Beyond Semidefinite Relaxtion: Basis Banks and Computationally Enhanced Guarantees

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***Abstract*—As a widely used tool in tackling general quadratic optimization problems, semidefinite relaxation (SDR) promises both a polynomial-time complexity and an *a priori* known sub-optimality guarantee for its approximate solutions. While attempts at improving the guarantees of SDR in a general sense have proven largely unsuccessful, it has been widely observed that the quality of solutions obtained by SDR is usually considerably better than the provided guarantees. In this paper, we propose a novel methodology that paves the way for obtaining improved data-dependent guarantees in a computational way. The derivations are dedicated to a specific quadratic optimization problem (called m-QP) which lies at the core of many communication and active sensing schemes; however, the ideas may be generalized to other quadratic optimization problems. The new guarantees are particularly useful in accuracy sensitive applications, including decision-making scenarios.**

I. INTRODUCTION

The NP-hard problem [1] of optimizing a quadratic form over the m-ary alphabet, viz.

$$(m\text{-QP}):\quad \max_{\boldsymbol{s}\in\Omega_m^n} \boldsymbol{s}^H\boldsymbol{R}\boldsymbol{s} \tag{1}$$

$$\text{with } \Omega_m = \left\{1, e^{j\frac{2\pi}{m}}, \cdots, e^{j\frac{2\pi}{m}(m-1)}\right\},$$

arises in a wide variety of communication and active sensing applications from signal design for transmission to signal processing at the receive side [1]-[7]. An interesting example of such applications is the maximum likelihood (ML) estimation of m-ary codes: consider a discrete-time linear multi-input multi-output (MIMO) channel modeled as

$$\boldsymbol{y} = \boldsymbol{Q}\boldsymbol{s} + \boldsymbol{v} \tag{2}$$

where $\boldsymbol{y} \in \mathbb{C}^n$ is the received signal, the matrix $\boldsymbol{Q} \in \mathbb{C}^{n\times n}$ represents the channel effect on the transmitted signal vector $\boldsymbol{s}$, and $\boldsymbol{v}$ denotes the additive white Gaussian noise. We assume that the entries of $\boldsymbol{s}$ belong to the m-ary constellation, i.e. $\boldsymbol{s}(k) \in \Omega_m$, for $1 \leq k \leq n$. The ML approximation of $\boldsymbol{s}$ may be stated as

$$\widehat{\boldsymbol{s}}_{ML} = \arg\min_{\boldsymbol{s}\in\Omega_m^n} \|\boldsymbol{y} - \boldsymbol{Q}\boldsymbol{s}\|_2. \tag{3}$$

It can be easily verified that the optimization problem in (3) is equivalent to the following m-QP [7]:

$$\max_{\overline{\boldsymbol{s}}\in\Omega_m^{n+1}} \overline{\boldsymbol{s}}^H(\lambda_{\max}(\boldsymbol{R})\,\boldsymbol{I} - \boldsymbol{R})\overline{\boldsymbol{s}} \tag{4}$$

where

$$\boldsymbol{R} = \begin{pmatrix} \boldsymbol{Q}^H\boldsymbol{Q} & -\boldsymbol{Q}^H\boldsymbol{y} \\ -\boldsymbol{y}^H\boldsymbol{Q} & 0 \end{pmatrix}, \overline{\boldsymbol{s}} = \begin{pmatrix} e^{j\frac{2\pi}{m}c}\boldsymbol{s} \\ e^{j\frac{2\pi}{m}c} \end{pmatrix} \tag{5}$$

with the integer c being a free auxiliary variable.

As the solution to (1) is invariant to diagonal loadings of $\boldsymbol{R}$, without loss of generality, we assume in the sequel that $\boldsymbol{R}$ belongs to the set of positive semidefinite matrices $\mathcal{H}_+^{n\times n}$. The authors in [8] show that if the matrix $\boldsymbol{R}$ is rank-deficient (more precisely, when rank d behaves like $\mathcal{O}(1)$ with respect to n), m-QP can be solved in polynomial-time and they propose a $\mathcal{O}((mn/2)^{2d})$-complexity algorithm to solve the problem. On the other hand, in general cases with no specific assumption on $\boldsymbol{R}$, one usually settles for approximation or local optimization algorithms. A well-known approximation approach to m-QP is semidefinite relaxation (SDR) which considers the following relaxed version of (1) (see [4] for details):

$$\begin{aligned} \max_{\boldsymbol{S}} \quad & \operatorname{tr}(\boldsymbol{R}\boldsymbol{S}) \\ \text{s.t.} \quad & \boldsymbol{S}(k,k) = 1,\ 1 \leq k \leq n, \\ & \boldsymbol{S}\text{: positive semidefinite.} \end{aligned} \tag{6}$$

