14.3 Prediction Error Methods

14.3.1 Exercises

Exercise 3.1: On the use of cross-correlation test for the LS model.

Consider an ARX model

$$A(q^{-1})y_t = B(q^{-1})u_t + e_t$$

with parameters $\theta = \left(a_1, \ldots, a_{n_a}, b_1, \ldots b_{n_b}\right)^T$. Assume that the estimate $\hat{\theta}$ is found by LS, show that $\hat{r}_{eu}(\tau) = \frac{1}{n}\sum_{t=1}^{n} u_t e_t = 0$ for all $\tau = 1, \ldots, n_b$.

Exercise 3.2: Identifiability results for ARX models.

Consider the system (with A_0, B_0 coprime, and degrees n_{a_0}, n_{b_0})

$$A_0(q^{-1})Y_t = B_0(q^{-1})u_t + D_t$$

where D_t is zero mean, white noise. Let $\{y_t\}_t$ be realizations of the stochastic process $\{Y_t\}$. Use the LS in the model structure

$$A(q^{-1})y_t = B(q^{-1})u_t + \epsilon_t$$

with degrees $n_a \geq n_{a_0}$ and $n_b \geq n_{b_0}$. Assume the system operates in open-loop and the input u_t is PE of order n_b. Let $\epsilon_t(\theta)$ denote the random variables modeling the residuals ϵ_t for given parameters $\theta = (A, B)$. Prove the following results:

(a) The asymptotic cost function $\mathbb{E}[\epsilon_t^2(\theta)]$ has a unique minimum.

(b) The estimated polynomials are coprime.

Exercise 3.3

Given the predictor

$$\hat{y}(t+1|t) = \frac{1-\alpha}{1-\alpha q^{-1}} y_t,$$

where $|\alpha| < 1$ is a constant.

1. For which ARMA-model is this the optimal predictor, and what assumptions do we make on the noise process so that this is L_2 optimal?

2. What is the optimal two-step ahead predictor $\hat{y}(t|t-2)$?

3. If using a numerical optimization procedure for implementing PEM, one wants to use the gradient of $\epsilon(t+1|t) = y_t - \hat{y}(t|t-1)$ with respect to α. Give a closed form expression of this gradient.

Exercise 3.4: An illustration of the Parsimony Principle.

Consider the following AR(1) process:

$$Y_t + a_0 Y_{t-1} = D_t,$$

with $|a_0| < 1$ unknown, and where $\{D_t\}_t$ is a white noise stochastic process with zero mean and variance λ^2. Let us have a realization $\{y_t\}_{t=1}^n$ of length n of this process. Then assume we fit this system with the following two candidate models:

$$\mathcal{M}_1 : y_t + a y_{t-1} = \epsilon_t$$

and

$$\mathcal{M}_2 : y_t + a_1 y_{t-1} + a_2 y_{t-2} = \epsilon'_t$$

Let $\hat{a}$ denote the LS estimate of a in $\mathcal{M}_1$, and let $\hat{a}_1, \hat{a}_2$ be the LS estimate in $\mathcal{M}_2$. What are the asymptotic variances $\sqrt{n}(\hat{a} - a_0)$ and $\sqrt{n}(\hat{a}_1 - a_0)$?