On Meeting the Peak Correlation Bounds

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***Abstract*—In this paper, we study the problem of meeting peak periodic or aperiodic correlation bounds for complex-valued sets of sequences. To this end, the Welch, Levenstein, and Exponential bounds on the peak inner-product of sequence sets are considered and used to provide compound peak correlation bounds in both periodic and aperiodic cases. The peak aperiodic correlation bound is further improved by using the intrinsic dimension deficiencies associated with its formulation. In comparison to the compound bound, the new aperiodic bound contributes an improvement of more than 35% for some specific values of the sequence length n and set cardinality m. We study the tightness of the provided bounds by using both analytical and computational tools. In particular, novel algorithms based on alternating projections are devised to approach a given peak periodic or aperiodic correlation bound. Several numerical examples are presented to assess the tightness of the provided correlation bounds as well as to illustrate the effectiveness of the proposed methods for meeting these bounds.**

***Index Terms*—Autocorrelation, correlation bound, cross-correlation, peak sidelobe level (PSL), sequence set, Welch bound.**

I. Introduction

SEQUENCE sets with impulse-like autocorrelation and small cross-correlation are required in many communication and active sensing applications. For example, such sets are used in asynchronous CDMA to separate different users while performing a synchronization operation at the same time [1]. As an active sensing example, such correlation properties of the probing sequences enable the multi-input multi-output (MIMO) radars to conveniently retrieve (via matched filters) the received signals from the range bin of interest while suppressing the probing signals backscattered from other range bins [2].

Let X be a set of m sequences of length n. We assume that the sequences in X have identical energy[1], i.e., $\|\boldsymbol{x}\|_2^2 = \sigma$ for all $\boldsymbol{x} \in X$. Let $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$ denote two sequences from the set X. The periodic $\{c_{u,v}(k)\}$ and aperiodic $\{r_{u,v}(k)\}$ cross-correlations of $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$ are defined as

$$c_{u,v}(k) = \sum_{l=1}^{n} \boldsymbol{x}_u(l)\boldsymbol{x}_v^*(l+k)_{mod\ n}, \tag{1}$$

$$r_{u,v}(k) = \sum_{l=1}^{n-k} \boldsymbol{x}_u(l)\boldsymbol{x}_v^*(l+k) = r_{v,u}^*(-k), \tag{2}$$

for $0 \leq k \leq (n-1)$. The periodic and aperiodic autocorrelations of any $\boldsymbol{x}_u \in X$ are obtained from the above definitions by using $\boldsymbol{x}_v = \boldsymbol{x}_u$. Moreover, the inner product of $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$ is given by $\boldsymbol{x}_v^H \boldsymbol{x}_u = c_{u,v}(0) = r_{u,v}(0)$.

The Welch bounds [3] are the most well-known theoretical limits on the collective smallness measures of both inner-products and correlations of sequence sets. Several such measures along with the associated Welch lower bounds are summarized in Table I. Briefly stated, the main objectives of this paper are:

- To update the peak correlation bounds based on the current sate-of-knowledge on peak inner-product bound, as well as to propose a scheme for improvement of the aperiodic correlation bound. The proposed scheme exploits the intrinsic low dimensional properties that appear in derivation of the peak aperiodic bound. The new aperiodic peak sidelobe level (PSL) bound can be significantly larger than the previously known aperiodic bound (by more than 35% for some $(m, n) \in [2, 1024] \times [2, 256]$).
- To determine how close we can get to the previously known or improved PSL correlation bounds. In order to achieve this goal, a computational method is devised to approach any given (feasible) PSL level for both periodic and aperiodic correlations. To the best of our knowledge, the provided computational method is the first (non-heuristic) algorithm to tackle the problem of achieving a given low PSL.

The rest of this paper is organized as follows. In Section II, the relationship between the inner-product and correlation bounds is studied and employed to provide a derivation of peak correlation bounds. The tightness of the provided bounds along with an improvement of the aperiodic correlation bound are discussed in Section III. In Section IV, a general framework is devised to approach a given (periodic or aperiodic) peak correlation bound. Section V is devoted to the numerical examples. Finally, Section VI concludes the paper.

