APPROACHING PEAK CORRELATION BOUNDS VIA ALTERNATING PROJECTIONS

Mojtaba Soltanalian+, Mohammad Mahdi Naghsh† and Petre Stoica+*

+Dept. of Information Technology, Uppsala University, Uppsala, Sweden
† ECE Dept., Isfahan University of Technology, Isfahan, Iran

ABSTRACT

In this paper, we study the problem of approaching peak periodic or aperiodic correlation bounds for complex-valued sets of sequences. In particular, novel algorithms based on alternating projections are devised to approach a given peak periodic or aperiodic correlation bound. Several numerical examples are presented to assess the tightness of the known correlation bounds as well as to illustrate the effectiveness of the proposed methods for meeting these bounds.

Index Terms— Autocorrelation, correlation bound, cross-correlation, peak sidelobe level (PSL), Welch bound.

1. INTRODUCTION

Sequence sets with impulse-like autocorrelation and small cross-correlation are required in many communication and active sensing applications. For example, such sets are used in asynchronous CDMA to separate different users while performing a synchronization operation [1]. As an active sensing example, such correlation properties of the probing sequences enable the multi-input multi-output (MIMO) radars to conveniently retrieve (via matched filters) the received signals from the range bin of interest while suppressing the probing signals backscattered from other range bins [2]-[4].

Let X be a set of m sequences of length n. We assume that the sequences in X have identical energy[1], i.e. $\|\boldsymbol{x}\|_2^2 = \sigma$ for all $\boldsymbol{x} \in X$. Let $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$ denote two sequences from the set X. The periodic $\{c_{u,v}(k)\}$ and aperiodic $\{r_{u,v}(k)\}$ cross-correlations of $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$ are defined as

$$c_{u,v}(k) = \sum_{l=1}^{n} \boldsymbol{x}_u(l)\boldsymbol{x}_v^*(l+k)_{mod\ n}, \qquad (1)$$

$$r_{u,v}(k) = \sum_{l=1}^{n-k} \boldsymbol{x}_u(l)\boldsymbol{x}_v^*(l+k) = r_{v,u}^*(-k), \qquad (2)$$

for $0 \leq k \leq (n-1)$. The periodic and aperiodic autocorrelations of any $\boldsymbol{x}_u \in X$ are obtained from the above definitions by using $\boldsymbol{x}_v = \boldsymbol{x}_u$. Moreover, the inner product of $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$ is given by $\boldsymbol{x}_v^H \boldsymbol{x}_u = c_{u,v}(0) = r_{u,v}(0)$.

The Welch bounds [5] are the most well-known theoretical limits on the collective smallness measures of both inner-products and correlations of sequence sets. A brief presentation of the correlation peak sidelobe level (PSL) metric (for both periodic and aperiodic cases) as well as the corresponding Welch bounds can be found in Table 1. The main objective of this work is to determine how close we can get to the known peak correlation bounds. In order to achieve this goal, a computational method is devised to approach any given (feasible) PSL level for both periodic and aperiodic correlations. To the best of our knowledge, the provided computational method is the first (non-heuristic) algorithm to tackle the problem of achieving a given low PSL value.

2. APPROACHING A CORRELATION BOUND

2.1. Problem Formulation

The *twisted product* [6] of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ of length n is defined as $\boldsymbol{x} \circlearrowleft \boldsymbol{y}^H \triangleq$

$$\begin{pmatrix} \boldsymbol{x}(1)\boldsymbol{y}^*(1) & \boldsymbol{x}(2)\boldsymbol{y}^*(2) & \cdots & \boldsymbol{x}(n)\boldsymbol{y}^*(n) \\ \boldsymbol{x}(1)\boldsymbol{y}^*(2) & \boldsymbol{x}(2)\boldsymbol{y}^*(3) & \cdots & \boldsymbol{x}(n)\boldsymbol{y}^*(1) \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{x}(1)\boldsymbol{y}^*(n) & \boldsymbol{x}(2)\boldsymbol{y}^*(1) & \cdots & \boldsymbol{x}(n)\boldsymbol{y}^*(n-1) \end{pmatrix} \qquad (3)$$

where $\boldsymbol{x}(k)$ and $\boldsymbol{y}(k)$ are the k^{th} entries of $\boldsymbol{x}$ and $\boldsymbol{y}$ respectively. In a more general context, we define the twisted product of two matrices $\boldsymbol{X} = (\boldsymbol{x}_1\, \boldsymbol{x}_2 \cdots \boldsymbol{x}_p)$ and $\boldsymbol{Y} = (\boldsymbol{y}_1\, \boldsymbol{y}_2 \cdots \boldsymbol{y}_q)$ as

