Localized Orthogonal Decomposition for Raviart–Thomas elements

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Abstract

We analyze a **multiscale** method for Poisson's equation with applications to multiphase flows in porous media. Porous media often exhibit highly varying and heterogeneous permeability without scale separation. We prove stability and **regularity independent** rate of convergence for Localized Orthogonal Decomposition (**LOD**) of the Raviart–Thomas mixed finite elements. We obtain a **modified basis** for the flux that can, for instance, be applied in a nonlinear iteration, where it can be **reused** for efficient evolution of an upscaled solution.

Standard discretization

We consider the mixed formulation of Poisson's equation over Ω, find $\mathbf{u}$ and p

$$\begin{aligned}(a^{-1}\mathbf{u},\mathbf{v}) + (\nabla\cdot\mathbf{v},p) &= 0\\ (\nabla\cdot\mathbf{u},q) &= -(f,q)\end{aligned}$$

for all $\mathbf{v}$ and q

a — highly varying coefficient
$\mathbf{u}$ — velocity solution

The lowest order Raviart–Thomas (RT) elements (V_h) yield a **mass conservative** discrete solution $\mathbf{u}_h$ with error estimate

$$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)} \lesssim h|\mathbf{u}|_{H^1(\Omega)}$$

For coefficients varying down to scale ϵ, we have $|\mathbf{u}|_{H_1(\Omega)} \sim \epsilon^{-1}$. Then we need a small mesh, $h \leq \epsilon$ for accurate solutions.

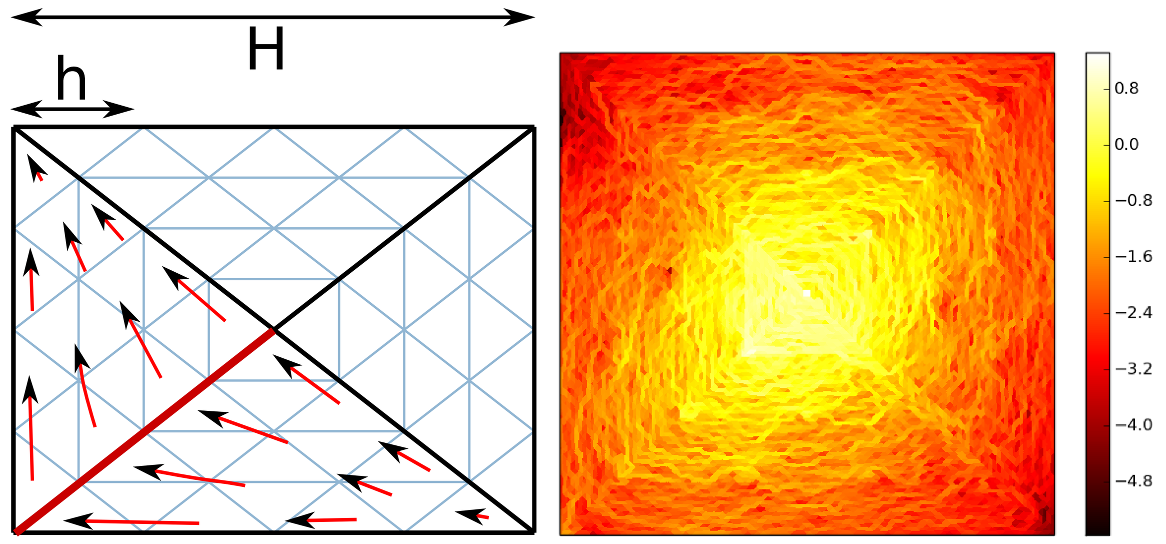

Figure 1: Left: Part of fine (h) and coarse (H) mesh. Coarse basis function ϕ_H in red. Right: 10-logarithm of mean flux in corrector function for noisy a.

