Chapter 14

Problem Solving Sessions

14.1 Dynamic Models

14.1.1 Exercises

Exercise 1.0: Hello world

1. Given a discrete system $G(z) = \frac{K}{\tau z}$ with $K = 1$ and $\tau = 2$. Is it BIBO stable? Why/not?

2. Given a system which outputs positive values for any input. Is it LTI? Why/not?

3. Can you solve a least squares estimate for $\theta$ for a system satisfying $x_i \theta = y_i$ for any $\{(x_i, y_i)\}_i$? Why/not?

4. Is the median estimate optimal in a least squares sense? Why/not?

5. If we are to model a certain behavior and we know some of the physics behind it - should we go for a black box model? Why/not?

6. If we have a very fast system (time constants smaller than $O(10^{-2})s$). Can we get away with slow sampling? Why/not?

7. Does a non-causal model allow an impulse representation? Why/not?

8. Is a sequence of two nontrivial LTIs identifiable from input-output observations? Why/not?

9. Is an ARMAX system linear in the parameters of the polynomials? Why/not?

10. Is an OE model LIP? Why/not?

Exercise 1.1: Stability boundary for a second-order system.

Consider the second-order AR model

$$y_t + a_1 y_{t-1} + a_2 y_{t-2} = \epsilon_t$$

Derive and plot the area in the $(a_1, a_2) \in \mathbb{R}^2$-plane for which the model is asymptotically stable.
Exercise 1.2: Least Squares with Feedback

Consider the second-order AR model

\[ y_t + a y_{t-1} = b u_{t-1} + e_t \]

where \( u_t \) is given by feedback as

\[ u_t = -K y_t. \]

Show that given realizations of this signal we cannot estimate \( a_0, b_0 \) separately, but we can estimate \( a_0 + b_0 k \).

Exercise 1.3: Determine the time constant \( T \) from a step response.

A first order system \( Y(s) = G(s)U(s) \) with

\[ G(s) = \frac{K}{1 + s T e^{-s \tau}} \]

or in time domain as a differential equation

\[ T \frac{dy(t)}{dt} + y(t) = K u(t - \tau) \]

derive a formula of the step response of an input \( u_t = I(t > 0) \).

Exercise 1.4: Step response as a special case of spectral analysis.

Let \( (y_t)_t \) be the step response of an LTI \( H(q^{-1}) \) to an input \( u_t = a I(t \geq 0) \). Assume \( y_t = 0 \) for \( t < 0 \) and \( y_t \approx c \) for \( t > N \). Justify the following rough estimate of \( H \)

\[ \hat{h}_k = \frac{y_k - y_{k-1}}{a}, \forall k = 0, \ldots, N \]

and show that it is approximatively equal to the estimate provided by the spectral analysis.

Exercise 1.5: Ill-conditioning of the normal equations in case of a polynomial trend model.

Given model

\[ y_t = a_0 + a_1 t + \cdots + a_r t^r + e_t \]

Show that the condition number of the associated matrix \( \Phi^T \Phi \) is ill-conditioned:

\[ \text{cond}(\Phi^T \Phi) \geq O(N^{2r}/(2r + 1)) \]

for large \( n \), and where \( r > 1 \) is the polynomial order. Hint. Use the relations for a symmetric matrix \( A \):

- \( \lambda_{\text{max}}(A) \geq \max_i A_{ii} \)
- \( \lambda_{\text{min}}(A) \leq \min_i A_{ii} \)
Exercise 1.6

Determine the covariance function for an AR(1) process

\[ y_t + ay_{t-1} = e_t \]

where \( e_t \) come from a white noise process with zero mean and unit variance. Determine the covariance function for an AR(2) process

\[ y_t + ay_{t-1} + ay(t-2) = e_t \]

Determine the covariance function for an MA(1) process

\[ y_t = e_t + be_{t-1} \]

Exercise 1.7

Given two systems

\[ H_1(z) = \frac{b}{z + a} \]

and

\[ H_2(z) = \frac{b_0 z + b_1}{z^2 + a_1 z + a_2} \]

(a) If those systems filters white noise \( \{e_t\} \) coming from a stochastic process \( \{D_t\} \), which is zero mean, and has unit variance. What is the variance of the filtered signal \( \{y_t\} \)?

(b) What happens to the output of the second system when you move the poles of \( H_2(z) \) towards the unit circle?

(c) Where to place the poles to get a 'low-pass' filter?

(d) Where to put the poles in order to have a resonance top at \( \omega = 1 \)?

(e) How does a resonant system appear on the different plots?

(f) What happens if \( H_2(z) \) got a zero close to the unit circle?

Exercise 1.8

Given an input signal \( V_t \) shaped by an ARMA filter,

\[ A(q^{-1})X_t = C(q^{-1})V_t, \]

where \( A \) and \( C \) are monomials of appropriate order, and where \( V_t \) white, zero mean and variance \( \sigma_v^2 \). Given noisy observations of this signal, or

\[ Y_t = X_t + E_t \]

where \( E_t \) follows a stochastic process with white, zero mean and variance \( \sigma_e^2 \) and uncorrelated to \( D_t \). Rewrite this as a ARMA process, what would be the corresponding variance of the 'noise'? How would the spectrum of \( Y_t \) look like?