If the solution $\boldsymbol{S}$ to the above is rank-one, with $\boldsymbol{s} \in \Omega_m^n$ such that $\boldsymbol{S} = \boldsymbol{s}\boldsymbol{s}^H$, then the relaxation has been tight and $\boldsymbol{s}$ is the solution to the original problem (1). Otherwise, a randomized procedure maps $\boldsymbol{S}$ to the space of m-ary signals in order to approximate $\boldsymbol{s}$ [1], [4]. In particular, it has been shown in [1], [12] that the expected value $v_e(SDR)$ of the quadratic objective in (1) associated with SDR randomized solutions satisfies

$$\frac{v_e(SDR)}{max_{\boldsymbol{s}\in\Omega_m^n} \boldsymbol{s}^H\boldsymbol{R}\boldsymbol{s}} \geq \gamma_{SDR} \triangleq \begin{cases} 2/\pi, & m = 2, \\ \frac{(m\sin(\pi/m))^2}{4\pi}, & m \geq 3. \end{cases} \tag{7}$$

The latter analytically derived sub-optimality guarantee has its own pros and cons: at the positive side, γ_{SDR} is *a priori* known and valid for all positive semidefinite $\boldsymbol{R}$. The drawback is, however, that the solutions obtained from SDR have been widely observed to possess considerably better quality compared to what is guaranteed by (7)—in fact, for some practical applications, rank-one SDR solutions are easily achievable (see e.g. [4], [14], [15] and the references therein). This is while it is evidenced that the *a priori* guarantees such as (7) may not be improvable due to worst-case scenarios. For example, for the continuous version of (1) (corresponding to $m \to \infty$) it is shown that the the quality of the SDR solution can be arbitrarily close to $\gamma_{SDR} = \pi/4$ [1].

In light of the above, in this paper, we propose a new approach by which for a given problem instance and corresponding solution to m-QP, one can calculate *a posteriori* case-dependent guarantees that might outperform (7).

II. Preliminaries: The Conic Structure

We begin with the following result originally shown in [5]:

Theorem 1. *Let $\mathcal{K}(\boldsymbol{s})$ represent the set of matrices $\boldsymbol{R}$ for which a given $\boldsymbol{s} \in \Omega_m^n$ is the global optimizer of m-QP. Then*

1) *$\mathcal{K}(\boldsymbol{s})$ is a convex cone.*
2) *For any two vectors $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \Omega_m^n$, the one-to-one mapping (where $\boldsymbol{s}_0 = \boldsymbol{s}_1^* \odot \boldsymbol{s}_2$)*
$$\boldsymbol{R} \in \mathcal{K}(\boldsymbol{s}_1) \Longleftrightarrow \boldsymbol{R} \odot (\boldsymbol{s}_0 \boldsymbol{s}_0^H) \in \mathcal{K}(\boldsymbol{s}_2) \tag{8}$$
holds among the matrices in $\mathcal{K}(\boldsymbol{s}_1)$ and $\mathcal{K}(\boldsymbol{s}_2)$.

Thanks to its conic structure, $\mathcal{K}(\boldsymbol{s})$ can be built based on its *prime elements*:

Definition 1. *We call a matrix $\boldsymbol{R}$ (with $\|\boldsymbol{R}\|_F = 1$) a **prime element** of $\mathcal{K}(\boldsymbol{s})$ if it cannot be written as a convex combination (i.e. a linear combination with non-negative weights) of the other elements of $\mathcal{K}(\boldsymbol{s})$. Moreover, we let $\mathcal{P}(\boldsymbol{s})$ denote the set of all prime elements of $\mathcal{K}(\boldsymbol{s})$.*

The prime elements of $\mathcal{K}(\boldsymbol{s})$ represent a specific subset of the *boundary* of the cone $\mathcal{K}(\boldsymbol{s})$:

Lemma 1. *Let $\boldsymbol{R} \in \mathcal{P}(\boldsymbol{s})$, and suppose $\boldsymbol{W} \in \mathbb{C}^{n\times n}$ is such that $\boldsymbol{R} - \boldsymbol{W} \in \mathcal{K}(\boldsymbol{s})$. Then $\boldsymbol{R} + \boldsymbol{W} \notin \mathcal{K}(\boldsymbol{s})$.*

Proof: If both $\boldsymbol{R} - \boldsymbol{W}$ and $\boldsymbol{R} + \boldsymbol{W}$ occur in $\mathcal{K}(\boldsymbol{s})$, then $\boldsymbol{R}$ can be written as
$$\boldsymbol{R} = \frac{1}{2}(\boldsymbol{R} - \boldsymbol{W}) + \frac{1}{2}(\boldsymbol{R} + \boldsymbol{W}) \tag{9}$$
which contradicts the primeness of $\boldsymbol{R}$. ■

Note that as $|\Omega_m^n|$ is finite, the n^2-dimensional volume of $\mathcal{K}(\boldsymbol{s})$ is non-zero[1], and hence $|\mathcal{P}(\boldsymbol{s})| \geq n^2$. The prime elements of $\mathcal{K}(\boldsymbol{s})$ have the following interesting properties (theorems 2 and 3):