Ideal multiscale basis

Introduce a coarse mesh with mesh size H and an RT element space V_H. Let Π_H be the projection onto V_H, preserving **normal fluxes** between coarse elements. Fine scale variations are "invisible" to Π_H, so we construct the **divergence free fine space**

$$K_h^{\mathrm{f}} = \{\mathbf{v}\in V_h : \nabla\cdot\mathbf{v} = 0, \Pi_H\mathbf{v} = 0\}$$

For every coarse scale RT-basis function ϕ_H, we have a divergence free **fine scale corrector** $G_h(\phi_H) \in K_h^{\mathrm{f}}$, for all $\mathbf{v}_h^{\mathrm{f}} \in K_h^{\mathrm{f}}$

$$(a^{-1}(G_h(\phi_H)-\phi_H),\mathbf{v}_h^{\mathrm{f}}) = 0$$

Define the **modified basis** $\phi_{H,h}^{\mathrm{ms}} = \phi_H - G_h(\phi_H)$. Solving with the modified basis, we get a solution $\mathbf{u}_{H,h}^{\mathrm{ms}}$ with error bound

$$\|\mathbf{u}-\mathbf{u}_{H,h}^{\mathrm{ms}}\|_{L^2(\Omega)} \lesssim H + h|\mathbf{u}|_{H^1(\Omega)}$$

with constant independent of $|\mathbf{u}|_{H_1(\Omega)}$.

Exponential decay of basis

The correctors $G_h(\phi_H)$ have global support but **decay exponentially** with distance from the support of coarse basis function ϕ_H:

$$\|G_h(\phi_H)\|_{L^2(\Omega\setminus U_k(\phi_H))} \lesssim \theta^{\gamma(H/h)k}\|G_h(\phi_H)\|_{L^2(\Omega)}$$

where $0<\theta<1$ and k is the number of coarse layers in **patch** $U_k(\phi_H)$.

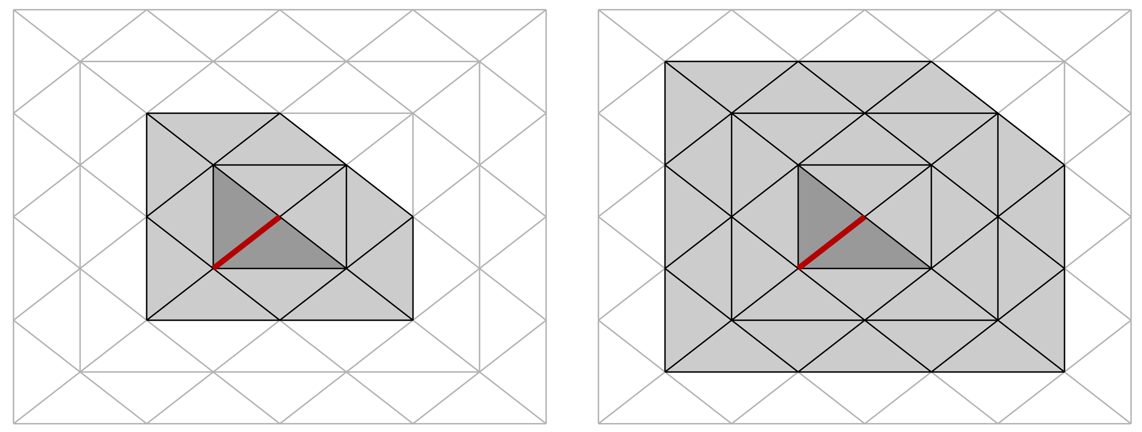

Figure 2: A one ($U_1(\phi_H)$) and two layer patch ($U_2(\phi_H)$) around support of coarse basis function ϕ_H.

Localization of modified basis

The exponential decay makes it possible to **localize** (on a k layer patch) the modified basis functions, i.e., $G_h^k(\phi_H) \in K_h^{\mathrm{f}}(U_k(\phi_H))$:

$$\phi_{H,h}^{\mathrm{ms},k} = \phi_H - G_h^k(\phi_H)$$

Solving with this basis gives error bound:

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_{H,h}^{\mathrm{ms},k}\|_{L^2(\Omega)} \lesssim H + h|\mathbf{u}|_{H^1(\Omega)}\\ + k^{d/2}\gamma(H/h)^{-1}\theta^k\end{aligned}$$

Here $\gamma(H/h) = (1+\log(H/h))^{-1/2}$. Constant depends on contrast. **Choose** $k \approx \log(1/H)$ to maintain order H.

Benefits

- Rate of convergence H is **regularity independent**
- Modified basis can be **reused** for instance in a **nonlinear iteration**
- Localized computations can be done in **parallel** and make it possible to handle very **large meshes**
- We keep **mass conservation**

References

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