$$\boldsymbol{X} \circlearrowleft \boldsymbol{Y}^H \triangleq \begin{pmatrix} \boldsymbol{x}_1 \circlearrowleft \boldsymbol{y}_1^H \\ \vdots \\ \boldsymbol{x}_1 \circlearrowleft \boldsymbol{y}_q^H \\ \boldsymbol{x}_2 \circlearrowleft \boldsymbol{y}_1^H \\ \vdots \\ \vdots \\ \boldsymbol{x}_p \circlearrowleft \boldsymbol{y}_q^H \end{pmatrix} \qquad (4)$$

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* Please address all the correspondence to Mojtaba Soltanalian, Phone: (+46) 18-471-3168; Fax: (+46) 18-511925; Email: mojtaba.soltanalian@it.uu.se

[1]For the sake of generality, an energy of σ is considered for sequences throughout the paper. We note that the typical values of σ suggested in the literature are $\sigma = 1$ for inner-product bounds, and $\sigma = n$ for correlation bounds. However, one can easily verify that using different values of σ leads to nothing but a scaling of the inner-product or correlation bounds.

Table 1. The correlation peak sidelobe level (PSL) metric and the associated Welch bounds

Metric	Metric definitions and Welch lower bounds ($\mathcal{W}$)
Periodic case: $\mathrm{PSL}^{\mathcal{P}}(X)$	$\mathrm{PSL}^{\mathcal{P}}(X) = \max\left(\{\lvert c_{u,v}(k)\rvert\}_{u\neq v;\,k} \cup \{\lvert c_{u,u}(k)\rvert\}_{u;\,k\neq 0}\right) \geq$ $\mathcal{W}^{\mathcal{P}}_{m,n} \triangleq \max_{s:\,\binom{n+s-1}{s}\leq mn} \sigma\left(\frac{1}{mn-1}\left[\frac{mn}{\binom{n+s-1}{s}}-1\right]\right)^{\frac{1}{2s}}$
Aperiodic case: $\mathrm{PSL}^{\mathcal{AP}}(X)$	$\mathrm{PSL}^{\mathcal{AP}}(X) = \max\left(\{\lvert r_{u,v}(k)\rvert\}_{u\neq v;\,k} \cup \{\lvert r_{u,u}(k)\rvert\}_{u;\,k\neq 0}\right) \geq$ $\mathcal{W}^{\mathcal{AP}}_{m,n} \triangleq \max_{s:\,\binom{2n+s-2}{s}\leq m(2n-1)} \sigma\left(\frac{1}{m(2n-1)-1}\left[\frac{m(2n-1)}{\binom{2n+s-2}{s}}-1\right]\right)^{\frac{1}{2s}}$

where all $\{\boldsymbol{x}_k\}$ and $\{\boldsymbol{y}_l\}$ are of length n. Interestingly, meeting a PSL bound can be formulated by using the concept of twisted product for both periodic and aperiodic correlations. Let $\mathbf{1}$ denote an all one vector/matrix. It should be observed that $\boldsymbol{X}$ meets a peak periodic correlation bound $B^{\mathcal{P}}$ if and only if the entries of

$$\boldsymbol{c} = (\boldsymbol{X} \circlearrowleft \boldsymbol{X}^H)\mathbf{1}_{n\times 1} \tag{5}$$

satisfy

$$\begin{cases} \boldsymbol{c}(t) = \sigma & t = l(m+1)n+1,\ 0\leq l\leq m-1, \\ \lvert \boldsymbol{c}(t)\rvert \leq \mathcal{B}^{\mathcal{P}} & \text{otherwise} \end{cases} \tag{6}$$

where the first condition corresponds to the energy constraints on $\{\boldsymbol{x}_k\}$.

Next note that for any two sequences $\boldsymbol{x}_u, \boldsymbol{x}_v \in \mathbb{C}^n$ the periodic cross-correlations $\{\widetilde{c}_{u,v}(k)\}$ of $\widetilde{\boldsymbol{x}}_u = (\boldsymbol{x}_u^T\ \mathbf{0}_{1\times(n-1)})^T$ and $\widetilde{\boldsymbol{x}}_v = (\boldsymbol{x}_v^T\ \mathbf{0}_{1\times(n-1)})^T$ are given by

$$\widetilde{c}_{u,v}(k) = \begin{cases} r_{u,v}(k) & 0\leq k\leq n-1, \\ r^*_{u,v}(2n-k-1) & n\leq k\leq 2n-2. \end{cases} \tag{7}$$