Notations

We use bold lowercase letters for vectors/sequences and bold uppercase letters for matrices. $(\cdot)^T$, $(\cdot)^*$ and $(\cdot)^H$ denote the vector/matrix transpose, the complex conjugate, and the Hermitian transpose, respectively. $\mathbf{1}$ and $\mathbf{0}$ are the all-one and

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[1]For the sake of generality, an energy of σ is considered for sequences throughout the paper. We note that the typical values of σ suggested in the literature are $\sigma = 1$ for inner-product bounds, and $\sigma = n$ for correlation bounds. However, one can easily verify that using different values of σ leads to nothing but a scaling of the inner-product or correlation bounds.

TABLE I
SUMMARY OF INNER-PRODUCT AND CORRELATION SMALLNESS MEASURES ALONG WITH THE ASSOCIATED WELCH LOWER BOUNDS

	Metric		Metric definitions and Welch lower bounds ($\mathcal{W}$)
Inner-product	Root-mean-square (RMS) inner-product level: $I_{rms}(X)$		$I_{rms}(X) = \left(\frac{1}{m(m-1)} \sum_{u\neq v} \lvert\boldsymbol{x}_v^H \boldsymbol{x}_u\rvert^2\right)^{\frac{1}{2}} \geq \overline{\mathcal{W}}_{m,n} \triangleq \sigma\sqrt{\frac{m-n}{(m-1)n}}$
	Peak inner-product level: $I_{max}(X)$		$I_{max}(X) \triangleq \max\left\{\lvert\boldsymbol{x}_v^H \boldsymbol{x}_u\rvert\right\}_{u\neq v} \geq \mathcal{W}_{m,n} \triangleq \max_{s:\ \binom{n+s-1}{s}\leq m} \sigma\left(\frac{1}{m-1}\left[\frac{m}{\binom{n+s-1}{s}}-1\right]\right)^{\frac{1}{2s}}$
Correlation	Integrated sidelobe level	Periodic case: $\mathrm{ISL}^{\mathcal{P}}(X)$	$\mathrm{ISL}^{\mathcal{P}}(X) = \sum_{u\neq v;\, k} \lvert c_{u,v}(k)\rvert^2 + \sum_{u;\, k\neq 0} \lvert c_{u,u}(k)\rvert^2 \geq \overline{\mathcal{W}}^{\mathcal{P}}_{m,n} \triangleq \sigma^2 m(m-1)$
		Aperiodic case: $\mathrm{ISL}^{\mathcal{AP}}(X)$	$\mathrm{ISL}^{\mathcal{AP}}(X) = \sum_{u\neq v;\, k} \lvert r_{u,v}(k)\rvert^2 + \sum_{u;\, k\neq 0} \lvert r_{u,u}(k)\rvert^2 \geq \overline{\mathcal{W}}^{\mathcal{AP}}_{m,n} \triangleq \sigma^2 m(m-1)$
	Peak side-lobe level	Periodic case: $\mathrm{PSL}^{\mathcal{P}}(X)$	$\mathrm{PSL}^{\mathcal{P}}(X) = \max\left(\{\lvert c_{u,v}(k)\rvert\}_{u\neq v;\, k} \cup \{\lvert c_{u,u}(k)\rvert\}_{u;\, k\neq 0}\right) \geq \mathcal{W}^{\mathcal{P}}_{m,n} \triangleq \max_{s:\ \binom{n+s-1}{s}\leq mn} \sigma\left(\frac{1}{mn-1}\left[\frac{mn}{\binom{n+s-1}{s}}-1\right]\right)^{\frac{1}{2s}}$
		Aperiodic case: $\mathrm{PSL}^{\mathcal{AP}}(X)$	$\mathrm{PSL}^{\mathcal{AP}}(X) = \max\left(\{\lvert r_{u,v}(k)\rvert\}_{u\neq v;\, k} \cup \{\lvert r_{u,u}(k)\rvert\}_{u;\, k\neq 0}\right) \geq \mathcal{W}^{\mathcal{AP}}_{m,n} \triangleq \max_{s:\ \binom{2n+s-2}{s}\leq m(2n-1)} \sigma\left(\frac{1}{m(2n-1)-1}\left[\frac{m(2n-1)}{\binom{2n+s-2}{s}}-1\right]\right)^{\frac{1}{2s}}$

