System Identification

Exercise session 1

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April 11, 2016
Exercise 1.0: Hello world

1. BIBO stability
   - Definition The system $y_t = Gu_t$ is BIBO stable if any bounded input $u_t$ produces a bounded output $y_t$.
   - A system is BIBO iff
     $|u_t| \leq B_u < \infty \Rightarrow |y_t| \leq B_y < \infty$
     for all $t$.
   - Stability and IR
     $\sum_{t=-\infty}^{\infty} |h_t| < \infty \iff \text{LTI system is BIBO stable}$
   - Stability of rational system function filter
     $G(z) = \sum_{t=-\infty}^{\infty} h_t z^{-t}$
     BIBO stability $\Rightarrow$ all the poles inside the unit circle
Exercise 1.0: Hello world

1. BIBO stability
   ▶ TF \( G(z) = \frac{-z^{-1}}{1 - 2z^{-1}} \)
   ▶ \( \gg zplane([0, -1], [1, -2]) \)

Figure: zero-pole plot.
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- Definition: LTI system
  - Linearity if the input $u_t$ produces $y_t$ and the input $\bar{u}_t$ produces output $\bar{y}_t$ then the scaled summed input $a_1 u_t + a_2 \bar{u}_t$ produces scaled summed output $a_1 y_t + a_2 \bar{y}_t$ ($a_1$, $a_2$ are real and scalar)

- Time invariance
  - The output does not depend explicitly on time. If the input $u_t$ produces output $y_t$ then the shifted time input $u_{t+\delta}$ produces shifted time output $y_{t+\delta}$ ($\delta$ is real and scalar)
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Ordinary least squares estimates given \( \{x_i, y_i\}_{i=1}^n \), estimating parameter \( \theta \) by minimizing the sum of squared residuals (errors), a residual being the difference between the observed value and the fitted value provided by the model:

\[
\hat{\theta} = \arg\min_{\theta \in \mathbb{R}} \sum_{i=1}^{n} (y_i - \theta x_i)^2
\]
(Average) find a variable $\theta$ which is 'close' to all datasamples $\{y_i\}_{i=1}^{n}$ taking values in $\mathbb{R}$.

$$\hat{\theta} = \arg \min_{\theta \in \mathbb{R}} \sum_{i=1}^{n} (y_i - \theta)^2$$

the mean estimate is optimal in the least square approximation of a bunch of samples

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} y_i$$
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▶ Causality

\[ y_t = \sum_{\tau = -\infty}^{\infty} h_\tau u_{t - \tau} = h_t \ast u_t \]

▶ A discrete-time LTI system is causal if the current value of the output depends on only the current value and past values of the inputs

\[ u_t, t < t_0 \iff y_t, t < t_0 \]

”no output before input is applied”

▶ A necessary and sufficient condition for causality is

\[ h_t = 0 \quad \forall t < 0 \]

where \( h_t \) is the impulse response.
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- identifiability of sequence of two LTIs

\[
S_1 = a_1 \frac{1}{1 - a_2 q^{-1}} \quad S_2 = b_1 (1 - b_2 q^{-1})
\]

\[
\bar{S}_1 = a_1 \frac{1}{(1 - a_2 q^{-1})(1 - c_1 q^{-1})} \quad \bar{S}_2 = b_1 (1 - b_2 q^{-1})(1 - c_1 q^{-1})
\]

- Sequence of LTIs systems are not identifiable if they share common factors.
Definition 2 Lecture Notes p. 22

Linear in the Parameters (LIP) A model \( \{y_i\}_i \) is linear in the unknown parameters \( \{\theta_j\}_{j=1}^d \) if for each \( y_i : i = 1, \ldots, n \), one has given values \( \{x_{ij}\}_{j=1}^d \) such that

\[
y_i = \sum_{j=1}^d x_{ij} \theta_j + e_i
\]

and \( e_i \) are unobserved noise terms.
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- **ARMAX**

\[ A(q^{-1})y_t = B(q^{-1})u_t + C(q^{-1})e_t \]