Consequently, a similar approach as in the case of the periodic correlation can be used to characterize the sequence sets meeting a peak aperiodic correlation bound $\mathcal{B}^{\mathcal{AP}}$. Let $\widetilde{\boldsymbol{X}} = \left(\boldsymbol{X}^T\ \mathbf{0}_{m\times(n-1)}\right)^T$. Now note that $\boldsymbol{X}$ meets $\mathcal{B}^{\mathcal{AP}}$ if and only if the entries of

$$\widetilde{\boldsymbol{c}} = (\widetilde{\boldsymbol{X}} \circlearrowleft \widetilde{\boldsymbol{X}}^H)\mathbf{1}_{(2n-1)\times 1} \tag{8}$$

satisfy

$$\begin{cases} \widetilde{\boldsymbol{c}}(t) = \sigma & t = l(m+1)(2n-1)+1,\ 0\leq l\leq m-1, \\ \lvert \widetilde{\boldsymbol{c}}(t)\rvert \leq \mathcal{B}^{\mathcal{AP}} & \text{otherwise.} \end{cases} \tag{9}$$

2.2. Computational Framework

In the sequel, we devise a computational framework based on alternating projections to approach the given bounds $\mathcal{B}^{\mathcal{P}}$ and $\mathcal{B}^{\mathcal{AP}}$. The case of approaching a periodic PSL bound will be discussed in detail. However, the generalization of the ideas to the aperiodic case is straightforward.

• *The Periodic Case:* Consider the convex set $\Gamma^{\mathcal{P}}_{n,m}$ of all matrices $\boldsymbol{Z}$ for which the entries of $\boldsymbol{c} = \boldsymbol{Z}\mathbf{1}_{n\times 1}$ satisfy the conditions in (6). Furthermore, consider the set

$$\Lambda^{\mathcal{P}}_{n,m} = \left\{\boldsymbol{Z} \mid \boldsymbol{Z} = \boldsymbol{X} \circlearrowleft \boldsymbol{X}^H,\ \boldsymbol{X}\in\mathbb{C}^{n\times m}\right\}. \tag{10}$$

Let $\Psi^{\mathcal{P}}_{m,n}(\mathcal{B}^{\mathcal{P}})$ denote the sequence sets with a peak periodic correlation equal to $\mathcal{B}^{\mathcal{P}}$. As there exists a one-to-one mapping between the two sets $\Psi^{\mathcal{P}}_{m,n}(\mathcal{B}^{\mathcal{P}})$ and $\Gamma^{\mathcal{P}}_{n,m}\cap\Lambda^{\mathcal{P}}_{n,m}$, a natural approach to find the elements of $\Psi^{\mathcal{P}}_{m,n}(\mathcal{B}^{\mathcal{P}})$ is to employ alternating projections onto the two sets $\Gamma^{\mathcal{P}}_{n,m}$ and $\Lambda^{\mathcal{P}}_{n,m}$.

Let $vec(\boldsymbol{X}) = (\boldsymbol{x}_1^T\,\boldsymbol{x}_2^T\cdots\boldsymbol{x}_m^T)^T$. Note that, as all the entries of $\boldsymbol{X} \circlearrowleft \boldsymbol{X}^H$ occur in $vec(\boldsymbol{X})\,vec^H(\boldsymbol{X})$ exactly once, there exists a unique re-ordering function that maps the two matrices to each other; more concretely, there exists $\mathcal{G}:\mathbb{C}^{m^2n\times n}\to\mathbb{C}^{mn\times mn}$ such that

$$\mathcal{G}(\boldsymbol{X} \circlearrowleft \boldsymbol{X}^H) = vec(\boldsymbol{X})\,vec^H(\boldsymbol{X}) \tag{11}$$

for all $\boldsymbol{X}\in\mathbb{C}^{n\times m}$, and also $\mathcal{G}^{-1}$ exists. As a result, the Frobenius norm projection $\boldsymbol{Z}^{\Lambda}_{\perp}$ of any $\boldsymbol{Z}\in\mathbb{C}^{m^2n\times n}$ on $\Lambda^{\mathcal{P}}_{n,m}$ can be obtained as the solution to the optimization problem

$$\begin{aligned} \min_{\boldsymbol{X}_\perp,\boldsymbol{Z}^{\Lambda}_\perp} \quad & \|\boldsymbol{Z}-\boldsymbol{Z}^{\Lambda}_\perp\|_F \\ \text{s.t.} \quad & \boldsymbol{Z}^{\Lambda}_\perp = \boldsymbol{X}_\perp \circlearrowleft \boldsymbol{X}^H_\perp \end{aligned} \tag{12}$$