Theorem 2. *For any $s \in \Omega_m^n$, $\mathcal{P}(\boldsymbol{s})$ can be obtained from any other $\mathcal{P}(\boldsymbol{s}')$ using the mapping in (8). In particular,*
$$\mathcal{P}(\boldsymbol{s}) = \{\boldsymbol{R} \odot (\boldsymbol{s}\boldsymbol{s}^H) : \ \boldsymbol{R} \in \mathcal{P}(\mathbf{1})\}. \tag{10}$$

Theorem 3. *Other than $\mathcal{K}(\boldsymbol{s})$, any $\boldsymbol{R} \in \mathcal{P}(\boldsymbol{s})$ is included in at least $n-1$ sets $\mathcal{K}(\boldsymbol{s}')$ (i.e. with $\boldsymbol{s}$ and all other $n-1$ vectors $\boldsymbol{s}' \in \Omega_m^n$ being distinct)[2].*

Most importantly, any element $\boldsymbol{R}$ in the convex cone $\mathcal{K}(\boldsymbol{s})$ can be written as a unique convex combination of the elements of $\mathcal{P}(\boldsymbol{s})$; more precisely, for any $\boldsymbol{R} \in \mathcal{K}(\boldsymbol{s})$ there exist unique and non-negative $\{\lambda_k\}$ such that
$$\boldsymbol{R} = \sum_{\boldsymbol{R}_k \in \mathcal{P}(\boldsymbol{s})} \lambda_k \boldsymbol{R}_k. \tag{11}$$

[1]Unlike $\mathbb{C}^{n\times n}$ whose elements can be characterized by $2n^2$ real-valued parameters, the linear space of Hermitian matrices in $\mathbb{C}^{n\times n}$ can be described by only n^2 independent real-valued parameters, and thus is n^2-dimensional.

[2]Due to invariance of the m-QP objective to the phase shifts of $\boldsymbol{s}$, we consider two vectors $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ from Ω_m^n *distinct*, if and only if $\boldsymbol{s}_1 \neq e^{j\frac{2\pi}{m}l}\boldsymbol{s}_2$, for all $0 \leq l \leq m-1$.

III. Computational Sub-Optimality Guarantees

Based on the above results, one may consider the following alternative of m-QP:
$$\min_{\boldsymbol{s},\ \lambda_k \geq 0} \left\| \boldsymbol{R} - \left(\sum_k \lambda_k \boldsymbol{R}_k \right) \odot (\boldsymbol{s}\boldsymbol{s}^H) \right\|_F \tag{12}$$
where all $\{\boldsymbol{R}_k\}$ belong to $\mathcal{K}(\mathbf{1})$. Note that if $\{\boldsymbol{R}_k\}$ include all elements of $\mathcal{P}(\mathbf{1})$, then the expression
$$\left(\sum_k \lambda_k \boldsymbol{R}_k \right) \odot (\boldsymbol{s}\boldsymbol{s}^H) \tag{13}$$
characterizes all the elements of $\mathcal{K}(\boldsymbol{s})$—otherwise, it can approximate $\mathcal{K}(\boldsymbol{s})$.

Definition 2. *We call a set $\{\boldsymbol{R}_k\}$, where all $\boldsymbol{R}_k \in \mathcal{K}(\boldsymbol{s})$ and $\|\boldsymbol{R}_k\|_F = 1$, a **basis bank** for $\mathcal{K}(\boldsymbol{s})$, if and only if $\{\boldsymbol{R}_k\}$ are **relatively prime**, i.e. they cannot be described as a convex combination of each other.*

Although constructing $\mathcal{K}(\boldsymbol{s})$ based on its prime elements would be optimal, yet determining whether an element of $\mathcal{K}(\boldsymbol{s})$ is prime appears to be difficult. Nevertheless, it is useful to observe that, to approximate $\mathcal{K}(\boldsymbol{s})$, the elements of $\{\boldsymbol{R}_k\}$ do not necessarily need to be prime. Indeed, the cone $\mathcal{K}(\boldsymbol{s})$ can be approximated well by a convex combination of several relatively prime elements (constituting a basis bank) on the boundary of $\mathcal{K}(\boldsymbol{s})$. This aspect is further studied in Section IV. We note again that m-QP is generally hard to solve. But if the m-QP solutions for several matrices are known, we might be able to use such information (that is indeed a valuable computational heritage) to tackle other m-QPs rather easily. Such a methodology requires considering (12) as is, with variable $\boldsymbol{s}$, which is an interesting problem that will be studied in a future publication. In this paper, we are particularly interested in using (12) when $\boldsymbol{s}$ is fixed. This is useful if the solution $\boldsymbol{s}$ is already approximated by another method such as SDR, and we are interested in bounding how close its cost is to the optimal cost. We note that m-QP basis banks can be designed/used in communication and active sensing systems in various ways, e.g.