  - \[ A(q^{-1}) = 1 + a_1 q^{-1} + \cdots + a_n a q^{-n a} \]
  - \[ B(q^{-1}) = b_1 q^{-1} + \cdots + b_{n b} q^{-n b} \]
  - \[ C(q^{-1}) = 1 + c_1 q^{-1} + \cdots + c_{n c} q^{-n c} \]

A simple example

\[ y_t + ay_{t-1} = bu_{t-1} + e_t + ce_{t-1} \]

- **OE model**

\[ y_t = \frac{B(q^{-1})}{F(q^{-1})} u_t + e_t \]

  - \[ B(q^{-1}) = b_1 q^{-1} + \cdots + b_{n b} q^{-n b} \]
  - \[ F(q^{-1}) = f_1 q^{-1} + \cdots + f_{n f} q^{-n f} \]

A simple example

\[ y_t = b_1 u_{t-1} + e_t + f_1 e_{t-1} \]

- ARMAX and OE models are nonlinear in the parameters because we have cross terms \( ce_{t-1} \) and \( fe_{t-1} \) of unknown quantities.
Exercixe 1.2 LS with Feedback

Consider the second order AR model

\[ y_t + a_0 y_{t-1} = b_0 u_{t-1} + e_t \]

where \( u_t \) is given by feedback as \( u_t = -K y_t \). Show that given realization of this signal we can not estimate \( a_0 \) and \( b_0 \) seperately but we can estimate \( a_0 + K b_0 \).

\[
\left( \begin{array}{cc}
\sum y_{t-1}^2 & - \sum y_{t-1} u_{t-1} \\
- \sum y_{t-1} u_{t-1} & \sum u_{t-1}^2 \\
\end{array} \right) \left( \begin{array}{c}
\hat{a} \\
\hat{b} \\
\end{array} \right) = \left( \begin{array}{c}
- \sum y_t y_{t-1} \\
\sum y_t u_{t-1} \\
\end{array} \right)
\]

\[
\sum y_{t-1}^2 \left( \begin{array}{cc}
1 & K \\
K & K^2 \\
\end{array} \right) \left( \begin{array}{c}
\hat{a} \\
\hat{b} \\
\end{array} \right) = - \sum y_t y_{t-1} \left( \begin{array}{c}
1 \\
K \\
\end{array} \right)
\]

\[ \text{only the linear combinaton } a_0 + b_0 K \text{ can be uniquely identified.} \]
Exercise 1.3 Step response

- A first order system $Y(s) = G(s)U(s)$ with

$$G(s) = \frac{K}{1 + sT}e^{-sT}$$

or in time domain

$$T \frac{dy_t}{dt} + y_t = Ku_{t-\tau}$$

derive a formula of the step response of an input $u_t = I(t > 0)$.

The step response:

$$y_t = \begin{cases} 
  0 & t < \tau \\
  K(1 - exp(-(t - \tau)/T)) & t \geq \tau
\end{cases}$$
Exercise 1.3 Step response

\[ G(s) = \frac{K}{1 + sT} e^{-sT} \]

Figure: step response.
Exercise 1.4: Step response as a special case of spectral analysis

Let \((y_t)_t\) be the step response of an LTI \(H(q^{-1})\) to an input \(u_t = aI(t \geq 0)\). Assume \(y_t = 0\) for \(t < 0\) and \(y_t \approx c\) for \(t > n\). Justify the following rough estimate of \(H\)

\[
\hat{h}_t = \frac{y_t - y_{t-1}}{a}, \quad \forall t = 0, \ldots, n
\]

and show that it is approximately equal to the estimate provided by the spectral analysis.

\[
H(e^{-i\omega}) = \sum_{k=0}^{\infty} h_k e^{-i\omega k}
\]
The end