ONE DIMENSIONAL TDM SIGNAL RECONSTRUCTION FROM ITS FOURIER TRANSFORM PHASE OR MAGNITUDE

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ABSTRACT

Unlike multidimensional signals, reconstruction of one dimensional signals from their Fourier phase or magnitude is faced with an inherent dilemma. In this paper, the reconstruction of 1D Time Division Multiplexing (TDM) signals (which are sparse in time) is discussed. We will show that such signals can be uniquely reconstructed from their Fourier phase; on the other hand, reconstruction from the Fourier magnitude results in a class of signals. For reconstruction of the mentioned signals, we propose a method followed by a modification inspired by the Hayes uniqueness theorems for recovery of the 2D signals from their Fourier phase or magnitude. In contrast to the previous works which consider criterion in frequency-domain, this work aims to recover the signals using the knowledge of the non-zero sample locations in time-domain. Simulation results confirm the performance of the proposed methods.

Index Terms— TDM signal, signal reconstruction, Fourier transform phase or magnitude.

1. INTRODUCTION

In the past couple of decades, reconstruction of the signals, solely from the phase or magnitude of their Fourier transform, has been the topic of extensive research works. Reconstruction from the magnitude (sometimes referred as phase retrieval) is required in a number of fields including optical astronomy, electron microscopy, crystallography and wavefront sensing. Also applications in antenna and filter design are available [1]. Importance of the reconstruction from the phase (known as magnitude retrieval) is revealed in the blind deconvolution of a mixture of signals in order to extract a desired signal [2]. Other applications are addressed in [3].

In 1982, Hayes Showed that multidimensional signals are reconstructable using either the phase or the magnitude of their Fourier transform [4]. He concluded that the reconstruction from the magnitude would result in a unique equivalence class of signals which we will review in section 3. In addition, besides a constant scaling factor, reconstruction from the phase results in a unique signal. Unfortunately, these results are not valid for one dimensional signals; i.e., samples of Fourier phase or magnitude do not produce a unique 1D signal (or an equivalence class of 1D signals). Nevertheless, a number of methods, whether direct or iterative, are proposed for reconstruction of multidimensional signals [1, 3].

Since only the samples of the Fourier transform are available in practice, the quality of the reconstructed signal, highly depends on the precision of the samples. Furthermore, it is shown that the problem is ill-conditioned for the large sequences [3]. Due to the lack of an almost one to one mapping between the 1D signals and their Fourier phase or magnitude, extra constraints (in addition to the phase or magnitude information) are usually considered to uniquely point out the signal. Common constraints are zero, minimum or maximum phase criterion [5]. In this paper, unlike the previous common constraints considered in the frequency-domain, we focus on the characteristics in the time domain; i.e., we aim to recover the signals with the knowledge of their non-zero sample locations in time-domain.

As well as the previous applications, the problem of blind deconvolution of multi-channel communication signals is sometimes solved by considering the Fourier phase or magnitude of the mixture [6]. In this paper, theoretic aspects of the reconstruction of a TDM signal from its Fourier phase or magnitude is discussed; TDM signals are the transport elements in most of the communication networks and their sparsity is due to the divided access of the users to the resources in time. The mentioned reconstruction problem arises in cases where the magnitude or the phase of the Fourier transform of the signal is either lost or impractical to measure. We employ the time sparsity of the mentioned class (TDM) of signals to overcome the inherent reconstruction problem of 1D signals from the phase or the magnitude of the Fourier transform.

This work has two major parts: In the first part, a transformation converting the 1D to the 2D Discrete Fourier Transform (DFT) is introduced and their relationship is investigated. In the second part, the primary objective is to use this

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relationship to reconstruct the 1D TDM signal. Analysis of the 1D TDM signal recovery is followed by proposing a rough direct and an accurate iterative reconstruction algorithm.

