Slow Quiz #3
Numerical Functional Analysis, 5.0 hp

Præparatus supervivet

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1. Consider the PDE

$$
\begin{aligned}
-\Delta u &= f, \quad x \in \Omega, \\
u &= 0, \quad x \in \partial\Omega,
\end{aligned}
$$

where reasonably $u \in C^2(\Omega)$. Define $(u, v) = \int_\Omega uv \, dx$. Multiplying the PDE by a test-function v and integrating using Green's formula we get the variational form

$$(\nabla v, \nabla u) = (v, f). \tag{1}$$

The use of Green's formula assumes $u, v \in V$, where

$$V = \left\{ v; \; \|v\|_V^2 := \|v\|^2 + \|\nabla v\|^2 < \infty, \, v|_{\partial\Omega} = 0 \right\}.$$

Show that V is a subspace of a Hilbert space H. What is the inner product $(\cdot, \cdot)_V = (\cdot, \cdot)_H$? The full variational formulation is *"find $u \in V$ s.t. (1) holds for $\forall v \in V$"*. -Identify the linear functional and the bilinear form for this problem. *Using the $\| \cdot \|_V$-norm*, prove boundedness and that the bilinear form is coercive. Deduce that the variational formulation is well-posed. *Hint:* the Poincaré inequality might come in handy,

$$\|v\| \leq C \|\nabla v\|, \quad v \in V,$$

with $C > 0$ a constant.