Task #1: “The Metric Lax”
Numerical Functional Analysis

*Divide et Impera*

Stefan Engblom

Division of Scientific Computing
Department of Information Technology
Uppsala University

Uppsala, September, 2014
Prerequisites

Please form exactly 4 groups of about 3–4 persons.
Task 1

⇒ Formulate the Lax (or Lax-Richtmyer) equivalence theorem.
Task 1

⇒ Formulate the Lax (or Lax-Richtmyer) equivalence theorem.

Something like: "A consistent method applied to a well-posed problem is convergent if and only if it is stable."
Some notation. We consider the mathematical problem (or M-problem for short) “find $x \in X$ s.t. $Tx = y$”. Assume metric spaces $(X, d)$ and $(Y, \tilde{d})$. The solution is thus $x = T^{-1}y$.
Tasks
To solve during 15 min of fika

Some notation. We consider the mathematical problem (or M-problem for short) “find $x \in X$ s.t. $Tx = y$”. Assume metric spaces $(X, d)$ and $(Y, \tilde{d})$. The solution is thus $x = T^{-1}y$.

The M-problem is to be approached numerically by solving (a sequence of) numerical (‘N’-) problems “find $x_n \in X$ s.t. $T_n x_n = y$”. We can think of $n$ as the resolution of the scheme.
Some notation. We consider the mathematical problem (or M-problem for short) \textit{“find }x \in X \text{ s.t. } Tx = y\text{”}. Assume metric spaces \((X, d)\) and \((Y, \tilde{d})\). The solution is thus \(x = T^{-1}y\).

The M-problem is to be approached numerically by solving (a sequence of) numerical (‘N’-) problems \textit{“find }x_n \in X \text{ s.t. } T_n x_n = y\text{”}. We can think of \(n\) as the resolution of the scheme.

Tasks (one per group): in this setting...

1. ...define what is meant by well-posed.
2. ...define what is meant by a consistent numerical method.
3. ...define what is meant by a convergent numerical method.
4. ...define what is meant by a stable numerical method.
Some notation. We consider the mathematical problem (or M-problem for short) “find \( x \in X \) s.t. \( Tx = y \)”. Assume metric spaces \((X, d)\) and \((Y, \tilde{d})\). The solution is thus \( x = T^{-1}y \).

\[
\Rightarrow \text{ Define what is meant by well-posed in this setting.}
\]
Some notation. We consider *the mathematical problem* (or M-problem for short) “find $x \in X$ s.t. $Tx = y$”. Assume metric spaces $(X, d)$ and $(Y, \tilde{d})$. The solution is thus $x = T^{-1}y$.

⇒ Define what is meant by **well-posed** in this setting.

The M-problem is well-posed if $T^{-1}$ is continuous in some neighborhood containing $y$. 
The M-problem is to be approached numerically by solving (a sequence of) numerical (‘N’- ) problems “find $x_n \in X$ s.t. $T_n x_n = y$”. We can think of $n$ as the resolution of the scheme.

⇒ Define what is meant by a consistent numerical method in this setting.
The M-problem is to be approached numerically by solving (a sequence of) numerical (‘N’-) problems “find $x_n \in X$ s.t. $T_n x_n = y$”. We can think of $n$ as the resolution of the scheme.

⇒ Define what is meant by a **consistent** numerical method in this setting.

A method is consistent if, for any $x$ in the domain of $T$, $T_n x \rightarrow T x$ as $n \rightarrow \infty$. 
Define what is meant by a **convergent** numerical method in this setting.
⇒ Define what is meant by a **convergent** numerical method in this setting.

A method is convergent if \( T_n^{-1}y = x_n \rightarrow x \) as \( n \rightarrow \infty \).
Define what is meant by a **stable** numerical method in this setting.
Define what is meant by a **stable** numerical method in this setting.

A method is stable if for all $n$, $T_n^{-1}$ is continuous in some neighborhood containing $y$. 
“The Metric Lax”

A formulation of the Lax principle in metric spaces

Suppose $T^{-1}$ is continuous and that $T_n x \to T x$ for any $x$. Then $T_n^{-1} y \to x$ as $n \to \infty$ if and only if for all $n$, $T_n^{-1}$ is continuous.
“The Metric Lax”
A formulation of the Lax principle in metric spaces

Suppose $T^{-1}$ is continuous and that $T_n x \to Tx$ for any $x$. Then $T_n^{-1} y \to x$ as $n \to \infty$ if and only if for all $n$, $T_n^{-1}$ is continuous.

“A consistent method applied to a well-posed problem is convergent if and only if it is stable.”
Suppose \( T^{-1} \) is continuous and that \( T_n x \rightarrow T x \) for any \( x \). Then \( T_n^{-1} y \rightarrow x \) as \( n \rightarrow \infty \) if and only if for all \( n \), \( T_n^{-1} \) is continuous.

“A consistent method applied to a well-posed problem is convergent if and only if it is stable.”

\( \implies \) Try to find weaknesses with this formulation. -Can you straightforwardly apply it in a setting with which you are familiar?
“The Metric Lax”
A formulation of the Lax principle in metric spaces

Suppose $T^{-1}$ is continuous and that $T_n x \to Tx$ for any $x$. Then $T^{-1}_n y \to x$ as $n \to \infty$ if and only if for all $n$, $T^{-1}_n$ is continuous.

“A consistent method applied to a well-posed problem is convergent if and only if it is stable.”

Try to find weaknesses with this formulation. -Can you straightforwardly apply it in a setting with which you are familiar?

- $y \mapsto y_n$ would be more realistic.
- Usually $T$ and $T_n$ are not defined on the same space.
- ...

S. Engblom (IT/TDB)