2. TRANSFORMATION BETWEEN 1D AND 2D DFTS

In order to relate the 1-D DFT to the 2-D DFT we define a transformation that maps a vector to a matrix. That is, for a given signal $x[n]$ of length N and a factorization of N such as $N = R \times C$, we define the $R \times C$ 2D signal corresponding to $x[n]$ by $x_2[k_1, k_2] = x[k_1 C + k_2]$, where $0 \leq k_1 \leq R-1$ and $0 \leq k_2 \leq C-1$. Also we denote the linear transformation converting the 1D signal into the 2D form by $T_{R,C}(.)$. In this paper, we represent the Fourier transform of the signals by uppercase letters; for instance, X_2 represents the Fourier transform of x_2:

$$\begin{aligned} X_2[p,q] &= \sum_{k_1=0}^{R-1}\sum_{k_2=0}^{C-1} e^{-jk_1\omega_1 p} e^{-jk_2\omega_2 q} x_2[k_1,k_2] \\ &= \sum_{k_1=0}^{R-1} e^{-jk_1\omega_1 p} \sum_{k_2=0}^{C-1} e^{-jk_2\omega_2 q} x_2[k_1,k_2] \\ &= \mathcal{F}_{y(p)}\{\mathcal{F}_{x(q)}\{x_2\}\} \end{aligned} \tag{1}$$

where $w_1 = \frac{2\pi}{R}$, $w_2 = \frac{2\pi}{C}$ and $\mathcal{F}_x$ and $\mathcal{F}_y$ are horizontal and vertical DFT operators, respectively.
In addition, X (the one dimensional DFT of x) is:

$$\begin{aligned} X[n] &= \sum_{k=0}^{N-1} x[k] e^{-jkw_0 n} \\ &= \sum_{r_0=0}^{R-1} \Big(\sum_{k=r_0 C}^{(r_0+1)C-1} x[k] e^{-jkw_0 n} \Big) \end{aligned} \tag{2}$$

Let M_x be the matrix obtained from x_2 by taking the DFT of its rows ($\mathcal{F}_x\{x_2\}$). Using the mentioned linear transformation ($T_{R,C}$), the relationship between X and M_x is given by:

$$\begin{aligned} X[n] &= \sum_{r_0=0}^{R-1} \Big(\sum_{k=0}^{C-1} x_2[r_0,k] e^{-jkw_2(\frac{n}{R})} \Big) e^{-jr_0 C w_0 n} \\ &= \sum_{r_0=0}^{R-1} M_x[r_0, \frac{n}{R}] e^{-jr_0 w_1 n} \end{aligned} \tag{3}$$

Assuming $n = Rq + p$ with $0 \leq p < R$, we have:

$$\begin{aligned} X[n] &= \sum_{r_0=0}^{R-1} M_x[r_0, q + \frac{p}{R}] e^{-jr_0 w_1 p} \\ &= \mathcal{F}_{y(p)}\{\mathcal{F}_{x(q+\frac{p}{R})}\{x_2\}\} \end{aligned} \tag{4}$$

The above equation indicates a non-uniform sampling over the continuous Fourier transform of x_2. From the basic equations of an m point DFT

$$\begin{aligned} X[k] &= X(e^{jw})|_{w=\frac{2\pi k}{m}} \\ X(e^{jw}) &= \frac{1-e^{-jwm}}{m} \cdot \sum_{k=0}^{m-1} \frac{X[k]}{1-e^{-jw} e^{j\frac{2\pi k}{m}}} \end{aligned} \tag{5}$$

Applying equations in (5) to all the rows of the matrix M_x, we conclude:

$$\mathcal{F}_{x(\frac{n}{R})} = \sum_{k=0}^{C-1} \frac{1-e^{-j2\pi\frac{n}{R}}}{C(1-e^{-j\frac{2\pi n}{N}} e^{j\frac{2\pi k}{C}})} \cdot \begin{pmatrix} M_x[0,k] \\ M_x[1,k] \\ \vdots \\ M_x[R-1,k] \end{pmatrix} \tag{6}$$