whose objective function may be recast as:

$$\begin{aligned} \|\boldsymbol{Z}-\boldsymbol{Z}^{\Lambda}_\perp\|_F &= \|\boldsymbol{Z}-\boldsymbol{X}_\perp \circlearrowleft \boldsymbol{X}^H_\perp\|_F \\ &= \|\mathcal{G}(\boldsymbol{Z}) - vec(\boldsymbol{X}_\perp)\,vec^H(\boldsymbol{X}_\perp)\|_F. \end{aligned} \tag{13}$$

Let $\eta(.)$ and $\boldsymbol{v}(.)$ represent the dominant eigenvalue, and respectively, the corresponding eigenvector of the Hermitian matrix argument. By using (13), the minimizer $\boldsymbol{X}_\perp$ of (12) can be obtained as $vec(\boldsymbol{X}_\perp) = \sqrt{\eta(\mathcal{G}(\boldsymbol{Z}))}\,\boldsymbol{v}\left(\mathcal{G}(\boldsymbol{Z})\right)$, which yields

$$\boldsymbol{Z}^{\Lambda}_\perp = \boldsymbol{X}_\perp \circlearrowleft \boldsymbol{X}^H_\perp \tag{14}$$

as the optimal projection on $\Lambda^{\mathcal{P}}_{n,m}$.

Next, we study the Frobenius norm projection $\boldsymbol{Z}^{\Gamma}_\perp$ of any $\boldsymbol{Z}\in\mathbb{C}^{m^2n\times n}$ on $\Gamma^{\mathcal{P}}_{n,m}$. Such a projection can be obtained by solving the optimization problem

$$\min_{\boldsymbol{Z}^{\Gamma}_\perp\in\Gamma^{\mathcal{P}}_{n,m}} \quad \|\boldsymbol{Z}-\boldsymbol{Z}^{\Gamma}_\perp\|_F. \tag{15}$$

We note that the condition (6) on $\boldsymbol{Z}_{\perp}^{\Gamma} \in \Gamma_{n,m}^{\mathcal{P}}$ is row-wise. Let $\boldsymbol{z}^T$ and $\boldsymbol{z}_{\perp}^T$ represent two generic rows of $\boldsymbol{Z}$ and $\boldsymbol{Z}_{\perp}^{\Gamma}$, respectively. Therefore, we consider the nearest-vector problem

$$\min_{\boldsymbol{z}_{\perp}} \quad \|\boldsymbol{z} - \boldsymbol{z}_{\perp}\|_2 \tag{16}$$

in which $\boldsymbol{z}_{\perp}$ is constrained either to have a given sum σ, i.e. $\boldsymbol{z}_{\perp}^T \mathbf{1} = \sigma$, or the absolute value of its sum is supposed to be upper bounded by $\mathcal{B}^{\mathcal{P}}$, viz. $|\boldsymbol{z}_{\perp}^T \mathbf{1}| \leq \mathcal{B}^{\mathcal{P}}$.

To tackle the above nearest-vector problem, assume $\boldsymbol{z}^T \mathbf{1} = \alpha_1 e^{j\theta_1}$ and $\boldsymbol{z}_{\perp}^T \mathbf{1} = \alpha_2 e^{j\theta_2}$ for some $\alpha_1, \alpha_2 \in \mathbb{R}_+$, $\theta_1, \theta_2 \in [0, 2\pi)$, and let $\boldsymbol{z}_d = \boldsymbol{z} - \boldsymbol{z}_{\perp}$. By using the Cauchy-Schwarz inequality we have that

$$\|\boldsymbol{z}_d\|_2^2 \geq \frac{|\boldsymbol{z}_d^T \mathbf{1}|^2}{\|\mathbf{1}\|_2^2} = \frac{|\alpha_1 e^{j\theta_1} - \alpha_2 e^{j\theta_2}|^2}{n} \tag{17}$$

where the equality is attained if and only if all the entries of $\boldsymbol{z}_d$ are identical:

$$\boldsymbol{z}_d(k) = \frac{\alpha_1 e^{j\theta_1} - \alpha_2 e^{j\theta_2}}{n}, \quad 1 \leq k \leq n. \tag{18}$$