all-zero vectors/matrices. $\|\boldsymbol{x}\|_n$ is the l_n-norm of the vector $\boldsymbol{x}$ defined as $\left(\sum_k |\boldsymbol{x}(k)|^n\right)^{\frac{1}{n}}$ where $\{\boldsymbol{x}(k)\}$ are the entries of $\boldsymbol{x}$. The Frobenius norm of a matrix $\boldsymbol{X}$ (denoted by $\|\boldsymbol{X}\|_F$) with entries $\{\boldsymbol{X}(k,l)\}$ is equal to $\left(\sum_{k,l} |\boldsymbol{X}(k,l)|^2\right)^{\frac{1}{2}}$. $\mathrm{tr}\{\boldsymbol{X}\}$ denotes the trace of the matrix $\boldsymbol{X}$. $\eta_k(\boldsymbol{X})$ and $\boldsymbol{v}_k(\boldsymbol{X})$ represent the k^{th} dominant eigenvalue and the corresponding eigenvector of the Hermitian matrix $\boldsymbol{X}$, respectively. The symbol $\odot$ stands for the Hadamard (element-wise) product of matrices, whereas $\otimes$ stands for the Kronecker product of matrices. $\boldsymbol{x}^{\otimes n}$ is equal to $\underbrace{\boldsymbol{x} \otimes \boldsymbol{x} \otimes \cdots \otimes \boldsymbol{x}}_{n}$. $[n]$ denotes the set $\{1, 2, \ldots, n\}$. For any $n_1, n_2 \in \mathbb{N}$, $[n_1, n_2]$ is equal to $[n_2] \backslash [n_1 - 1]$. $\binom{n}{k}$, often read as "n choose k", is the coefficient of the x^k-term in the polynomial expansion of the binomial power $(1+x)^n$. Finally, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ represent the set of natural, integer, real and complex numbers, respectively.

II. A STUDY OF THE INNER-PRODUCT AND CORRELATION BOUNDS

In the following, a study of the currently known inner-product and correlation bounds is accomplished. The provided background lays the ground for tightness assessments as well as the bound improvements suggested in the paper.

A. Inner-Product Bounds

The collective smallness of the inner products of $\{\boldsymbol{x}_u\}$ can be measured by using the *peak inner-product level* metric:

$$I_{\max}(X) = \max \left\{|\boldsymbol{x}_v^H \boldsymbol{x}_u|\right\}_{u \neq v} \tag{3}$$

as well as the *root-mean-square (RMS) inner-product level* metric,

$$I_{rms}(X) = \left(\frac{1}{m(m-1)} \sum_{u\neq v} |\boldsymbol{x}_v^H \boldsymbol{x}_u|^2\right)^{\frac{1}{2}} \tag{4}$$

where clearly $I_{\max}(X) \geq I_{rms}(X)$. In [3], Welch derived lower bounds on the above collective smallness measures of the inner-product levels associated with X; the *Welch lower bounds* on $I_{\max}(X)$ and $I_{rms}(X)$ are given (assuming $m > n$) by

$$I_{rms}(X) \geq \sigma\sqrt{\frac{m-n}{(m-1)n}} \triangleq \overline{\mathcal{W}}_{m,n} \tag{5}$$

and

$$I_{\max}(X) \geq \max_{s:\binom{n+s-1}{s}\leq m} \sigma\left(\frac{1}{m-1}\left[\frac{m}{\binom{n+s-1}{s}}-1\right]\right)^{\frac{1}{2s}} \triangleq \mathcal{W}_{m,n}. \tag{6}$$

Note that both $\overline{\mathcal{W}}_{m,n}$ and $\mathcal{W}_{m,n}$ are zero for $m \leq n$.