This yields

$$X[n] = \frac{1-e^{-j2\pi(q+\frac{p}{R})}}{C} \cdot \sum_{k=0}^{C-1} \frac{X_2[p,k]}{1-e^{-j\frac{2\pi n}{N}} e^{j\frac{2\pi k}{C}}} \tag{7}$$

Therefore, for every $0 \leq p < R$, the following linear system could be defined:

$$\begin{pmatrix} X[p] \\ X[R+p] \\ \vdots \\ X[R(C-1)+p] \end{pmatrix} = A_{(p)\ C\times C} \cdot \begin{pmatrix} X_2[p,0] \\ X_2[p,1] \\ \vdots \\ X_2[p,C-1] \end{pmatrix} \tag{8}$$

where $A_{(p)\ k,l} = \frac{1}{C} \cdot \frac{1-e^{-j2\pi(k+\frac{p}{R})}}{1-e^{-j2\pi(\frac{kR+p}{RC})} e^{-j2\pi(\frac{l}{C})}}$.

The above equality provides us with the possibility to directly obtain X from X_2. Now we show the invertibility of the above transformation. Consider the following sequential process flow:

$$X \longrightarrow x \longrightarrow x_2 \longrightarrow X_2 \tag{9}$$

The transformation of X to x is linear. Therefore, since $x_2[k_1,k_2] = x[k_1 C + k_2]$, the value of $x_2[k_1,k_2]$ could be represented linearly by $X[n]$:

$$\exists\ e_{k_1,k_2,n} :\ x_2[k_1,k_2] = \sum_{n=0}^{N-1} e_{k_1,k_2,n} X[n] \tag{10}$$

Suppose $\widehat{X}_2 = T^{-1}_{R,C}(X_2)$, $\widehat{x}_2 = T^{-1}_{R,C}(x_2)$ and $\widehat{X} = T^{-1}_{C,R}\big(T_{R,C}(X)^T\big)$. Due to the definition of the 2D DFT we have:

$$\exists\ R_{N\times N} :\ \hat{X}_2 = R.\hat{x}_2 \tag{11}$$

Therefore, (10) implies the existence of a square matrix $Q_{N\times N}$ that relates $\hat{X}_2$ and $\hat{X}$ by the equation $\hat{X}_2 = Q \cdot \hat{X}$. On the other hand, from the linear system in (8), we conclude that there exists a square matrix P that satisfies the equation $\hat{X} = P \cdot \hat{X}_2$ where $P_{N\times N}$ is formed by diagonal arrangement of the matrices $A_{(p)}$. Since $Q = P^{-1}$ exists, all matrices $A_{(p)}$ are invertible.

3. 1D TDM SIGNAL RECONSTRUCTION

Knowing the phase or magnitude of the Fourier transform of a 1D signal, we wish to reconstruct the original signal. To benefit from the existing reconstruction algorithms for 2D signals we use the transformation discussed in the previous section to transform this 1D signal to a 2D signal. To accomplish 1D reconstruction we will first review the theoretical background for 2D reconstruction of the signals and then describe our idea.

Suppose we want to reconstruct a $r_x \times c_x$ matrix x'_2 from the magnitude or phase of a $R \times C$ DFT-domain matrix $\langle X_2 \rangle$. Under the following conditions, Hayes theorems guarantee the uniqueness of the reconstructed signal:

$$\begin{cases} R \geq 2r_x - 1 \\ C \geq 2c_x - 1 \end{cases} \tag{12}$$

As discussed before, reconstruction from Fourier phase results in a unique signal while reconstruction from the magnitude results in a unique equivalence class of signals. This equivalence class is defined as follows:

$$y[n] \equiv x[n] \Longleftrightarrow y[n] = \pm x[k \pm n] \tag{13}$$

where k and n are 2D indices. Given a phase or magnitude matrix $\langle X_2 \rangle$, by adding the lost information (magnitude or phase respectively), x_2 (the inverse DFT of X_2) is a $R \times C$ matrix which includes x'_2 and zero sub-matrices:

$$x_{2(R\times C)} = \begin{pmatrix} x'_{2(r_x \times c_x)} & 0_{(r_x \times (C - c_x))} \\ 0_{((R-r_x)\times c_x)} & 0_{((R-r_x)\times(C-c_x))} \end{pmatrix} \tag{14}$$

Considering the nature of TDMA systems, TDMs are signals with constant length frames where a specific part of each frame is dedicated to a user's data and values of other indices are zero. The basic idea is that when T^{-1} acts on $x_{2(R\times C)}$ it results in a TDM signal followed by some zeros.

