



Abstract

Perfect phase-quantized unimodular sequences with entries in $\{x \in \mathbb{C} \mid x^m = 1\}$ have optimal peak-to-average-power ratio (PAR); however, they are extremely rare. For active sensing or communication systems which are able to tolerate sub-optimal PAR values, we show how to construct phase-quantized sequences possessing both virtually perfect periodic autocorrelation and low PAR. Numerical examples are provided to illustrate the performance of the proposed methods.

Introduction

The periodic autocorrelation of \mathbf{x} is defined as

$$R_u = \sum_{l=0}^{n-1} x_l x_{l+u}^*, \quad 0 \leq u < n$$

The sequence \mathbf{x} is called *perfect* iff

$$R_u = \begin{cases} E & u \equiv 0 \pmod{n}, \\ 0 & \text{otherwise} \end{cases} \quad \left| \begin{array}{l} \text{ISL} \triangleq \sum_{u=1}^{n-1} |R_u|^2 \\ \text{PSL} \triangleq \max\{|R_u|\}_{u=1}^{n-1} \end{array} \right.$$

The PAR metric is defined as

$$\text{PAR} \triangleq \frac{\max_l |x_l|^2}{\frac{1}{n} \sum_{l=0}^{n-1} |x_l|^2} \rightarrow \begin{cases} \text{unimodular sequences } x_l = e^{j\phi_l} \\ \text{phase-quantized sequences } x_l = e^{j\frac{2\pi}{m}k_l} \end{cases}$$

The key contributions of the proposed methods are:

(i) they allow small alphabet sizes $2 \leq m \ll n$, (ii) they do not restrict the length or alphabet sizes (in contrast to almost all perfect sequence construction methods), and (iii) they provide many phase-quantized sequences therefore circumventing the rareness dilemma of perfect phase-quantized unimodular sequences.

Phase Selection

A phase-quantized sequence is of the form $\mathbf{x} = \{\alpha_l e^{j\frac{2\pi}{m}k_l}\}_{l=0}^{n-1}$ where $0 \leq k_l \leq m-1$ and $\{\alpha_l\}$ are non-negative real numbers.

$$\text{PAR} = \max\{\alpha_l^2\}_{l=0}^{n-1}$$

Clearly, the PAR metric attains lower values when $\alpha = \{\alpha_l\}$ is closer to $\mathbf{1}$ (in the sense of l_∞ -norm).

PECAN method

a perfect unimodular sequence

$$\mathbf{x}^u = \{e^{j\phi_l}\}_{l=0}^{n-1}$$

m -level phase-quantized version of \mathbf{x}^u

$$k_l \equiv \left[m \left(\frac{\phi_l}{2\pi} \right) \right] \pmod{m}$$

In the following, we discuss the design of α for the integer phases $\{k_l\}$.

The Proposed methods

The modified PECAN

Generate a perfect unimodular sequence and consider the m -level phase-quantized version of it.

Compute $\{\psi_l\}$
 $\psi_l = \arg(X_l), \quad 0 \leq l < n$
where $\{X_l\}$ denotes the discrete Fourier transform (DFT) of $\{x_l\}$

Repeat the cyclic minimization until a stop criterion is satisfied.

Compute $\{\alpha_l\}$

$$\alpha_l = \begin{cases} \beta_l, & \beta_l \in [1 - \varepsilon_1, 1 + \varepsilon_2] \\ 1 + \varepsilon_2, & \beta_l > 1 + \varepsilon_2 \\ 1 - \varepsilon_1, & \beta_l < 1 - \varepsilon_1 \end{cases}$$

$$\beta_l = |z_l| \cos\left(\arg(z_l) - \frac{2\pi}{m}k_l\right), \quad 0 \leq l < n$$

where $\{z_l\}$ represents the Inverse DFT of $\{e^{j\psi_l}\}$

Alternating Projections

Generate a perfect unimodular sequence and consider the m -level phase-quantized version of it.

