1. Prove that if a weak solution of a hyperbolic conservation law is smooth, then it is also a classical solution.

2. Consider traffic flow on a road with varying number of lanes, (1 or 2). Use the ideas in chapter 11 to formulate a hyperbolic conservation law. Make sure your model conserves cars.

3. Consider the Riemann problem for the inviscid Burgers’ equation with initial data $u(x, 0) = \begin{cases} \frac{u_L}{u_R}, & x < 0 \\ 1, & 0 < x < 2 \\ 0, & x > 2 \end{cases}$

Construct one example with discontinuities of each of the following type, A) violates the entropy condition but satisfies the Rankine-Hugoniot condition.
B) Satisfies the entropy condition, but violates the Rankine-Hugoniot condition.
C) Are your examples weak solutions? Show by an example that on its own the weak solution concept does not imply uniqueness.

4. Solve the inviscid Burgers’ equation by sketching the characteristics and shock paths in the $x$-$t$ plane when initial data is $u(x, 0) = \begin{cases} 2, & x < 0 \\ 1, & 0 < x < 2 \\ 0, & x > 2 \end{cases}$

Show that Lax’s and Oleinik’s entropy conditions are satisfied by the solution.

5. Consider the hyperbolic PDE $u_t + Au_x = 0$. Solve the Riemann problem when

$$A = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}$$

and A) $u_L = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, u_R = \begin{pmatrix} 1 \end{pmatrix}$ B) $u_L = \begin{pmatrix} 1 \end{pmatrix}, u_R = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$