Now we explain, how uniqueness of reconstruction in 2D case results in the uniqueness in 1D case. Consider the 1D signal in the 2D form:

$$\langle \widetilde{X}_2 \rangle = T_{C,R}(\langle X \rangle)^T \tag{15}$$

Using equation (4), $\langle \widetilde{X}_2 \rangle$ represents the phase or magnitude of another Fourier transform which is relatively rotated and its sampling rate is changed in comparison to Fourier transform of x_2. Since the Fourier transform of this signal is uniquely recoverable, the elements of X_2 (which are samples of 2D Fourier transform of x_2) and (as a direct outcome) the desired signal (x) can be recovered uniquely.

It is clear that TDM signals have some kind of sparsity in which non-zero elements are gathered in clusters. Therefore, we obtained the uniqueness in 1D case owing to this kind of sparsity. Supposing the 2D reconstruction algorithm to be perfect, due to uniqueness constrains discussed in (12), following constrains for 1D unique reconstruction will be obtain:

1. For c_x (number of non-zero samples in each frame) and C (the length of each frame), the constraint inequality is $C \geq 2c_x - 1$. The concept of time multiplexing satisfies this constraint because only multiplexing of time between two users is needed.

2. The corresponding constraint for $r_x C$ (length of the TDM signal) and $R \times C$ (length of the known DFT phase or magnitude) is $R \geq 2r_x - 1$. This inequality is a generalization of the similar constraint in 2D (and also multidimensional) case; i.e., if the length of the desired signal is N, for unique reconstruction, we need $2N - 1$ samples of its Fourier phase or magnitude.

It is interesting to notice that because of the predicted effect of shift in the time-domain signal on its Fourier transform phase or magnitude, the idea could be used for reconstruction of all TDM channel signals.

3.1. Proposed Reconstruction Methods

In this subsection we propose a basic and an improved method for 1D TDM signal reconstruction. Clearly, $\langle \widetilde{X}_2 \rangle = T_{C,R}(\langle X \rangle)^T$ is an approximation of $\langle X_2 \rangle$, because by using zero-order interpolation, we obtain:

$$\mathcal{F}_{x(q+\frac{p}{R})}\{\widetilde{x}_2\} \approx \mathcal{F}_{x(q)}\{\widetilde{x}_2\} \tag{16}$$

Therefore $X[n] \approx X_2[p,q]$ and

$$\langle X[n] \rangle \approx \langle X_2[p,q] \rangle \tag{17}$$

A basic method (which we call it zero-order reconstruction) is to use this approximation as an input for the 2D reconstruction algorithm which leads to a rough reconstruction of the signal. The idea which improved the reconstruction is to use a mechanism for vanishing phase or magnitude distortion caused by zero-order interpolation. For the rows of M_x and its corresponding estimation $\widetilde{M_x}$, we have:

$$\widetilde{M_x}[p,q] = M_x[p,q].e^{-j2\pi\frac{pq}{RC}} \tag{18}$$

Therefore X_2 can be calculated using these steps: (i) calculate the inverse DFT of the rows of $\widetilde{X}_2$, (ii) multiply the resulted matrix elements by $e^{+j2\pi\frac{pq}{RC}}$ and (iii) finally calculate the DFT of the rows. We call this sequential operation Zero-Order Correction (ZOC) and the similar operation which lead from X_2 to $\widetilde{X}_2$ INVerse ZOC (INV-ZOC). The block diagram of corrected zero-order method is depicted in figure 1. In this method, in each iteration, with successive use of INV-ZOC and ZOC the known phase or magnitude is replaced in its correct location in frequency-domain. For initial phase or magnitude attribution to X_2, its zero-order estimation $\langle \widetilde{X}_2 \rangle$ can be used.