- The device manufacturer can design an efficient m-QP basis bank as a part of the device startup package.
- The device can use its "spare" time or resources to design or upgrade such basis banks.
- The m-QP basis banks can be created or updated by the manufacturer as an after-sale service.

If $\boldsymbol{s}$ is given, then the objective of (12) becomes
$$\begin{aligned} f(\{\lambda_k\}) &\triangleq \left\| \boldsymbol{R} - \left(\sum_k \lambda_k \boldsymbol{R}_k \right) \odot (\boldsymbol{s}\boldsymbol{s}^H) \right\|_F \\ &= \left\| \boldsymbol{R} \odot (\boldsymbol{s}^* \boldsymbol{s}^{*H}) - \left(\sum_k \lambda_k \boldsymbol{R}_k \right) \right\|_F . \end{aligned} \tag{14}$$
Specifically, (14) is a non-negative least squares (NNLS) problem and is convex with respect to $\{\lambda_k\}$. Hence, the global

minimizer $\{\lambda_k\}$ of (14) can be obtained very efficiently (in polynomial-time).

We show that considering (12) in lieu of (1) lays the ground for a novel type of sub-optimality guarantees. Assume that $\{\lambda_k\}$ are already obtained, and let

$$\boldsymbol{E} \triangleq \boldsymbol{R} - \underbrace{\left(\sum_k \lambda_k \boldsymbol{R}_k\right) \odot (\boldsymbol{s}\boldsymbol{s}^H)}_{\boldsymbol{R}_{\boldsymbol{s}}}. \tag{15}$$

By construction, the global optimum of the m-QP associated with $\boldsymbol{R}_{\boldsymbol{s}}$ is $\boldsymbol{s}$. We have that

$$\begin{aligned} \max_{\boldsymbol{s}' \in \Omega_m^n} \boldsymbol{s}'^H \boldsymbol{R} \boldsymbol{s}' &\leq \max_{\boldsymbol{s}' \in \Omega_m^n} \boldsymbol{s}'^H \boldsymbol{R}_{\boldsymbol{s}} \boldsymbol{s}' + \max_{\boldsymbol{s}' \in \Omega_m^n} \boldsymbol{s}'^H \boldsymbol{E} \boldsymbol{s}' \qquad (16) \\ &\leq \max_{\boldsymbol{s}' \in \Omega_m^n} \boldsymbol{s}'^H \boldsymbol{R}_{\boldsymbol{s}} \boldsymbol{s}' + n\, \lambda_{\max}(\boldsymbol{E}) \\ &= \boldsymbol{s}^H \boldsymbol{R}_{\boldsymbol{s}} \boldsymbol{s} + n\, \lambda_{\max}(\boldsymbol{E}). \end{aligned}$$

Furthermore,

$$\begin{aligned} \max_{\boldsymbol{s}' \in \Omega_m^n} \boldsymbol{s}'^H \boldsymbol{R} \boldsymbol{s}' &\geq \max_{\boldsymbol{s}' \in \Omega_m^n} \boldsymbol{s}'^H \boldsymbol{R}_{\boldsymbol{s}} \boldsymbol{s}' + \min_{\boldsymbol{s}' \in \Omega_m^n} \boldsymbol{s}'^H \boldsymbol{E} \boldsymbol{s}' \qquad (17) \\ &\geq \max_{\boldsymbol{s}' \in \Omega_m^n} \boldsymbol{s}'^H \boldsymbol{R}_{\boldsymbol{s}} \boldsymbol{s}' + n\, \lambda_{\min}(\boldsymbol{E}) \\ &= \boldsymbol{s}^H \boldsymbol{R}_{\boldsymbol{s}} \boldsymbol{s} + n\, \lambda_{\min}(\boldsymbol{E}). \end{aligned}$$

As a result, an upper bound and a lower bound on the objective function for the global optimum of (1) can be obtained for any given $\boldsymbol{s}$. As to the sub-optimality guarantee, we obtain

$$\delta = \frac{\boldsymbol{s}^H \boldsymbol{R} \boldsymbol{s}}{\max_{\boldsymbol{s}' \in \Omega_m^n} \boldsymbol{s}'^H \boldsymbol{R}_{\boldsymbol{s}} \boldsymbol{s}'} \geq \gamma \tag{18}$$

where

$$\gamma \triangleq \frac{\boldsymbol{s}^H \boldsymbol{R} \boldsymbol{s}}{\boldsymbol{s}^H \boldsymbol{R}_{\boldsymbol{s}} \boldsymbol{s} + n\, \lambda_{\max}(\boldsymbol{E})} = \frac{\boldsymbol{s}^H \boldsymbol{R}_{\boldsymbol{s}} \boldsymbol{s} + \boldsymbol{s}^H \boldsymbol{E} \boldsymbol{s}}{\boldsymbol{s}^H \boldsymbol{R}_{\boldsymbol{s}} \boldsymbol{s} + n\, \lambda_{\max}(\boldsymbol{E})}. \tag{19}$$

Note that the quality of (19) depends on both problem instance and the basis bank. In fact, it is numerically observed that (i) in some cases, γ is actually smaller that γ_{SDR}, and (ii) we can usually achieve better sub-optimality guarantees than γ_{SDR}—more on this later.