Knowledge of inner-product bounds is essential to the derivation of both periodic and aperiodic correlation bounds. Let $\boldsymbol{x}_v^H \boldsymbol{x}_u = \sigma \cos(\phi_{u,v})$ where $\phi_{u,v}$ denotes the angle between the two vectors $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$. From a geometrical point of view, the Welch peak inner-product bound provides a lower bound on the maximum of the angles $\{\phi_{u,v}\}$ among the set of m equi-norm vectors $\{\boldsymbol{x}_u\}$ in $\mathbb{C}^n$. A direct algebraic derivation of the inner-product bound (which appears to be simpler than that in [3]) is as follows. Let $\boldsymbol{X} \in \mathbb{C}^{n\times m}$ (with $m > n$) represent the matrix whose columns are $\{\boldsymbol{x}_u\}$. Then we have that

$$\begin{aligned}\sum_{u,v} |\boldsymbol{x}_v^H \boldsymbol{x}_u|^2 &= \|\boldsymbol{X}^H\boldsymbol{X}\|_F^2 = \sum_{k=1}^{n} \lambda_k^2 \\ &\geq \frac{1}{n}\left(\sum_{k=1}^{n}\lambda_k\right)^2 = \frac{1}{n}\left(\mathrm{tr}(\boldsymbol{X}^H\boldsymbol{X})\right)^2 \\ &= \frac{m^2\sigma^2}{n} \end{aligned} \tag{7}$$

where $\{\lambda_k\}$ are the non-zero eigenvalues of $\boldsymbol{X}^H\boldsymbol{X}$. As a result,

$$\begin{aligned} I_1(X) &\triangleq \sum_{u\neq v} |\boldsymbol{x}_v^H \boldsymbol{x}_u|^2 \\ &= \left(\sum_{u,v}|\boldsymbol{x}_v^H \boldsymbol{x}_u|^2\right) - m\sigma^2 \geq m\sigma^2\left(\frac{m}{n}-1\right) \end{aligned} \tag{8}$$

which implies

$$I_{\max}(X) \geq \left(\frac{I_1(X)}{m(m-1)}\right)^{\frac{1}{2}} \geq \sigma\left(\frac{1}{m-1}\left(\frac{m}{n}-1\right)\right)^{\frac{1}{2}}. \tag{9}$$

As an aside remark, it is straightforward to verify that for $m \leq n$, (7)–(9) yield a trivial lower bound i.e., zero.

Next observe that for any $s \in \mathbb{N}$, one can verify that $(\boldsymbol{x}_v^H\boldsymbol{x}_u)^s = (\boldsymbol{x}_v^{\otimes s})^H \boldsymbol{x}_u^{\otimes s}$. However, even though $\{\boldsymbol{x}_u^{\otimes s}\}$ are of length n^s, they lie in a lower dimensional subspace of $\mathbb{C}^{n^s}$. To see this, we count the number of distinct entries in any general vector $\boldsymbol{x}^{\otimes s}$. Note that any entry of $\boldsymbol{x}^{\otimes s}$ is of the form

$$(\boldsymbol{x}(1))^{\nu_1}(\boldsymbol{x}(2))^{\nu_2}\cdots(\boldsymbol{x}(n))^{\nu_n} \tag{10}$$

where $\nu_1+\nu_2+\cdots+\nu_n = s$, and $\nu_l \in \mathbb{N}\cup\{0\}$. The number of possible combinations of $(\nu_1,\nu_2,\ldots,\nu_n)$ which satisfy this same condition is given by $d = \binom{n+s-1}{s}$. Let $\boldsymbol{X}_s$ denote a matrix whose columns are $\{\boldsymbol{x}_u^{\otimes s}\}$. Based on the above argument, there exist a semi-unitary matrix $\boldsymbol{U} \in \mathbb{C}^{n^s\times d}$ and a rank–d matrix $\boldsymbol{Y}_s \in \mathbb{C}^{d\times m}$ such that $\boldsymbol{X}_s = \boldsymbol{U}\boldsymbol{Y}_s$. By using the same approach as in (7) we have that

$$\sum_{u,v} |\boldsymbol{x}_v^H\boldsymbol{x}_u|^{2s} = \|\boldsymbol{X}_s^H\boldsymbol{X}_s\|_F^2 = \|\boldsymbol{Y}_s^H\boldsymbol{Y}_s\|_F^2 \geq \frac{m^2\sigma^{2s}}{d}. \tag{11}$$

It follows from (11) that

$$I_s(X) \triangleq \sum_{u\neq v}|\boldsymbol{x}_v^H\boldsymbol{x}_u|^{2s} \geq m\sigma^{2s}\left(\frac{m}{d}-1\right) \tag{12}$$

which yields

$$I_{\max}(X) \geq \sigma\left(\frac{1}{m-1}\left(\frac{m}{\binom{n+s-1}{s}}-1\right)\right)^{\frac{1}{2s}}. \tag{13}$$

The above dimension reduction scheme, which lies at the core of the higher order (i.e., with $s>1$) Welch bounds, emphasizes the usefulness of considering hidden dimension deficiencies of the vector sets. Such dimension deficiencies play a main role in improving the peak aperiodic correlation bound in Section III-B.

