Isar — A language for structured proofs

Apply scripts

- unreadable

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No structure!

Apply scripts versus Isar proofs

Apply script = assembly language program

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Isar proof = structured program with comments

Apply scripts versus Isar proofs

Apply script = assembly language program

Isar proof = structured program with comments

But: **apply** still useful for proof exploration

A typical Isar proof

proof

  assume $formula_0$

  have $formula_1$ **by** *simp*

  $\vdots$

  have $formula_n$ **by** *blast*

  show $formula_{n+1}$ **by** $\ldots$

qed

A typical Isar proof

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proves $formula_0 \Longrightarrow formula_{n+1}$

Overview

- Basic Isar
- Propositional logic
- Predicate logic

Isar core syntax

proof	=	**proof** [method] statement* **qed**
	|	**by** method

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| **by** method

method = *(simp* . . . *)* | *(blast* . . . *)* | *(rule* . . . *)* | . . .

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statement = **fix** variables ($\bigwedge$)
| **assume** proposition ($\Longrightarrow$)
| [**from** name$^+$] (**have** | **show**) proposition proof

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| **next** (separates subgoals)

Isar core syntax

proof = **proof** [method] statement* **qed**
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method = *(simp ...)* | *(blast ...)* | *(rule ...)* | ...

statement = **fix** variables ($\bigwedge$)
| **assume** proposition ($\Longrightarrow$)
| [**from** name$^+$] (**have** | **show**) proposition proof
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proposition = [name:] formula

Demo: propositional logic, introduction rules

Basic proof methods

Basic atomic proof:

by $method$
apply $method$, then prove all subgoals by assumption

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Basic proof method:

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apply a rule in $\vec{a}$;
if $\vec{a}$ is empty: apply a standard elim or intro rule.

Abbreviations:

.	=	**by** do-nothing
..	=	**by** *rule*

Demo: propositional logic, elimination rules

Elimination rules / forward reasoning

- Elim rules are triggered by facts fed into a proof:
  from $\vec{a}$ **have** $formula$ **proof**

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- *rule*: tries elim rules first (if there are incoming facts $\vec{a}$!)
- **from** $\vec{a}$ **have** $formula$ **proof** *(rule* $rule$*)*

  $\vec{a}$ must prove the first n premises of $rule$, in the right order

  the others are left as new subgoals

Abbreviations

this	=	the previous proposition proved or assumed
then	=	**from** *this*
thus	=	**then show**
hence	=	**then have**
with $\vec{a}$	=	**from** $\vec{a}$ *this*

using

First the what, then the how:

(**have**|**show**) proposition **using** facts

using

First the what, then the how:

(**have**|**show**) proposition **using** facts

=

from facts (**have**|**show**) proposition

using

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(**have**|**show**) proposition **using** facts

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from facts (**have**|**show**) proposition

Can be mixed:

from major-facts (**have**|**show**) proposition **using** minor-facts

using

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(**have**|**show**) proposition **using** facts
=
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Can be mixed:

from major-facts (**have**|**show**) proposition **using** minor-facts
=
from major-facts minor-facts (**have**|**show**) proposition

Demo: avoiding duplication

Schematic term variables

$?A$

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- Defined by pattern matching:

$$x = 0 \wedge y = 1 \ (\textbf{is}\ ?A \wedge \_)$$

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$$?A$$

- Defined by pattern matching:

  $x = 0 \wedge y = 1$ (**is** $?A \wedge \_$)

- Predefined: *?thesis*
  The last enclosing **show** formula

Demo: predicate calculus

obtain

Syntax:

obtain variables **where** proposition proof

Mixing proof styles

```
from ...
have ...
  apply -        make incoming facts assumptions
  apply(...)
  ⋮
  apply(...)
  done
```

Advanced Isar

Overview

- Case distinction
- Induction
- Calculational reasoning

Case distinction

Boolean case distinction

proof *cases*

assume $formula$

⋮

next

assume $\neg formula$

⋮

qed

Boolean case distinction

proof *cases*
  assume $formula$
  ⋮
next
  assume $\neg formula$
  ⋮
qed

proof *(cases* $formula$*)*
  case *True*
  ⋮
next
  case *False*
  ⋮
qed

Boolean case distinction

proof *cases*
  assume $formula$
  ⋮
next
  assume $\neg formula$
  ⋮
qed

proof *(cases* $formula$*)*
  case *True*
  ⋮
next
  case *False*
  ⋮
qed

case *True* $\equiv$
assume *True:* $formula$

Demo: case distinction

Datatype case distinction

proof (*cases* $term$)

  case *Constructor*$_1$

  ⋮

next

⋮

next

  case (*Constructor*$_k$ $\vec{x}$)