IV. Cone Approximation Methodology

As indicated earlier, the cone $\mathcal{K}(\boldsymbol{s})$ can be approximated via a convex combination of several relatively prime elements lying at the boundary of $\mathcal{K}(\boldsymbol{s})$. It is worth mentioning that the sub-optimality guarantee and bounds derived above are applicable even if $\{\boldsymbol{R}_k\}$ are not prime. Moreover, according to Theorem 2 and the discussions afterward, we can focus on designing the basis bank of $\mathcal{K}(\boldsymbol{s})$ for solely one element $\boldsymbol{s}$ of Ω_m^n; a trivial choice would be $\boldsymbol{s} = \mathbf{1}$.

A. Basis Bank Design

A basis bank $\mathcal{B}$ for m-QP can be designed in a blind way. Suppose the communication or active sensing system solves m-QP for any $\boldsymbol{R} \in \mathcal{H}_+^{n\times n}$, leading to a solution $\boldsymbol{s} = \boldsymbol{s}_\star$. Then according to the one-to-one mapping in (8), the matrix

$$\boldsymbol{R} \odot (\boldsymbol{s}_\star^* \boldsymbol{s}_\star^{*\,H}) \tag{20}$$

can be added to the matrix bank with an associated m-QP solution $\boldsymbol{s} = \mathbf{1}$. On the contrary, one can employ a constructive approach to build $\mathcal{B}$. To describe our constructive approach in the following, we first observe that the function $g_{\boldsymbol{s}}(\boldsymbol{R}) = \boldsymbol{s}^H \boldsymbol{R} \boldsymbol{s}$ is symmetric around the symmetry axis $\boldsymbol{s}\boldsymbol{s}^H \in \mathcal{K}(\boldsymbol{s})$:

Lemma 2. *Let $\overline{\boldsymbol{R}}$ be the image of $\boldsymbol{R}$ with respect to $\boldsymbol{s}\boldsymbol{s}^H$ (for some $\boldsymbol{s} \in \Omega_m^n$). Then $g_{\boldsymbol{s}}(\overline{\boldsymbol{R}}) = g_{\boldsymbol{s}}(\boldsymbol{R})$.*

Moreover, for any given $\widetilde{\boldsymbol{R}}$ and sufficiently small λ, we have that

$$\boldsymbol{R} = \boldsymbol{s}\boldsymbol{s}^H + \lambda \widetilde{\boldsymbol{R}} \in \mathcal{K}(\boldsymbol{s}). \tag{21}$$

Therefore, a natural way to approximate the cone $\mathcal{K}(\boldsymbol{s})$ is via a convex combination of matrices $\boldsymbol{R}$ formulated as in (21). However, an efficient approximation of $\mathcal{K}(\boldsymbol{s})$ is possible only if λ of (21) is maximized; in which case (21) represents a matrix $\boldsymbol{R}$ on the boundary of $\mathcal{K}(\boldsymbol{s})$ (assuming $\widetilde{\boldsymbol{R}} \notin \mathcal{K}(\boldsymbol{s})$).

In order to efficiently construct $\mathcal{B}$, we consider the matrices obtained from the formula

$$\boldsymbol{R} = \boldsymbol{s}\boldsymbol{s}^H + \lambda \boldsymbol{R}_\perp \tag{22}$$

where $\boldsymbol{r}_\perp = vec(\boldsymbol{R}_\perp)$ is orthogonal to $\boldsymbol{s}_{vec} = vec(\boldsymbol{s}\boldsymbol{s}^H)$ (which is equivalent to $\boldsymbol{s}^H \boldsymbol{R}_\perp \boldsymbol{s} = 0$), and $\lambda \geq 0$. Note that $\boldsymbol{s}^H \boldsymbol{R}_\perp \boldsymbol{s} = 0$ if and only if $\boldsymbol{s} \in \ker(\boldsymbol{R}_\perp)$. Therefore, the matrices $\boldsymbol{R}_\perp$ with the property $\boldsymbol{s}^H \boldsymbol{R}_\perp \boldsymbol{s} = 0$ can be characterized (via an eigenvalue decomposition structure) as in (23) where $\boldsymbol{U}$ is a semi-unitary matrix spanning the $(n-1)$-dimensional space orthogonal to $\boldsymbol{s}$ (obtained efficiently via the Gram-Schmidt process), and $\boldsymbol{D}$ is a diagonal real-valued matrix that may be considered as the design variable. The diagonal matrix $\boldsymbol{D}$ can be chosen in different ways:

- <u>*Computationally:*</u>

  We choose $\boldsymbol{D}$ randomly, with the condition that its diagonal entries should not be all negative (as then $\boldsymbol{R}_\perp$ occurs in $\mathcal{K}(\boldsymbol{s})$).