Due to applications in compressive sensing and synchronous CDMA, meeting the Welch bounds on the inner-products associated with sequence sets (also referred to as measurement matrices [4], codebooks [5]–[7], or Grassmannian frames [8] depending on the application) has been studied widely. It is known that the Welch bound on $I_{rms}(X)$ can be met for many (m,n) (see, e.g., [1] and the references therein). An X meeting the Welch bound on I_{rms} is called Welch-bound-equality (WBE) set [1]. On the other hand, sequence sets meeting the Welch bound on the peak inner-product level (known as maximum-Welch-bound-equality (MWBE) sets, see [1]) are hard to obtain either analytically or numerically. Examples of and some conditions for the existence of MWBE sets for given (m,n) were presented in [8], [9]. Particularly, if MWBE sets do not exist[2] for $s=1$ in (6), then they do not exist for any $s>1$ [10]. Note that $\mathcal{W}_{m,n}$ in (6) associated with $s=1$ is equal to $\overline{\mathcal{W}}_{m,n}$. These facts not only emphasize the importance of the Welch peak inner-product level bound for $s=1$ but also imply that if the peak inner-product level of a sequence set meets the Welch bound (i.e., the Welch bound is tight) then all the inner products among the sequences in the set have the same absolute value which is equal to $\overline{\mathcal{W}}_{m,n}$. Furthermore, let the maximum of the functions in (6) occur for $s=s_0$. Then a necessary condition for the existence of MWBE sets for given (m,n) is [10]

$$\binom{n+s_0-1}{s_0} \leq n^2. \tag{14}$$

It is also known that the Welch inner-product bound can be tight only if $m \leq n^2$ [9].

Two other bounds on $I_{\max}(X)$ were derived in the literature which are tighter than $\mathcal{W}_{m,n}$ for some (m,n). The latter bounds, which are not discussed in the literature as much as the Welch bound, are the *Levenstein bound* [12], [13],

$$\mathcal{L}_{m,n} \triangleq \sigma\sqrt{\frac{2m-n^2-n}{(n+1)(m-n)}} \tag{15}$$

for $m > n(n+1)/2$, and the *Exponential bound* [5],

$$\mathcal{E}_{m,n} \triangleq \sigma\left(1-2m^{\frac{-1}{n-1}}\right) \tag{16}$$

for $m > 2^{n-1}$. The above bounds can be combined with the Welch bound to yield

$$\mathcal{I}_{m,n} \triangleq \max\{\mathcal{W}_{m,n},\mathcal{L}_{m,n},\mathcal{E}_{m,n}\} \tag{17}$$

that encapsulates the current state-of-knowledge on the lower bounds for the peak inner-product level. Note that some bounds in (17) might not be useful (i.e., >0) for a specific (m,n).

B. Correlation Bounds

Excluding the *in-phase* (i.e., for $k=0$) lags of the autocorrelations of $\{\boldsymbol{x}_u\}$ (which equal the energy of sequences), one can measure the level of the *out-of-phase* correlations of sequences in X by using the *integrated sidelobe level* (ISL) metric:

$$\mathrm{ISL}^{\mathcal{P}}(X) = \sum_{u\neq v;\,k}|c_{u,v}(k)|^2 + \sum_{u;\,k\neq 0}|c_{u,u}(k)|^2 \tag{18}$$

$$\mathrm{ISL}^{\mathcal{AP}}(X) = \sum_{u\neq v;\,k}|r_{u,v}(k)|^2 + \sum_{u;\,k\neq 0}|r_{u,u}(k)|^2 \tag{19}$$