  $\cdots \; \vec{x} \; \cdots$

qed

Datatype case distinction

proof (*cases* $term$)

  case *Constructor*$_1$

  ⋮

next

⋮

next

  case (*Constructor*$_k$ $\vec{x}$)

  $\cdots\ \vec{x}\ \cdots$

qed

case (*Constructor*$_i$ $\vec{x}$) $\equiv$

fix $\vec{x}$ **assume** *Constructor*$_i$: $term$ = (*Constructor*$_i$ $\vec{x}$)

Induction

Overview

- Structural induction
- Rule induction
- Induction with fun

Structural induction for type nat

show $P(n)$
proof *(induction n)*
  case *0*
  . . .
  show *?case*
next
  case *(Suc n)*
  . . .
  $\cdots$ *n* $\cdots$
  show *?case*
qed

Structural induction for type nat

show $P(n)$
proof *(induction n)*
  case *0* $\equiv$ **let** *?case = $P(0)$*
  . . .
  show *?case*
next
  case *(Suc n)*
  . . .
  $\cdots$ *n* $\cdots$
  show *?case*
qed

Structural induction for type nat

show $P(n)$
proof *(induction n)*
    case *0* $\equiv$ **let** *?case* = $P(0)$
    . . .
    show *?case*
next
    case *(Suc n)* $\equiv$ **fix** *n* **assume** *Suc:* $P(n)$
    . . . **let** *?case* = $P(Suc\ n)$
    $\cdots$ *n* $\cdots$
    show *?case*
qed

Demo: structural induction

Structural induction with $\Longrightarrow$ and $\bigwedge$

show $\bigwedge x.\ A(n) \Longrightarrow P(n)$

proof *(induction n)*

case *0*

. . .

show *?case*

next

case *(Suc n)*

. . .

$\cdots$ *n* $\cdots$

. . .

show *?case*

qed

Structural induction with $\Longrightarrow$ and $\bigwedge$

show $\bigwedge x.\ A(n) \Longrightarrow P(n)$
proof *(induction n)*
  case *0* $\equiv$ **fix** *x* **assume** *0: A(0)*
  . . . **let** *?case = P(0)*
  show *?case*
next
  case *(Suc n)*
  . . .
  $\cdots$ *n* $\cdots$
  . . .
  show *?case*
qed

Structural induction with $\Longrightarrow$ and $\bigwedge$

show $\bigwedge x.\ A(n) \Longrightarrow P(n)$
proof *(induction n)*
  case *0* $\equiv$ **fix** x **assume** $0\colon A(0)$
  ... **let** $?case = P(0)$
  show $?case$
next
  case *(Suc n)* $\equiv$ **fix** $n\ x$
  ... **assume** $Suc\colon \bigwedge x.\ A(n) \Longrightarrow P(n)$
  ... n ... $A(Suc\ n)$
  ... **let** $?case = P(Suc\ n)$
  show $?case$
qed

A remark on style

- **case** *(Suc n)* . . . **show** *?case*
  is easy to write and maintain

A remark on style

- **case** *(Suc n)* . . . **show** *?case*
  is easy to write and maintain
- **fix** *n* **assume** $formula$ **. . . show** $formula'$
  is easier to read:
  - all information is shown locally
  - no contextual references (e.g. *?case*)

Demo: structural induction with $\Longrightarrow$ and $\bigwedge$

Rule induction

Inductive definition

inductive_set *S*

intros

rule$_1$: $[\![$ *s* $\in$ *S*; *A* $]\!] \Longrightarrow$ *s'* $\in$ *S*

$\vdots$

rule$_n$: $\ldots$

Rule induction

show *x* $\in$ *S* $\Longrightarrow$ *P(x)*

proof *(induct rule: S.induct)*

  case $rule_1$

  . . .

  show *?case*

next

⋮

next

  case $rule_n$

  . . .

  show *?case*

qed

Implicit selection of induction rule

assume *A*: $x \in S$

⋮

show *P(x)*

using *A* **proof** *induct*

⋮

qed

Implicit selection of induction rule

assume *A:* $x \in S$
⋮
show *P(x)*
using *A* **proof** *induct*
⋮
qed

lemma assumes *A:* $x \in S$ **shows** *P(x)*
using *A* **proof** *induct*
⋮
qed

Renaming free variables in rule

case *(rule$_i$ x$_1$ $\ldots$ x$_k$)*

Renames the (alphabetically!) first k variables in *rule$_i$* to *x$_1$ $\ldots$ x$_k$*.