Moreover, the equality in (17) can be achieved for any given α_2 and θ_2 via (18). As a result, to minimize $\|\boldsymbol{z} - \boldsymbol{z}_{\perp}\|_2 = \|\boldsymbol{z}_d\|_2$, it is sufficient to minimize $|\alpha_1 e^{j\theta_1} - \alpha_2 e^{j\theta_2}|^2$ with respect to α_2 and θ_2. For any fixed α_2, the minimizer θ_2 of the latter criterion is given by $\theta_2 = \theta_1$. On the other hand, the optimal α_2 depends on the constraint imposed on $\boldsymbol{z}_{\perp}$. In particular, for the constraint $\boldsymbol{z}_{\perp}^T \mathbf{1} = \sigma$ we have the optimum $\alpha_2 = \sigma$. In the case of the constraint $|\boldsymbol{z}_{\perp}^T \mathbf{1}| \leq \mathcal{B}^{\mathcal{P}}$, the minimizer α_2 is given by

$$\alpha_2 = \begin{cases} \alpha_1 & \alpha_1 \leq B^{\mathcal{P}}, \\ B^{\mathcal{P}} & \alpha_1 > B^{\mathcal{P}}. \end{cases} \tag{19}$$

Table 2 summarizes the steps of the proposed algorithm for approaching a given periodic PSL bound.

- *The Aperiodic Case:* To approach a given aperiodic PSL bound $\mathcal{B}^{\mathcal{AP}}$, similar (modified) alternating projections can be devised. We consider the set

$$\Lambda_{n,m}^{\mathcal{AP}} = \left\{ \boldsymbol{Z} \mid \boldsymbol{Z} = \widetilde{\boldsymbol{X}} \circlearrowright \widetilde{\boldsymbol{X}}^H,\ \widetilde{\boldsymbol{X}} = \begin{pmatrix} \boldsymbol{X} \\ \mathbf{0}_{(n-1)\times m} \end{pmatrix} ,\boldsymbol{X} \in \mathbb{C}^{n\times m} \right\}. \tag{20}$$

Let us define the masking matrix $\boldsymbol{M}$ as

$$\boldsymbol{M} = \begin{pmatrix} \boldsymbol{M}' & \cdots & \boldsymbol{M}' \\ \vdots & \ddots & \vdots \\ \boldsymbol{M}' & \cdots & \boldsymbol{M}' \end{pmatrix}, \tag{21}$$

$$\boldsymbol{M}'_{(2n-1)\times(2n-1)} = \begin{pmatrix} \mathbf{1}_{n\times n} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$

and in addition consider the convex set $\Gamma_{n,m}^{\mathcal{AP}}$ of all matrices $\boldsymbol{Z}$ such that

$$\boldsymbol{Z} \odot \mathcal{G}^{-1}(\boldsymbol{M}) = \boldsymbol{Z}, \tag{22}$$

Table 2. The Proposed Algorithm for approaching a given periodic/aperiodic PSL bound

Step 0: Initialize $\boldsymbol{X}$ with a random matrix in $\mathbb{C}^{n\times m}$; (i) in the periodic case: set $\boldsymbol{Z}_{\perp}^{\Lambda} = \boldsymbol{X} \circlearrowright \boldsymbol{X}^H$, (ii) in the aperiodic case: set $\boldsymbol{Z}_{\perp}^{\Lambda} = \widetilde{\boldsymbol{X}} \circlearrowright \widetilde{\boldsymbol{X}}^H$.
Step 1: Compute the optimal projection $\boldsymbol{Z}_{\perp}^{\Gamma}$ of $\boldsymbol{Z}_{\perp}^{\Lambda}$, (i) in the periodic case: find $\boldsymbol{Z}_{\perp}^{\Gamma} \in \Gamma_{n,m}^{\mathcal{P}}$ by using (15)-(19). (ii) in the aperiodic case: find $\boldsymbol{Z}_{\perp}^{\Gamma} \in \Gamma_{n,m}^{\mathcal{AP}}$ by using (24).
Step 2: Compute the optimal projection $\boldsymbol{Z}_{\perp}^{\Lambda}$ of $\boldsymbol{Z}_{\perp}^{\Gamma}$, (i) in the periodic case: find $\boldsymbol{Z}_{\perp}^{\Lambda} \in \Lambda_{n,m}^{\mathcal{P}}$ by using (14). (ii) in the aperiodic case: find $\boldsymbol{Z}_{\perp}^{\Lambda} \in \Lambda_{n,m}^{\mathcal{AP}}$ by using (23).
Step 3: Repeat steps 1 and 2 until a pre-defined stop criterion is satisfied, e.g. $\|\boldsymbol{Z}_{\perp}^{\Lambda} - \boldsymbol{Z}_{\perp}^{\Gamma}\|_F \leq \xi$, or $\|\boldsymbol{X}_{\perp}^{(t+1)} - \boldsymbol{X}_{\perp}^{(t)}\|_F \leq \xi$, for some $\xi > 0$, in which t denotes the iteration number.

where $\mathcal{G}$ is as defined in the periodic case but with dimension parameter $2n-1$ in lieu of n, and for which the entries of $\widetilde{\boldsymbol{c}} = \boldsymbol{Z}\mathbf{1}_{(2n-1)\times 1}$ satisfy the conditions in (9).