$$\mathbf{A} = \alpha\alpha^T$$

Compute the optimal projection on Γ

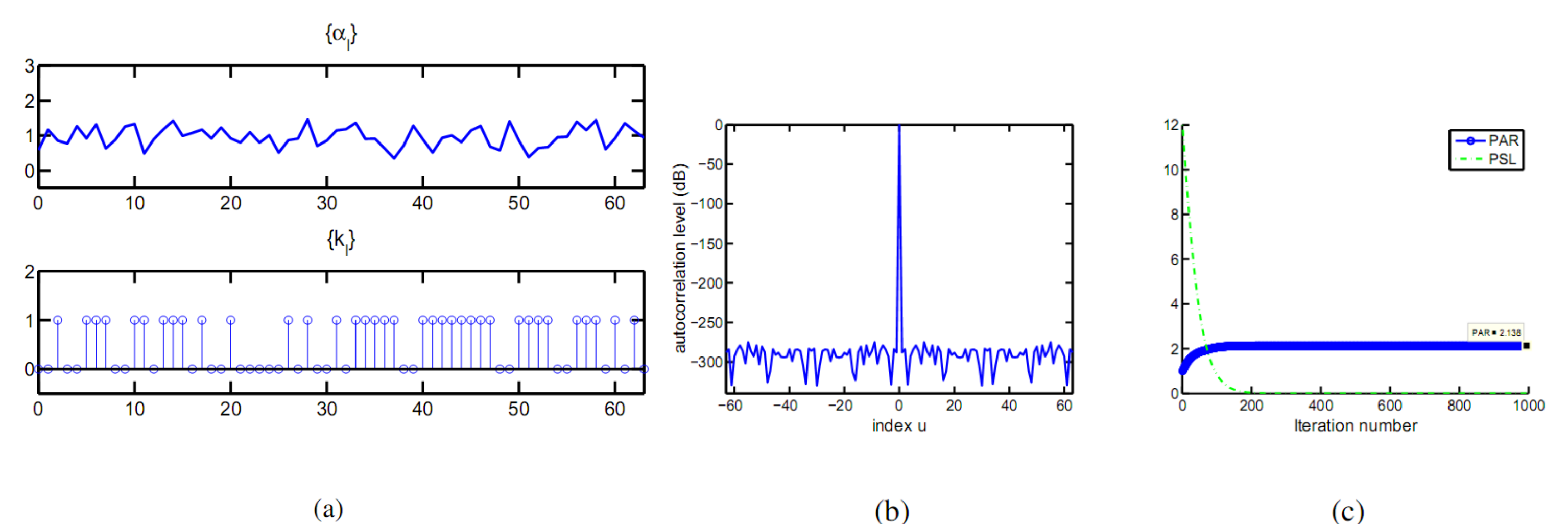
$$\min_{\mathbf{A}_\perp \in \Gamma} \|\mathbf{A} - \mathbf{A}_\perp\|_F^2$$

Compute the optimal projection on Λ

$$\min_{\mathbf{A}_\perp \in \Lambda} \|\mathbf{A} - \mathbf{A}_\perp\|_F^2$$

Repeat the projections.

Numerical Examples



$n \backslash m$	2	4	8	16	32	64
10	1.842	1.386	1.085	1.085	1.022	1.022
	1.842	1.250	1.250	1.022	1.022	1.022
	1.842	1.842	1.085	1.022	1.022	1.022
25	1.904	1.904	1.904	1.895	1.290	1.104
	1.992	1.992	1.992	1.610	1.289	1.245
	1.960	1.960	1.960	1.717	1.447	1.142
50	1.999	1.999	1.999	1.999	1.507	1.382
	1.999	1.999	1.999	1.999	1.655	1.350
	2.250	2.250	2.250	2.250	1.600	1.290
100	2.907	2.748	2.739	2.739	2.324	1.878
	3.095	3.095	3.095	2.097	2.027	1.515
	2.250	2.250	2.250	2.250	1.844	1.444

Fig. 1. Design of a phase-quantized sequence of length $n = 64$ with low PAR via the alternating projection method. The phase quantization level is $m = 2$. (a) the sequence amplitudes $\{\alpha_l\}$ and integer phases $\{k_l\}$. (b) the autocorrelation levels (in dB) of the obtained sequence. (c) the PSL and PAR vs. the iteration number for the obtained sequence. The sequence achieves practically perfect periodic correlation properties and $\text{PAR} = 2.138$.

Table 3. The lowest PAR values obtained for different lengths (n) and alphabet sizes (m) by running the two proposed methods 50 times; alternating projections (see the first row for each length) and the extended PECAN method with $\varepsilon_1 = 1$ and $\varepsilon_2 = \infty$ (see the second row for each length). The third row for each length presents the results of running the extended PECAN method 10 times for the same values of (n, m) and $\varepsilon_1 = \varepsilon_2 = \varepsilon \in \{0, 0.05, 0.1, 0.15, 0.3, 0.4, 0.5, 0.7, 0.9\}$. For each length/alphabet size, the bold font is used to indicate the lowest obtained PAR using the proposed methods.