14.2 Statistical Aspects of Least Squares

14.2.1 Exercises

Exercise 2.1: Convergence rates for consistent estimators.

For most consistent estimators of the parameters of stationary processes, the estimation error $\hat{\theta} - \theta_0$ tends to zero as $1/n$ when $n \to \infty$. For non-stationary processes, faster convergence rates may be expected. To see this, derive the variance of the least squares estimate in the model

$$Y_t = \alpha t + D_t,\ t = 1, \ldots, N,$$

with $\{D_t\}_t$ white noise, zero mean and variance λ^2.

Exercise 2.2

Illustration of unbiasedness and consistency properties. Let $\{X_i\}_i$ be a sequence of i.i.d. Gaussian random variables with mean μ and variance σ. Both are unknown. Let $\{x_i\}_{i=1}^n$ be a realization of this process of length n. Consider the following estimate of μ:

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{N} x_i,$$

and the following two estimates of σ^2:

$$\begin{cases} \hat{\sigma}_1^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \hat{\mu})^2 \\ \hat{\sigma}_2^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \hat{\mu})^2. \end{cases}$$

Determine the mean and the variance of the estimates $\hat{\mu}, \hat{\sigma}_1$ and $\hat{\sigma}_2$. Discuss their bias and consistency properties. Compare $\hat{\sigma}_1$ and $\hat{\sigma}_2$ in terms of their Mean Square Error (mse).