ARMA model gives $\{u_t\}_t$ such that*

$$A(q^{-1})u_t = B(q^{-1})e_t. \tag{3.80}$$

such that one may tune the properties of the filter by design of appropriate A, B polynomials

Example 27 (An ARMA Process) *Let $\{D_t\}$ be a white, zero mean stochastic process. Then this process filtered by an ARMA model gives $\{U_t\}_t$ such that*

$$A(q^{-1})U_t = B(q^{-1})D_t. \tag{3.81}$$

such that one may tune the properties of the filter by design of appropriate A, B polynomials

Then we have the property that

Example 28 (PE for an ARMA process) *A stochastic process following a nontrivial ARMA system is PE of any order.*

Another example which is often used is

Example 29 (A Sum of Sinusoids) *The following deterministic signal $\{u_t\}_t$ is often used:*

$$u_t = \sum_{j=1}^{m} a_j \sin\left(\omega_j t + \varphi_j\right), \tag{3.82}$$

with amplitudes $\mathbf{a} = (a_1, \ldots, a_m)^T$, frequencies $\omega = (\omega_1, \ldots, \omega_m)^T$ and phases $\varphi = (\varphi_1, \ldots, \varphi_m)^T$. A term with $\omega_j = 0$ will give a constant contribution $a_j \sin(\varphi_j)$, a term with $\omega_j = \pi$ gives a contribution which will oscillate in two sampling intervals, or

$$a_j \sin(\omega_j(t+1) + \varphi_j) = -a_j \sin(\omega_j(t+1) + \varphi_j). \tag{3.83}$$

for any t.

finally, it is often useful to design the input signals such that the resulting identified model is adequate w.r.t. a certain frequency range. In most cases, the input signal must emphasize the low-frequency properties during the modeling. There are different ways of obtaining such inputs, including

- Standard Filtering. This can be done by pre-filtering the input signals such that the resulting signal has the desirable property in frequency domain. An example is given during the computer labs.
- Increasing the clock period. If keeping the input signal constant over an increased amount of time exceeding the sampling interval, it must be clear that in that way one reduces the rapid fluctuations (high-frequencies) present in the original signal. This reasoning makes it clear that if the given sampling interval of the case at hand is relatively large, there is not so much hope to recover the dynamics of the system corresponding to the high-frequencies.
- Decreasing the probability of changing level. Consider the case of a binary *Stochastic sequence* $\{U_t\}_t$ taking values in $\{-a, a\}$ for $a > 0$, which has the stochastic model for $0 < p \leq 1$ as

$$U_t = \begin{cases} -U_{t-1} & \text{with probability} p \\ U_{t-1} & \text{else.} \end{cases} \tag{3.84}$$

  and $U_0 = a$. Then by increasing p, the signal reflects more rapid fluctuations. By decreasing p, the signal has a larger power in the low-frequency area.