14.4 Recursive Identification

14.4.1 Exercises

Exercise 4.1: Derivation of the real-time RLS algorithm.

Show that the weighted RLS algorithm

$$\begin{cases} \hat{\theta}_t = \hat{\theta}_{t-1} + K_t\epsilon_t \\ \epsilon_t = y_t - \varphi_t^T\hat{\theta}_{t-1} \\ K_t = \mathbf{P}_t\varphi_t \\ \mathbf{P}_t = \frac{1}{\lambda_t}\left[\mathbf{P}_{t-1} - \frac{\mathbf{P}_{t-1}\varphi_t\varphi_t^T\mathbf{P}_{t-1}}{\lambda_t + \varphi_t^T\mathbf{P}_{t-1}\varphi_t}\right] \end{cases}$$

solves in each step the problem

$$\theta_t = \underset{\theta}{\operatorname{argmin}} \sum_{s=1}^{t} \lambda^{t-s}\epsilon_s^2(\theta)$$

where $\epsilon_s^2(\theta) = y_s - \varphi_s^T\hat{\theta}$ for all $s = 1, \ldots, t$.

Exercise 4.2: Influence of forgetting factor on consistency properties of parameter estimates.

Consider the static-gain system

$$y_t = bu_t + D_t, \quad \forall t = 1, 2, \ldots$$

where

$$\mathbb{E}[D_t] = 0, \ \mathbb{E}[D_sD_t] = \delta_{t,s}$$

and u_t is a persistently exciting nonrandom signal. The unknown parameter b is estimated as

$$\hat{b} = \underset{b}{\operatorname{argmin}} \sum_{t=1}^{n} \lambda^{n-t}\Big(y_t - bu_t\Big)^2$$

where n denotes the number of datapoints, and the forgetting factor λ satisfies $0 < \lambda \leq 1$. Determine $\operatorname{var}(\hat{b})$. Show that for $n \to \infty$ one has $\operatorname{var}(\hat{b}) = 0$. Also, show that for $\lambda < 1$ there are signals u_t for which consistence is not obtained.

Hint. Consider the signal where u_t is constant.

Exercise 4.3: Convergence properties and dependence on initial conditions of the RLS estimate.

Consider the model

$$y_t = \varphi_t^T\theta_0 + \epsilon_t$$

Let the *offline* weighted LS estimate of θ_0 up to instant t be

$$\hat{\theta}_t = \left(\sum_{s=1}^{t} \lambda^{t-s}\varphi_s\varphi_s^T\right)^{-1}\left(\sum_{s=1}^{t} \lambda^{t-s}\varphi_s y_s\right)$$

Consider also the online weighted RLS estimates $\{\bar{\theta}_s\}_s$

(i) Derive the difference equations for $\mathbf{P}_t^{-1}$ and $\mathbf{P}_t^{-1}\hat{\theta}_t$. Solve this equations to find how $\hat{\theta}_t$ depends on the initial values $\hat{\theta}_0$ and $\mathbf{P}_0$ and on the forgetting factor λ.

(ii) Let $\mathbf{P}_0 = \rho I_n$, then prove that for every t where $\bar{\theta}_t$ exists

$$\lim_{\rho\to\infty} \hat{\theta}_t = \bar{\theta}_t$$

(iii) Suppose that $\bar{\theta}_t$ is bounded, and suppose that $\lambda^t \mathbf{P}_t \to 0$ as $t \to \infty$. Prove that

$$\lim_{t\to\infty} (\hat{\theta}_t - \bar{\theta}_t) = 0$$

Exercise 4.4: An RLS algorithm with a sliding window.

Consider the parameter estimate

$$\hat{\theta}_t = \operatorname*{argmin}_{\theta} \sum_{s=t-m+1}^{t} \epsilon_s^2(\theta)$$

where $\epsilon_s(\theta) = y_s - \varphi_s^T\theta$. The number m is the size of the sliding window. Show that such $\hat{\theta}_t$ can be computed recursively as

$$\begin{cases} \hat{\theta}_t = \hat{\theta}_{t-1} + \mathbf{K}_1\epsilon(t, \hat{\theta}_{t-1}) - \mathbf{K}_2\epsilon(t-m, \hat{\theta}_{t-1}) \\ \mathbf{K}_1 = \mathbf{P}_{t-1}\varphi_t \left(I + \begin{bmatrix} \varphi_t^T \\ -\varphi_{t-m}^T \end{bmatrix} \mathbf{P}_{t-1} \begin{bmatrix} \varphi_t & \varphi_{t-m} \end{bmatrix} \right)^{-1} \\ \mathbf{K}_2 = \mathbf{P}_{t-1}\varphi_{t-m} \left(I + \begin{bmatrix} \varphi_t^T \\ -\varphi_{t-m}^T \end{bmatrix} \mathbf{P}_{t-1} \begin{bmatrix} \varphi_t & \varphi_{t-m} \end{bmatrix} \right)^{-1} \\ \mathbf{P}_t = \mathbf{P}_{t-1} - \begin{bmatrix} \mathbf{K}_1 & \mathbf{K}_2 \end{bmatrix} \begin{bmatrix} \varphi_t^T \\ -\varphi_{t-m}^T \end{bmatrix} \mathbf{P}_{t-1} \end{cases}$$