Demo: rule induction

Induction with fun

Definition:

fun *f*

⋮

Induction with fun

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fun *f*

⋮

Proof:

show . . . *f*(. . .) . . .

proof *(induction* x_1 . . . x_k *rule: f.induct)*

Induction with fun

Definition:

fun *f*

⋮

Proof:

show $\ldots$ *f(* $\ldots$ *)* $\ldots$

proof *(induction x* $_1$ $\ldots$ *x* $_k$ *rule: f.induct)*

case *1*

⋮

Induction with fun

Definition:

fun *f*

⋮

Proof:

show $\ldots$ *f*($\ldots$) $\ldots$

proof *(induction* $x_1 \ldots x_k$ *rule: f.induct)*

  case *1*

  ⋮

Case i refers to equation i in the definition of *f*

Induction with fun

Definition:

fun *f*

⋮

Proof:

show $\ldots$ *f(* $\ldots$ *)* $\ldots$

proof *(induction x*$_1$ $\ldots$ *x*$_k$ *rule: f.induct)*

  case *1*

  ⋮

Case i refers to equation i in the definition of *f*
More precisely: to equation i in *f.simps*

Demo: induction with fun

Calculational Reasoning

Overview

- Accumulating facts
- Chains of equations and inequations

moreover

have $formula_1$ **. . .**

moreover

have $formula_2$ **. . .**

moreover

⋮

moreover

have $formula_n$ **. . .**

ultimately show . . .

— pipes facts $formula_1 \ldots formula_n$ into the proof

proof

⋮

also

have $t_0 = t_1$ . . . .

also

have $\ldots = t_2$ . . . .

also

⋮

also

have $\ldots = t_n$ . . . .

also

have $t_0 = t_1$ . . . .

also

have $\ldots = t_2$ . . . . $\qquad \ldots \equiv t_1$

also

$\vdots$

also

have $\ldots = t_n$ . . . .

also

have $t_0 = t_1$ . . . .

also

have $\ldots = t_2$ . . . . $\qquad \ldots \equiv t_1$

also

$\vdots$

also

have $\ldots = t_n$ . . . . $\qquad \ldots \equiv t_{n-1}$

also

have $t_0 = t_1$ . . . .

also

have $\ldots = t_2$ . . . . $\ldots \equiv t_1$

also

$\vdots$

also

have $\ldots = t_n$ . . . . $\ldots \equiv t_{n-1}$

finally **show** . . . .

— pipes fact $t_0 = t_n$ into the proof

proof

$\vdots$

...

"..." is merely an abbreviation

Demo: moreover and also

Variations on also

Transitivity:

have t_0 = t_1 . . . .

also have $\ldots$ = t_2 . . . .

also/finally $\leadsto$

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Variations on also

Transitivity:

have t_0 = t_1 **. . . .**

also have $\ldots$ = t_2 **. . . .**

also/finally $\leadsto$ t_0 = t_2

Substitution:

have $P(s)$ **. . . .**

also have s = t **. . . .**

also/finally $\leadsto$

Variations on also

Transitivity:

have $t_0 = t_1$ **. . . .**

also have $\ldots = t_2$ **. . . .**

also/finally $\leadsto t_0 = t_2$

Substitution:

have $P(s)$ **. . . .**

also have $s = t$ **. . . .**

also/finally $\leadsto P(t)$

From $=$ *to* $\leq$ *and* $<$

Transitivity:

have $t_0 \leq t_1$ **. . . .**

also have $\ldots \leq t_2$ **. . . .**

also/finally $\leadsto$

From $=$ to $\leq$ and $<$

Transitivity:

have $t_0 \leq t_1$ **. . . .**

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From $=$ to $\leq$ and $<$

Transitivity:

have $t_0 \leq t_1$ **. . . .**

also have $\ldots \leq t_2$ **. . . .**

also/finally $\leadsto t_0 \leq t_2$

Substitution:

have $r \leq f(s)$ **. . . .**

also have s **<** t **. . . .**

also/finally $\leadsto$

From $=$ ***to*** $\leq$ ***and*** $<$

Transitivity:

have $t_0 \leq t_1$ **. . . .**

also have $\ldots \leq t_2$ **. . . .**

also/finally $\leadsto t_0 \leq t_2$

Substitution:

have $r \leq f(s)$ **. . . .**

also have $s < t$ **. . . .**

also/finally $\leadsto (\bigwedge x.\ x < y \Longrightarrow f(x) < f(y)) \Longrightarrow r < f(t)$

From $=$ to $\leq$ and $<$

Transitivity:

have $t_0 \leq t_1$ **. . . .**

also have $\ldots \leq t_2$ **. . . .**

also/finally $\leadsto t_0 \leq t_2$

Substitution:

have $r \leq f(s)$ **. . . .**

also have $s < t$ **. . . .**

also/finally $\leadsto (\bigwedge x.\ x < y \Longrightarrow f(x) < f(y)) \Longrightarrow r < f(t)$

Similar for all other combinations of $=$, $\leq$ and $<$.

All about also

To view all combinations in Proof General:

Isabelle/Isar $\rightarrow$ Show me $\rightarrow$ Transitivity rules

Demo: monotonicity reasoning