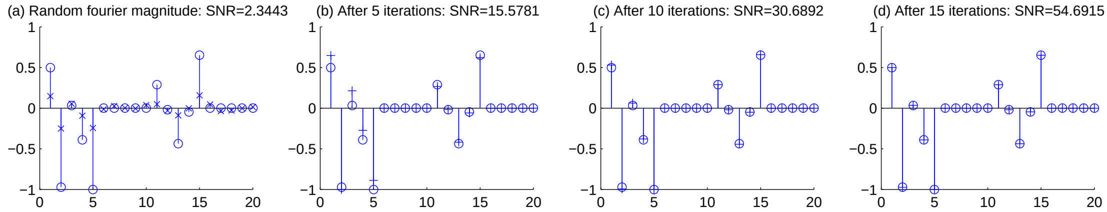

Fig. 2. Reconstruction of a random TDM signal ($R = 16, C = 10, r_x = 2, c_x = 5$) from its Fourier phase using: (a) a random Fourier transform magnitude denoted by ($\times$) and (b-d) corrected zero-order method denoted by ($+$).

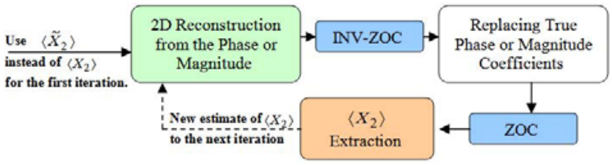

Fig. 1. Block diagram of corrected zero-order method

4. SIMULATION RESULTS

In this section, we present some simulation results to validate the convergence and accuracy of the proposed methods. Since the signal can be uniquely reconstructed from the phase of the Fourier transform, experiments are done using phase only reconstruction. For 2D signal reconstruction an internal simple iterative algorithm is used, however, experiments show that the overall algorithm achieves adequate results.

An intuitive view of the reconstruction of a random TDM signal is depicted in figure 2 using the corrected zero-order method with 32 internal iterations. Also figure 3 plots the SNR curves resulting from both zero-order and corrected zero-order methods. It seems, the corrected method has a great improvement in comparison to the basic method.

5. CONCLUSIONS

A DFT transformation between 1D and 2D spaces is suggested and uniqueness conditions for reconstruction of 1D TDM signal from its 2D counterpart is obtained. We provide some methods in which with the usage of these methods and such a transformation, every 2D signal reconstruction algorithm from phase or magnitude can be used for 1D TDM signal reconstruction. Experimental results have shown that the proposed method exhibits good performance from visual and objective points of view.

6. REFERENCES

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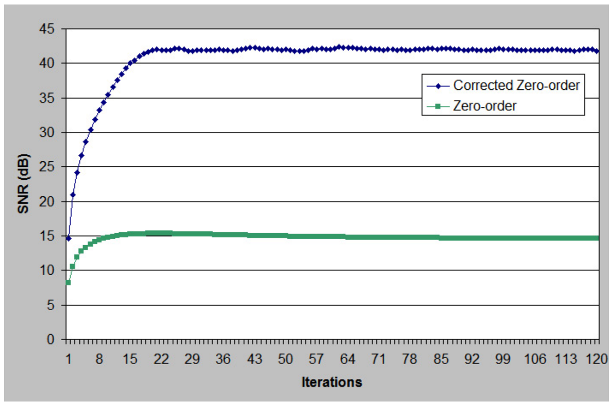

Fig. 3. SNR curves resulted by zero-order and corrected zero-order methods for reconstruction from the phase. The number of iterations in zero-order method and the internal and external cycles in corrected zero-order method is 120. The SNR curves are obtained by averaging over reconstruction SNR values of 32 random TDM signals with ($R = 64, C = 64, r_x = 8, c_x = 8$). Final SNR values are 14.593 dB and 41.797 dB, respectively.

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