We note that the projections onto the two sets $\Gamma_{n,m}^{\mathcal{AP}}$ and $\Lambda_{n,m}^{\mathcal{AP}}$ can be obtained in (almost) the same manner as in the periodic case. Particularly, the Frobenius norm projection $\boldsymbol{Z}_{\perp}^{\Lambda}$ of any $\boldsymbol{Z} \in \mathbb{C}^{m^2(2n-1)\times(2n-1)}$ on $\Lambda_{n,m}^{\mathcal{AP}}$ can be obtained as

$$\boldsymbol{Z}_{\perp}^{\Lambda} = \widetilde{\boldsymbol{X}}_{\perp} \circlearrowright \widetilde{\boldsymbol{X}}_{\perp}^H \tag{23}$$

where $vec(\boldsymbol{X}_{\perp}) = \sqrt{\eta(\mathcal{M}(\mathcal{G}(\boldsymbol{Z})))}\, \boldsymbol{v}\left(\mathcal{M}(\mathcal{G}(\boldsymbol{Z}))\right)$, with the operator $\mathcal{M}(.)$ collecting the entries of the matrix argument corresponding to the non-zero entries of the masking matrix $\boldsymbol{M}$. To compute the Frobenius norm projection $\boldsymbol{Z}_{\perp}^{\Gamma}$ of any $\boldsymbol{Z} \in \mathbb{C}^{m^2(2n-1)\times(2n-1)}$ on $\Gamma_{n,m}^{\mathcal{AP}}$, the variables α_2 and θ_2 can be obtained using the same approach as for $\Gamma_{n,m}^{\mathcal{P}}$ but with

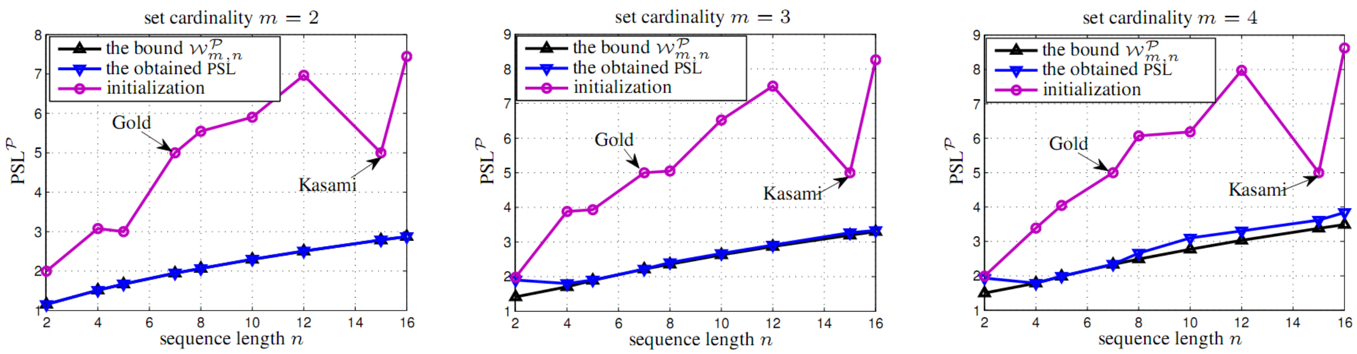
$$\boldsymbol{z}_d(k) = \begin{cases} \frac{\alpha_1 e^{j\theta_1} - \alpha_2 e^{j\theta_2}}{|\boldsymbol{\mu}^T \mathbf{1}|} & k \in \text{supp}(\boldsymbol{\mu}), \\ 0 & \text{otherwise} \end{cases} \tag{24}$$

where $\boldsymbol{\mu}$ represents the corresponding row in $\mathcal{G}^{-1}(\boldsymbol{M})$, and supp(.) denotes the set of non-zero locations in the vector argument.

