Numerical methods in stochastic modeling and simulations:

Miniproject 1

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In this miniproject you will basically investigate the convergence behavior of numerical methods for SDEs driven by Wiener processes. You are to submit the (draft) report on the mandatory part before the next seminar.

If you wish to also take the extended part you are to present your results towards the end of the course (but feel invited to actively help me schedule this presentation on a suitable occasion where the material still feels fresh!).

Note: to pass the course you need to take *one* extended part from one of the three miniprojects during the course.

Mandatory part

Reading

- The paper Higham. [2001] is a concise overview and contains working Matlab code that will certainly aid in the experiments. I suggest you start with this and turn directly to the practical experiments below.
- In the book Kloeden and Platen [1992]: the *Brief Survey of Stochastic Numerical Methods*, pp. XXI–XXXVI is a useful and quick starter. Chapter 9 introduces various concepts and Chapter 10.2–10.3 some basic schemes. I suggest you turn to this source when seeking support in the writing of the report.

Suitable SDEs Suitable SDEs to start testing with include *Geometrical Brownian motion*, the *Ornstein-Uhlenbeck process*, and the *Cox-Ingersoll-Ross model* (you can easily Google these terms). Other examples are found

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in [B. Øksendal, 2005, Exercises in Chap. 5], [Kloeden and Platen, 1992, Chap. 7], as well as on the web.

After starting with a 'basic' well understood model, *please use your own imagination and curiosity to find another test model that you personally take an interest in*!

In your report, aim for at least two distinct models. Make the report more interesting by offering a summary of the models selected: where do they come from, what do they model, what mathematical properties do they have? How could a numerical method be of use in the context?

Also try to inspect your numerical results with this overview in mind: does the numerical method tend to increase the (mean) population/interest rate/temperature/... in a biased way? What about the noise (variance)? Does the numerical method seem to be useful in the context of the model? Does it preserve some qualities of the mathematical model?

Suitable schemes Take *the Euler* and *the Milstein* schemes, see for example [Kloeden and Platen, 1992, Chap. 10.2–10.3]. Other methods exist but they are more complicated and model-dependent, and this is left for the extended part below.

Task 1: strong convergence With your selected model SDEs and the numerical methods, study the strong convergence of the numerical solution $Y_t^{(h)} \to X_t$ as the numerical step-size $h \to 0$. You typically do this by monitoring an error like

$$E[|Y_t^{(h)} - X_t|] \text{ or } E[|Y_t^{(h)} - X_t|^2],$$

and either a fixed end-value of t, or integrating over the integration interval, say $[0, t]$. In turn, to estimate the expected value you need to condition both $Y_t^{(h)}$ and X_t on the *same* Wiener process, and then you can use a sample mean,

$$E[|Y_t^{(h)} - X_t|] \approx \frac{1}{N} \sum_{i=1}^{N} |Y(t^{(h)}, \omega_i) - X(t, \omega_i)|,$$

with ω_i denoting different samples. Plot this estimator as a function of h. The error estimator above has a certain sample error associated with it. Include this in your plots, for example, by also plotting $\pm 2\sigma/\sqrt{N}$ with σ the standard deviation.

Task 2: a case-study For one of the models, pretend that you are going to use a numerical method to solve some practical problem. Then you are likely interested in an expected value of some function of the trajectory $f(X_t)_{t\geq 0}$, like the strike price, expected efficiency, expected variability, mean survival time, or similar.

Evaluate the method's performance *with respect to this property alone.* First try to determine the "true" value of the target function $E[f(X_t)_{t\geq 0}]$ as accurately as possible (maybe even analytically). Then look at the efficiency (for example the number of random numbers needed) of each method as a function of the achieved error. Try to ensure that the sample error from computing the average in

$$E[f(Y(t^{(h)}))_{t\geq 0}] \approx \frac{1}{N}\sum_{i=1}^{N} E[f(Y(t^{(h)},\omega_i))_{t\geq 0}],$$

is about equal to the numerical error in,

$$E[f(X(t))_{t\geq 0}] \approx E[f(Y(t^{(h)}))_{t\geq 0}].$$

If you cannot invent a function which is relevant in your context, take the mean first exit time, $E[\tau]$, where

$$\tau(\omega) = \inf_{t\geq 0} X(t,\omega) \notin D,$$

and D some suitable set with $X(0) \in D$.

Extended part

For the extended part, perform and report on *at least one* of the tasks below.

The tasks are very openly described and the intent here is that you undertake a bit of research and take the role of a critical reviewer. *Take your job seriously!*

Task 3: exact simulation Test and evaluate the exact simulation method described in Beskos and Roberts [2005], starting with Geometric Brownian motion as a sample model, then moving on to the example in eq. (10) of the paper (p. 2433). Critically discuss the suggested method and the paper itself, for example by inventing a numerical test yourself or following up papers that have cited it.

Task 4: weak convergence Test and evaluate at least one of the weakly convergent methods in [Kloeden and Platen, 1992, Chap. 14–15]. Set up a suitable case study yourself. Compare to the strongly convergent methods tested in the mandatory part above and offer a critical discussion.

Task 5: implicit methods vs. multiscale methods The paper Li et al. [2008] suggests that implicit methods applied to *stiff* problems is not the right way to go: rather should an appropriate multiscale method be designed. Critically discuss the message of the paper by repeating a selected few of the numerical experiments therein and preferably also look for some other test-problems. There *are* (semi-) implicit methods in active use, are they doing the wrong thing?

References

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