- <u>*Analytically:*</u>

  To ensure maximum efficiency in designing $\mathcal{B}$, we may employ a diverse set of angles for spinning off from $\boldsymbol{s}\boldsymbol{s}^H$. Examples of such geometrical structures are studied in the literature (see e.g. regular simplex in [16]). Herein, we propose the following simple and symmetric matrix sets to build $\boldsymbol{B}$. Let

$$\begin{aligned} \mathcal{D}_1 &= \{\boldsymbol{D}:\ \boldsymbol{D} = \mathbf{Diag}(\boldsymbol{e}_{k_1})\} \qquad (24) \\ \mathcal{D}_2 &= \{\boldsymbol{D}:\ \boldsymbol{D} = \mathbf{Diag}(\boldsymbol{e}_{k_1} \pm \boldsymbol{e}_{k_2})\} \\ &\vdots \\ \mathcal{D}_t &= \{\boldsymbol{D}:\ \boldsymbol{D} = \mathbf{Diag}(\boldsymbol{e}_{k_1} \pm \boldsymbol{e}_{k_2} \cdots \pm \boldsymbol{e}_{k_t})\} \end{aligned}$$

  where $t \leq n-1$. Note that $|\mathcal{D}_l| = 2^{l-1}\binom{n-1}{l}$ for $1 \leq l \leq t$, which implies that for $t = \mathcal{O}(1)$,

$$\left| \bigcup_{l=1}^{t} \mathcal{D}_l \right| \tag{25}$$

$$\boldsymbol{R}_{\perp}=\left(\begin{array}{cc}\boldsymbol{U}_{n\times(n-1)} & \boldsymbol{s}/\sqrt{n}\end{array}\right)\left(\begin{array}{cc}\boldsymbol{D}_{(n-1)\times(n-1)} & \boldsymbol{0}_{(n-1)\times 1}\\ \boldsymbol{0}_{1\times(n-1)} & 0\end{array}\right)\left(\begin{array}{cc}\boldsymbol{U}_{n\times(n-1)} & \boldsymbol{s}/\sqrt{n}\end{array}\right)^{H}. \tag{23}$$

behaves as $\mathcal{O}(n^t)$.

Next, we calculate the maximal λ of (22), denoted by $\lambda_\star$. In particular, we seek to maximize λ subject to the constraint:

$$\begin{aligned}\boldsymbol{s}^H\boldsymbol{R}\boldsymbol{s} &= n^2+\lambda\,\boldsymbol{s}^H\boldsymbol{R}_\perp\boldsymbol{s} \\ &\geq |\boldsymbol{s}'^H\boldsymbol{s}|^2+\lambda\,\boldsymbol{s}'^H\boldsymbol{R}_\perp\boldsymbol{s}'^H=\boldsymbol{s}'^H\boldsymbol{R}\boldsymbol{s}'^H\end{aligned} \tag{26}$$

for all $\boldsymbol{s}'\in\Omega_m^n\backslash\left\{e^{j2\pi l/m}\boldsymbol{s}\right\}$. Let

$$\xi=\{\boldsymbol{s}'\in\Omega_m^n:\ \boldsymbol{s}'^H\boldsymbol{R}_\perp\boldsymbol{s}'^H>\boldsymbol{s}^H\boldsymbol{R}_\perp\boldsymbol{s}\}. \tag{27}$$

Then, it follows from (26) that

$$\lambda_\star=\min_{\boldsymbol{s}'\in\xi}\left(\frac{n^2-|\boldsymbol{s}'^H\boldsymbol{s}|^2}{\boldsymbol{s}'^H\boldsymbol{R}_\perp\boldsymbol{s}'^H-\boldsymbol{s}^H\boldsymbol{R}_\perp\boldsymbol{s}}\right). \tag{28}$$

The candidate basis to be added to $\mathcal{B}$ thus becomes

$$\boldsymbol{R}_\star=\frac{\boldsymbol{s}\boldsymbol{s}^H+\lambda_\star\boldsymbol{R}_\perp}{\|\boldsymbol{s}\boldsymbol{s}^H+\lambda_\star\boldsymbol{R}_\perp\|_F}. \tag{29}$$

Ultimately, the addition of $\boldsymbol{R}_\star$ to $\mathcal{B}$ will be done if it passes a final step, i.e. if it cannot be represented as a convex combination of the current elements of $\mathcal{B}$.