Finally, the steps of the proposed alternating projections, in the periodic and aperiodic cases, are summarized in Table 2. Note that in both cases, each iteration of the algorithms has a $\mathcal{O}(m^2 n^2)$-complexity. The obtained complexity measure is a direct consequence of the generally large cardinality (i.e. mn) of the data that the algorithms should handle as well as the hardness of the original problem (with $m^2 n$ constraints, which should be compared to the fewer constraints (i.e. m^2) for achieving a given peak inner-product level).

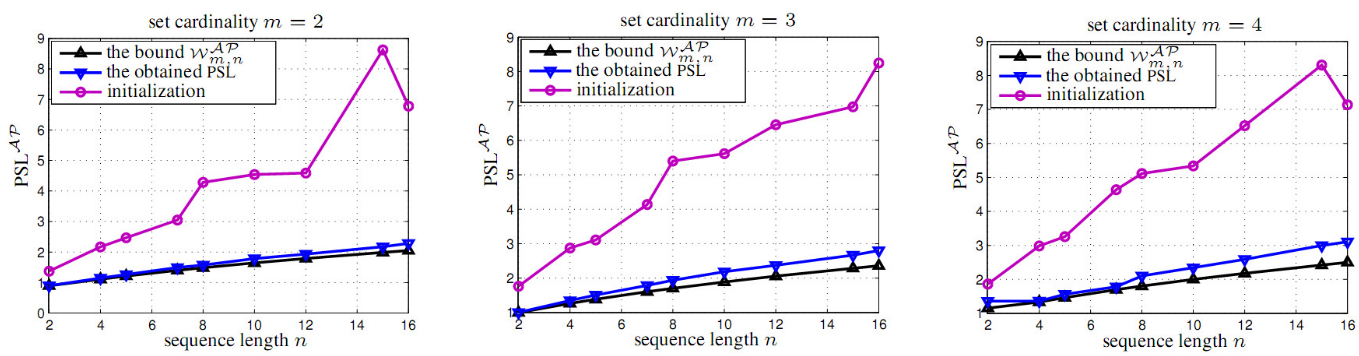
3. NUMERICAL RESULTS AND DISCUSSION

Both periodic and aperiodic correlations are employed in active sensing and communication applications. In particular, sequences with good periodic correlation are typically used



(a) (b) (c)

Fig. 1. $\mathrm{PSL}^{\mathcal{P}}$ of the obtained sequence sets by using the algorithm in Table 2 (the periodic case), versus sequence length n, and for different set cardinalities m. Gold, Kasami, and Weil sequence sets are used to initialize the algorithm when they exist (pinpointed by arrows).



(a) (b) (c)

Fig. 2. $\mathrm{PSL}^{\mathcal{AP}}$ of the obtained sequence sets by using the algorithm in Table 2 (aperiodic case), versus sequence length n, and for different set cardinalities m.

when each sequence can be transmitted several times in succession, whereas sequences with good aperiodic correlation are required when each sequence can be used only once . As a result, we consider numerical investigations of the proposed method in both periodic and aperiodic cases. We employ the suggested algorithm in Table 2 for different values of (n, m). The aim of these examples will be to approach the Welch bounds in Table 1.

Fig. 1 shows the peak periodic correlation ($\mathrm{PSL}^{\mathcal{P}}$) values corresponding to the initializations and the obtained sequence sets along with the bound $\mathcal{W}^{\mathcal{P}}_{m,n}$, for $m \in \{2, 3, 4\}$ and $n \in \{2, 4, 5, 7, 8, 10, 12, 15, 16\}$. Note that due to the non-convexity of $\Lambda_{n,m}$, the problem is multi-modal (i.e. it may have many convergence points), and hence, many random initial points might be needed to achieve a certain low peak correlation level. In this example, different random initializations are considered for 40 experiments, and the resultant $\mathrm{PSL}^{\mathcal{P}}$ of the proposed algorithm represents the best outcome of the 40 experiments. To examine the sensitivity to choosing the initial set, well-known sequence sets including Gold [18], Kasami [19], and Weil [20] are used as initializing sets for the (m, n) values for which they exist. Such cases are also reported in Fig. 1. It can be observed that $\mathcal{W}^{\mathcal{P}}_{m,n}$ can be practically met in several cases, e.g. for all n with $m = 2$. Furthermore, a considerable decrease of the peak periodic correlation obtained by using the proposed algorithm can be observed in all cases (even for the cases with well-known sets as initialization).

We conduct a similar numerical investigation in the aperiodic case. Fig. 2 illustrates the achieved $\mathrm{PSL}^{\mathcal{AP}}$ values by using the proposed alternating projections along with the Welch bound. The bound is met for the case $(m, n) = (2, 2)$. For other cases, in which the aperiodic bound cannot be met exactly, significant reductions in the obtained $\mathrm{PSL}^{\mathcal{AP}}$ can be observed compared to the $\mathrm{PSL}^{\mathcal{AP}}$ values corresponding to the initial sets.

