Exercises, Module III

Hand in individual solutions; state name and personal number. Discussions between students are encouraged, but write your solutions individually. Do not copy solutions from others. I prefer answering questions by phone (instead of by email). It is easiest to reach me on my cellular at 070-732 8111.

1. Some optimization problems involve a sequence of calculations in order to proceed from design variable $\phi$ to objective $J$: $\phi \rightarrow u \rightarrow v \rightarrow J$. Assume that there are two state equations:

$$ Au = \phi, $$

$$ Bu = u, $$

where $A$ and $B$ are square nonsingular matrices. Derive an expression for the gradient of the objective function $J = c^T v$ as a function of $\phi$ in terms of adjoint equations (note the order of computations!).

2. Assume a nonlinear state equation

$$ a(u) = b(\phi), $$

where $u, a \in \mathbb{R}^n$ and $\phi \in \mathbb{R}^m$. A least-squares objective function is common in many application:

$$ J = \frac{1}{2} \sum_{k=1}^{K} f_k^2. $$

Now assume that $f_k = c_k^T u - y_k$, for $c_k \in \mathbb{R}^n$ and $y_k \in \mathbb{R}$.

(a) Derive an expression for the gradient of $J$ as a function of $\phi$ in terms of an adjoint equation.

(b) Standard algorithms for nonlinear least-squares problems (such as the Levenberg–Marquard method) require gradients, not of $J$, but of the individual $f_k$’s as a function of $\phi$. Derive these expressions with the adjoint-equations approach.

3. Many material-design problems generate a state equation of the form

$$ A(\phi) u = b, $$

where the square matrix $A$ is given by

$$ A(\phi) = \sum_{n=1}^{N} \phi_n A_n $$

for each design variable $\phi = (\phi_1, \ldots, \phi_N)^T \in U \subset \mathbb{R}^N$; we assume that matrix $A(\phi)$ is nonsingular for each $\phi \in U$. Let the objective function have the form

$$ J(\phi) = c^T u. $$

(a) Derive an expression for $\nabla J$ in terms of the solution of an adjoint equation.
(b) Assume that $A(\phi)^T = A(\phi)$ and $b = c$. Show that the $n$th component of $\nabla J$ is given by $-u^TA_nu$ (so that no adjoint equation is needed in this case).

(c) Again assuming that $A(\phi)^T = A(\phi)$ in state equation (i), derive an expression for the gradient of the objective function $j(\phi) = \frac{1}{2}u^TAu$.

4. Assume we want to approximate the derivative of the smooth function $f : \mathbb{R} \to \mathbb{R}$ using the central difference formula

$$\frac{df}{dx} \approx \frac{f(x + \delta) - f(x - \delta)}{2\delta}$$

Derive the formula for the optimal step length $\delta$ in finite-precision arithmetic characterized by unit round-off $\epsilon_u$. (“Unit round-off” means that for any real $a$ within over- and underflow, there is a finite-precision number $\tilde{a}$ and a $\Delta_a$ satisfying $|\Delta_a| \leq \epsilon_u$ such that $\tilde{a} = (1 + \Delta_a)a$.

5. Consider the PDE

$$-\Delta u + u = 0 \quad \text{in } \Omega,$$

$$\frac{\partial u}{\partial n} = g \quad \text{on } \Gamma_i,$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_o,$$

which models the steady temperature field in a solid that is absorbing heat (say in the direction normal to plane). Heating is applied at the inner boundary $\Gamma_i$ and the outer boundary $\Gamma_o$ is insulated.

Assume that we want to control $u|_{\Gamma_o}$ by varying the function $g$, and we are thus interested in the properties of the mapping $\mathcal{A} : g \mapsto u|_{\Gamma_o}$ ($\mathcal{A}$ is a bounded linear mapping from $L^2(\Gamma_i)$ into $L^2(\Gamma_o)$).

(a) Show that $u|_{\Gamma_i}$ spans a dense subset of $L^2(\Gamma_o)$ as $g$ spans $L^2(\Gamma_i)$.

Hint: You may use the following uniqueness theorem. If $p$ satisfies $-\Delta p + p = 0$ in an open, bounded and connected region $\Omega$, and both $p$ and $\partial p/\partial n$ vanishes on an open smooth portion of the boundary, then $p \equiv 0$ throughout $\Omega$.

(b) Based on the result from part a), what could be the problem with the optimal-control problem consisting finding a $g \in L^2(\Gamma_i)$ that minimizes

$$\int_{\Gamma_i} (u - z)^2,$$

for some arbitrary target function $z \in L^2(\Gamma_i)$? Suggest a stabilized formulation of the optimal-control problem and derive the optimality system.

6. The discretized topology optimization problem using SIMP can be written

$$\min_{\rho} c^T u \quad \text{subject to}$$

$$\sum_{k=1}^{K} \rho_k^p A_k u = c$$

$$b^T \rho \leq r$$

$$0 \leq \rho \leq 1,$$
where $u, c \in \mathbb{R}^n; A_k$ is $n$-by-$n$ for $k = 1, \ldots, K; b, \rho = (\rho_1, \ldots, \rho_K)^T \in \mathbb{R}^K$; and $r > 0$. Give the first-order necessary optimality conditions.

7. Let $p$ be a smooth function on a region $\Omega \subset \mathbb{R}^d$, and let $\delta_m p$ and $\delta p$ denote the material and shape derivative, respectively, with respect to the domain variation $\delta \Omega : \overline{\Omega} \to \mathbb{R}^d$. Show the formulas

(a) $\delta_m (\nabla p) = \nabla (\delta_m p) - (\nabla \delta \phi)^T \nabla p$

(b) $\delta_m (\nabla q \cdot \nabla p) = \nabla \delta_m q \cdot \nabla p + \nabla q \cdot \nabla \delta_m p - \nabla q \cdot (\nabla \delta \phi + (\nabla \delta \phi)^T) \nabla p.$

Notation: The $i$th component of $(\nabla \delta \phi)^T \nabla p$ is $\sum_{j=1}^{d} \frac{\partial \delta \phi}{\partial x_i} \frac{\partial p}{\partial x_j}$. 
