Chapter 14

Problem Solving Sessions

14.1 Dynamic Models

14.1.1 Exercises

Exercise 1.0: Hello world

1. Given a discrete system $G(z) = \frac{K}{\tau - z}$ with $K = 1$ and $\tau = 2$. Is it BIBO stable? Why/not?
2. Given a system which outputs positive values for any input. Is it LTI? Why/not?
3. Can you solve a least squares estimate for θ for a system satisfying $x_i\theta = y_i$ for any $\{(x_i, y_i)\}_i$? Why/not?
4. Is the *median* estimate optimal in a least squares sense? Why/not?
5. If we are to model a certain behavior and we know some of the physics behind it - should we go for a black box model? Why/not?
6. If we have a very fast system (time constants smaller than $O(10^{-2})s$). Can we get away with slow sampling? Why/not?
7. Does a non-causal model allow an impulse representation? Why/not?
8. Is a sequence of two nontrivial LTIs identifiable from input-output observations? Why/not?
9. Is an ARMAX system linear in the parameters of the polynomials? Why/not?
10. Is an OE model LIP? Why/not?

Exercise 1.1: Stability boundary for a second-order system.

Consider the second-order AR model

$$y_t + a_1 y_{t-1} + a_2 y_{t-2} = e_t$$

Derive and plot the area in the $(a_1, a_2) \in \mathbb{R}^2$-plane for which the model is asymptotically stable.

Exercise 1.2: Least Squares with Feedback

Consider the second-order AR model

$$y_t + ay_{t-1} = bu_{t-1} + e_t$$

where u_t is given by feedback as

$$u_t = -Ky_t.$$

Show that given realizations of this signal we cannot estimate a_0, b_0 separately, but we can estimate $a_0 + b_0 K$.

Exercise 1.3: Determine the time constant T from a step response.

A first order system $Y(s) = G(s)U(s)$ with

$$G(s) = \frac{K}{1+sT} e^{-s\tau}$$

or in time domain as a differential equation

$$T\frac{dy(t)}{dt} + y(t) = Ku(t-\tau)$$

derive a formula of the step response of an input $u_t = I(t > 0)$.

Exercise 1.4: Step response as a special case of spectral analysis.

Let $(y_t)_t$ be the step response of an LTI $H(q^{-1})$ to an input $u_t = aI(t \geq 0)$. Assume $y_t = 0$ for $t < 0$ and $y_t \approx c$ for $t > N$. Justify the following rough estimate of H

$$\hat{h}_k = \frac{y_k - y_{k-1}}{a}, \ \forall k = 0, \ldots, N$$

and show that it is approximatively equal to the estimate provided by the spectral analysis.

Exercise 1.5: Ill-conditioning of the normal equations in case of a polynomial trend model.

Given model

$$y_t = a_0 + a_1 t + \cdots + a_r t^r + e_t$$

Show that the condition number of the associated matrix $\Phi^T\Phi$ is ill-conditioned:

$$\text{cond}(\Phi^T\Phi) \geq O(N^{2r}/(2r+1))$$

for large n, and where $r > 1$ is the polynomial order. Hint. Use the relations for a symmetric matrix A:

- $\lambda_{\max}(A) \geq \max_i A_{ii}$
- $\lambda_{\min}(A) \leq \min_i A_{ii}$

Exercise 1.6

Determine the covariance function for an AR(1) process

$$y_t + ay_{t-1} = e_t$$

where e_t come from a white noise process with zero mean and unit variance. Determine the covariance function for an AR(2) process

$$y_t + ay_{t-1} + ay(t-2) = e_t$$

Determine the covariance function for an MA(1) process

$$y_t = e_t + be_{t-1}$$

Exercise 1.7

Given two systems

$$H_1(z) = \frac{b}{z+a}$$

and

$$H_2(z) = \frac{b_0 z + b_1}{z^2 + a_1 z + a_2}$$

(a) If those systems filters white noise $\{e_t\}$ coming from a stochastic process $\{D_t\}_t$ which is zero mean, and has unit variance. What is the variance of the filtered signal $\{y_t\}$?

(b) What happens to the output of the second system when you move the poles of $H_2(z)$ towards the unit circle?

(c) Where to place the poles to get a 'low-pass' filter?

(d) Where to put the poles in order to have a resonance top at $\omega = 1$?

(e) How does a resonant system appear on the different plots?

(f) What happens if $H_2(z)$ got a zero close to the unit circle?

Exercise 1.8

Given an input signal $\{X_t\}_t$ shaped by an ARMA filter,

$$A(q^{-1})X_t = C(q^{-1})V_t,$$

where A and C are polynomials of appropriate order with the constant term equal to 1, and where is V_t a white noise source, zero mean and variance σ_v^2. Given noisy observations $\{Y_t\}_t$ of this signal, or

$$Y_t = X_t + E_t$$

where E_t follows a stochastic process with white, zero mean and variance σ_e^2 and uncorrelated to D_t. Rewrite this as a ARMA process, what would be the corresponding variance of the 'noise'? How would the spectrum of Y_t look like?