Localized Orthogonal Decomposition (LOD) is a multiscale method for solving elliptic multiscale problems. It aims to construct a divergence-free fine-scale basis that is localized around the support of a coarse basis function. The method is designed to be mass conservative and to have exponential decay of basis functions, which makes it suitable for problems with highly varying coefficients.

### Abstract

We analyze a multiscale method for Poisson’s equation with applications to multiphase flows in porous media. Porous media often exhibit highly varying and heterogeneous permeability without scale separation. We prove stability and regularity independent rate of convergence for Localized Orthogonal Decomposition (LOD) of the Raviart–Thomas mixed finite elements. We obtain a modified basis for the flux that can, for instance, be applied in a nonlinear iteration, where it can be reused for efficient evolution of an upscaled solution.

### Standard discretization

We consider the mixed formulation of Poisson’s equation over $\Omega$, find $u$ and $p$

$$(a^{-1}u, v) + (\nabla \cdot v, p) = 0$$

$$(\nabla \cdot u, q) = -(f, q)$$

for all $v$ and $q$

$a$ — highly varying coefficient

$u$ — velocity solution

The lowest order Raviart–Thomas (RT) elements $(V_h)$ yield a mass conservative discrete solution $u_h$ with error estimate

$$\|u - u_h\|_{L^2(\Omega)} \lesssim h|u|_{H^1(\Omega)}$$

For coefficients varying down to scale $\epsilon$, we have $|u|_{H^1(\Omega)} \sim \epsilon^{-1}$. Then we need a small mesh, $h \leq \epsilon$ for accurate solutions.

### Ideal multiscale basis

Introduce a coarse mesh with mesh size $H$ and an RT element space $V_H$. Let $\Pi_H$ be the projection onto $V_H$, preserving normal fluxes between coarse elements. Fine scale variations are “invisible” to $\Pi_H$, so we construct the divergence free fine space

$$K^I_h = \{ v \in V_h : \nabla \cdot v = 0, \Pi_H v = 0 \}$$

For every coarse scale RT-basis function $\phi_H$, we have a divergence free fine scale corrector $G_h(\phi_H) \in K^I_h$, for all $v^I_h \in K^I_h$

$$(a^{-1}(G_h(\phi_H) - \phi_H), v^I_h) = 0$$

Define the modified basis $\phi_{H,k} = \phi_H - G_h(\phi_H)$. Solving with this basis, we get a solution $u_{H,k}$ with error bound

$$\|u - u_{H,k}\|_{L^2(\Omega)} \lesssim H + h|u|_{H^1(\Omega)}$$

with constant independent of $|u|_{H^1(\Omega)}$.

### Exponential decay of basis

The correctors $G_h(\phi_H)$ have global support but decay exponentially with distance from the support of coarse basis function $\phi_H$:

$$\|G_h(\phi_H)\|_{L^2(\Pi_H(\phi_H))} \lesssim \theta (H/h)^k \|G_h(\phi_H)\|_{L^2(\Omega)}$$

where $0 < \theta < 1$ and $k$ is the number of coarse layers in patch $U_k(\phi_H)$.

### Localization of modified basis

The exponential decay makes it possible to localize (on a $k$ layer patch) the modified basis functions, i.e., $G_h^{\ast}(\phi_H) \in K^I_h(U_k(\phi_H))$:

$$\phi_{H,k} = \phi_H - G_h(\phi_H)$$

Solving with this basis gives error bound:

$$\|u - u_{H,k}\|_{L^2(\Omega)} \lesssim H + h|u|_{H^1(\Omega)} + k^{d/2} \gamma (H/h)^{-1} \theta^k$$

Here $\gamma (H/h) = (1 + \log(H/h)/h)^{-1/2}$. Con- stant depends on contrast. Choose $k \approx \log(1/H)$ to maintain order $H$.

### Benefits

- Rate of convergence $H$ is regularity independent
- Modified basis can be reused for instance in a nonlinear iteration
- Localized computations can be done in parallel and make it possible to handle very large meshes
- We keep mass conservation

### References