B. How Good is a Basis Bank Design?

In the following, we address the latter question by discussing three different (although related) type of measures.

1) Induced Sub-optimality Guarantees (γ)

A practical approach to assess the quality of a given basis bank ($\mathcal{B}$) would be to compute the m-QP sub-optimality guarantees associated with $\mathcal{B}$ for various real-world or random matrices $\boldsymbol{R}$.

We provide an example of such a quality investigation for the analytical/computational basis bank designs proposed in IV-A with $(n,m)=(10,3)$. In the analytical case, we have considered basis banks designed by employing $\boldsymbol{D}\in\cup_{l=1}^{t}\mathcal{D}_l$. For all t, the same number of basis matrices were generated via the alternative computational approach. Random matrices $\boldsymbol{R}\in\mathcal{H}_+^{n\times n}$ were generated using the formula $\boldsymbol{R}=\boldsymbol{Q}\boldsymbol{Q}^H$ where $\boldsymbol{Q}\in\mathbb{C}^n$ is a random matrix whose real-part and imaginary-part elements are i.i.d. with a standard Gaussian distribution $\mathcal{N}(0,1)$. The solutions $\boldsymbol{s}$ to the related m-QPs were approximated by SDR (with 30 randomizations [4]). Moreover, the obtained values of γ were averaged over 30 realizations of $\boldsymbol{R}$. The results are shown in Fig. 1. Note that γ can be smaller than γ_{SDR} as one can observe in the computational case for $t=2$. Nevertheless, it appears that, for larger cardinalities of the basis bank, γ can surpass γ_{SDR}. In addition, a generally growing γ with the cardinality of the basis bank is interesting, and somewhat expected.

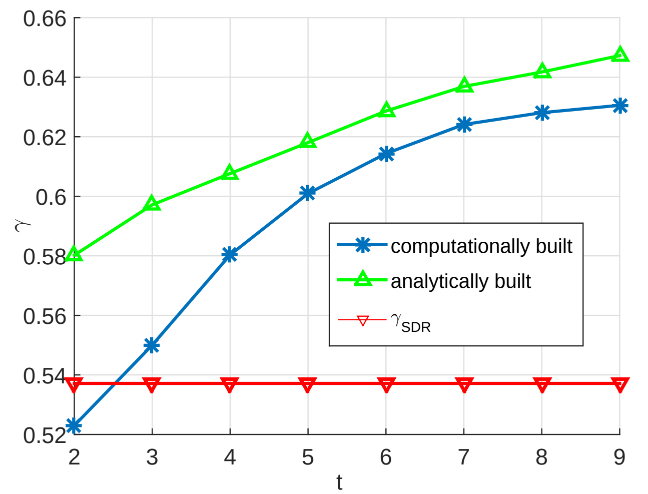

Fig. 1. Comparison of sub-optimality guarantees obtained by constructing the basis banks using the computational and analytical approaches proposed in Section IV, and that of SDR ($n=10$, $m=3$). The obtained values of γ are averaged over 30 realizations of $\boldsymbol{R}$.

2) How Much of $\mathcal{K}(\boldsymbol{s})$ is Spanned by $\mathcal{B}$?

An interesting way to measure the goodness of $\mathcal{B}$ would be to investigate which ratio of $\mathcal{K}(\boldsymbol{s})$ is spanned (or *covered*) by the cone associated with $\mathcal{B}$. This in fact represents the probability of achieving $\gamma=1$, provided that the global solution of m-QP is given. Moreover, from an intuitive point of view, a larger coverage of $\mathcal{K}(\boldsymbol{s})$ by $\mathcal{B}$ would lead to a smaller (14) and generally larger γ values.

As the volume, a key tool in our analysis, is well-defined in the real field, we resort to a transformation of the complex variables to their real-valued counterparts. More concretely, we define the operator $\mathcal{M}_{\mathcal{H}^{n\times n}\to\mathbb{R}^{n^2\times 1}}(\boldsymbol{X}_\mathcal{H})$ whose n^2-length vector output comprises the independent parameters of the Hermitian matrix argument $\boldsymbol{X}_\mathcal{H}$, namely

$$\{\Re\{[\boldsymbol{X}_\mathcal{H}]_{k,l}\}\}_{k\leq l}\bigcup\{\Im\{[\boldsymbol{X}_\mathcal{H}]_{k,l}\}\}_{k<l}, \tag{30}$$

characterizing the linear space of complex Hermitian matrices $\mathcal{H}^{n\times n}$. We consider the unit-radius n^2-ball defined as

$$\boldsymbol{O}_\mathbb{R}=\{\boldsymbol{x}\in\mathbb{R}^{n^2}:\ \|\boldsymbol{x}_\mathbb{R}\|_2\leq 1\}. \tag{31}$$