4. REFERENCES

[1] D. V. Sarwate, “Meeting the Welch bound with equality,” in *Sequences and their Applications (SETA)*. New York: Springer, 1999, pp. 79–102.

[2] H. He, P. Stoica, and J. Li, “Designing unimodular sequence sets with good correlations—including an application to MIMO radar,” *IEEE Transactions on Signal Processing*, vol. 57, no. 11, pp. 4391–4405, November 2009.

[3] J. Li and P. Stoica, Eds., *MIMO Radar Signal Processing*. Hoboken, NJ: John Wiley & Sons, Inc., 2008.

[4] E. Fishler, A. Haimovich, R. Blum, L. Cimini, D. Chizhik, and R. Valenzuela, “Spatial diversity in radars—models and detection performance,” *IEEE Transactions on Signal Processing*, vol. 54, no. 3, pp. 823–838, March 2006.

[5] L. Welch, “Lower bounds on the maximum cross correlation of signals,” *IEEE Transactions on Information Theory*, vol. 20, no. 3, pp. 397–399, 1974.

[6] M. Soltanalian and P. Stoica, “Computational design of sequences with good correlation properties,” *IEEE Transactions on Signal Processing*, vol. 60, no. 5, pp. 2180–2193, 2012.

[7] M. Soltanalian, M. M. Naghsh, and P. Stoica, “A fast algorithm for designing complementary sets of sequences,” *Signal Processing*, vol. 93, no. 7, pp. 2096–2102, 2013.

[8] H. Rauhut, “Compressive sensing and structured random matrices,” in *Theoretical foundations and numerical methods for sparse recovery*, ser. Radon Series on Computational and Applied Mathematics, M. Fornasier, Ed., vol. 9. De Gruyter, 2010, pp. 1–92.

[9] T. Strohmer and R. W. Heath, “Grassmannian frames with applications to coding and communication,” *Applied and Computational Harmonic Analysis*, vol. 14, no. 3, pp. 257–275, 2003.

[10] S. Datta, S. Howard, and D. Cochran, “Geometry of the Welch bounds,” *Linear Algebra and its Applications*, vol. 437, pp. 2455–2470, 2012.

[11] J. Marcus, S. Imtel, and P. Spasojevic, “On the size of binary MWBE sequence sets,” in *International Symposium on Modeling and Optimization in Mobile, Ad Hoc and Wireless Networks (WiOpt)*. Princeton, NJ, USA: IEEE, 2011, pp. 374–374.

[12] P. Pad, M. Faraji, and F. Marvasti, “Constructing and decoding GWBE codes using Kronecker products,” *IEEE Communications Letters*, vol. 14, no. 1, pp. 1–3, 2010.

[13] T. Strohmer, R. W. Heath, and A. J. Paulraj, “On the design of optimal spreading sequences for CDMA systems,” in *Asilomar Conference on Signals, Systems and Computers*, vol. 2. CA, USA: IEEE, 2002, pp. 1434–1438.

[14] P. Stoica, H. He, and J. Li, “Sequence sets with optimal integrated periodic correlation level,” *IEEE Signal Processing Letters*, vol. 17, no. 1, pp. 63–66, 2010.

[15] H. He, P. Stoica, and J. Li, “On aperiodic-correlation bounds,” *IEEE Signal Processing Letters*, vol. 17, no. 3, pp. 253–256, March 2010.

[16] V. Sidelnikov, “On mutual correlation of sequences,” *Soviet Math. Dokl*, vol. 12, no. 1, pp. 197–201, 1971.

[17] S. Kocabas and A. Atalar, “Binary sequences with low aperiodic autocorrelation for synchronization purposes,” *IEEE Communications Letters*, vol. 7, no. 1, pp. 36–38, January 2003.

[18] R. Gold, “Optimal binary sequences for spread spectrum multiplexing,” *IEEE Transactions on Information Theory*, vol. 13, no. 4, pp. 619 –621, October 1967.

[19] T. Kasami, “Weight distribution formula for some class of cyclic codes,” Coordinated Science Laboratory, University of Illinois, Tech. Rep., April 1966.

[20] J. Rushanan, “Weil sequences: A family of binary sequences with good correlation properties,” in *IEEE International Symposium on Information Theory*, Seattle, Washington, USA, July 2006, pp. 1648–1652.