We also let $\boldsymbol{B}_\mathbb{R}$ be a matrix whose columns comprise the vectorized versions of $\mathcal{M}_{\mathcal{H}^{n\times n}\to\mathbb{R}^{n^2\times 1}}(\boldsymbol{B}_\mathcal{H})$ where $\{\boldsymbol{B}_\mathcal{H}\}$ are the basis matrices in $\mathcal{B}$, and define

$$\widetilde{\mathcal{K}}(\boldsymbol{s})=\mathcal{M}_{\mathcal{H}^{n\times n}\to\mathbb{R}^{n^2\times 1}}(\mathcal{K}(\boldsymbol{s})\cap\boldsymbol{O}_\mathbb{R}). \tag{32}$$

As a result, the *probability* or *coverage factor* suggested earlier can be formulated as

$$P_g \quad \triangleq \quad \frac{vol(\mathtt{cone}(\boldsymbol{B}_{\mathbb{R}}) \cap \boldsymbol{O}_{\mathbb{R}})}{vol(\widetilde{\mathcal{K}}(\boldsymbol{s}))} \tag{33}$$

where $vol(.)$ is the volume or the *Lebesgue measure* of $\boldsymbol{R}^{n^2}$, and $\mathtt{cone}(.)$ denotes the cone generated by non-negative combinations of the columns of the matrix argument. We note that finding the volumes associated with convex cones is typically deemed to be very difficult unless for some *simplicial* cones [17], [18]. Several analytical and computational approaches are studied in [17]. In general scenarios, a random vector generation scheme may be used to estimate the cone volumes, for which the random sample must be *huge* (see [17] for details). In what follows, we show that at least the denominator of (33) can be computed analytically, according to the following result:

Theorem 4. *For any integer $t > 1$, and distinct $\boldsymbol{s}_{l_1}, \boldsymbol{s}_{l_2}, \cdots, \boldsymbol{s}_{l_t} \in \Omega_m^n$,*

$$vol\left(\widetilde{\mathcal{K}}(\boldsymbol{s}_{l_1}) \cap \widetilde{\mathcal{K}}(\boldsymbol{s}_{l_2}) \cap \cdots \cap \widetilde{\mathcal{K}}(\boldsymbol{s}_{l_t})\right) = 0. \tag{34}$$

Note that, based on Theorem 4, the n^2-dimensional volume of $\widetilde{\mathcal{K}}(\boldsymbol{s})$ is directly given by dividing the volume of $\boldsymbol{O}_{\mathbb{R}}$ by the number of distinct elements of Ω_m^n, viz.

$$vol(\widetilde{\mathcal{K}}(\boldsymbol{s})) = \left(\frac{\pi^{\frac{n^2}{2}}}{\Gamma(\frac{n^2}{2}+1)}\right) / m^{n-1}. \tag{35}$$

3) How Does a New Basis Contributes to the Basis Bank?

Answering this question is beneficial, in particular to see when we can stop adding new candidates to $\mathcal{B}$ without considerably degrading the obtained guarantees. A possible approach to determine the contribution of a new basis to the basis bank would be to calculate the value at the global minimum of the criterion:

$$\left\| \boldsymbol{R}_{new} - \left(\sum_k \lambda_k \boldsymbol{R}_k\right) \right\|_F \tag{36}$$

for $\{\lambda_k \geq 0\}$, where $\boldsymbol{R}_{new}$ denotes the new basis candidate to be added to $\mathcal{B}$. Clearly, the larger the criterion in (36), the more beneficial adding $\boldsymbol{R}_{new}$ to $\mathcal{B}$ become. On the contrary, if (36) is zero, then $\boldsymbol{R}_{new}$ can already be described by the current elements of $\mathcal{B}$ and adding it to $\mathcal{B}$ does not lead to any improvement in terms of guarantees.

An alternative approach to the above, is to consider the volume of the simplex built by the basis matrices in $\mathcal{B}$ and the origin:

$$\mathcal{S} = \left\{ c_1 \boldsymbol{b}_1 + \cdots + c_l \boldsymbol{b}_h : \sum_{l=1}^{h} c_l \leq 1; c_l \geq 0, \forall\, l \right\} \tag{37}$$

where $\boldsymbol{b}_l$ denotes the l^{th} column of $\boldsymbol{B}_{\mathbb{R}}$, with maximum column index $h = |\mathcal{B}|$. We have that

$$vol(\mathcal{S}) = \frac{1}{(n^2)!} \sqrt{det(\boldsymbol{B}_{\mathbb{R}} \boldsymbol{B}_{\mathbb{R}}^T)}. \tag{38}$$

Consequently, the contribution of a new basis can be measured by the resulting difference in $vol(\mathcal{S})$.

V. Conclusion

A novel methodology was proposed to derive data-dependent sub-optimality guarantees for approximate solutions to quadratic optimization (over the m-ary constellation). It was shown that the new guarantees might outperform the *a priori* known SDR guarantees, and various aspects related to deriving the new guarantees were discussed.

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