[2]Note that MWBE sets exist for $s=1$ iff (6) is maximized with $s=1$ and there exist a sequence set with peak inner-product $I_{\max}(X)$ equal to the obtained value of $\mathcal{W}_{m,n}$.

where $\mathcal{P}$ and $\mathcal{AP}$ stand for *periodic* and *aperiodic* correlations, respectively. Lower bounds on the above ISL metrics are given by [16], [17]

$$\mathrm{ISL}^{\mathcal{P}}(X) \geq \sigma^2 m(m-1) \triangleq \overline{\mathcal{W}}^{\mathcal{P}}_{m,n} \tag{20}$$

$$\mathrm{ISL}^{\mathcal{AP}}(X) \geq \sigma^2 m(m-1) \triangleq \overline{\mathcal{W}}^{\mathcal{AP}}_{m,n}. \tag{21}$$

Note that the ISL metric can be related to the RMS inner-product level defined in (4). Particularly, similar to $I_{rms}(X)$, the ISL bounds can be (nearly) met even for sequence sets with constrained alphabet [16], [17].

A different criterion for measuring the collective smallness of the out-of-phase correlations is the PSL metric:

$$\mathrm{PSL}^{\mathcal{P}}(X) = \max\left(\{|c_{u,v}(k)|\}_{u\neq v;\, k} \cup \{|c_{u,u}(k)|\}_{u;\, k\neq 0}\right) \tag{22}$$

$$\mathrm{PSL}^{\mathcal{AP}}(X) = \max\left(\{|r_{u,v}(k)|\}_{u\neq v;\, k} \cup \{|r_{u,u}(k)|\}_{u;\, k\neq 0}\right) \tag{23}$$

The PSL criteria have a close relationship with the peak inner-product level metric. In particular, Welch [3] used (6) to derive the following lower bounds on the periodic, and respectively, aperiodic PSL metrics:

$$\mathcal{W}^{\mathcal{P}}_{m,n} \triangleq \max_{s:\binom{n+s-1}{s}\leq mn} \sigma\left(\frac{1}{mn-1}\left[\frac{mn}{\binom{n+s-1}{s}}-1\right]\right)^{\frac{1}{2s}},$$

$$\mathcal{W}^{\mathcal{AP}}_{m,n} \triangleq \max_{s:\binom{2n+s-2}{s}\leq m(2n-1)} \sigma\left(\frac{1}{m(2n-1)-1}\left[\frac{m(2n-1)}{\binom{2n+s-2}{s}}-1\right]\right)^{\frac{1}{2s}}$$

with the bounds being non-trivial for $m > 1$.

We continue this section noting that the Welch peak correlation bounds are a direct consequence of the Welch bound on inner-products. To observe this fact, let $\{\boldsymbol{Q}_k\}$ be the periodic shifting matrices defined by

$$\boldsymbol{Q}_k = \boldsymbol{Q}^H_{-k} \triangleq \begin{pmatrix} \mathbf{0}_{(n-k)\times k} & \boldsymbol{I}_{n-k} \\ \boldsymbol{I}_k & \mathbf{0}_{k\times(n-k)} \end{pmatrix}. \tag{24}$$

Given a sequence set $\{\boldsymbol{x}_u\}_{u=1}^m$ with sequences of length n and energy σ, it is straightforward to verify that the inner-products of the mn sequences $\{\boldsymbol{Q}_v\boldsymbol{x}_u\}_{u,v\in[m]^2}$ become the out-of-phase periodic correlations of the set $\{\boldsymbol{x}_u\}_{u=1}^m$. Therefore, by using the Welch inner-product bound we obtain the following lower bound on $\mathrm{PSL}^{\mathcal{P}}(X)$:

$$\mathrm{PSL}^{\mathcal{P}}(X) \geq \mathcal{W}_{mn,n} = \mathcal{W}^{\mathcal{P}}_{m,n} \tag{25}$$

The Welch correlation bound in the aperiodic case can be derived by additionally observing that the periodic out-of-phase correlations of $\{\widetilde{\boldsymbol{x}}_u\}_{u=1}^m$ where $\widetilde{\boldsymbol{x}}_u = (\boldsymbol{x}_u^T\ \mathbf{0}_{1\times(n-1)})^T$ are identical to the aperiodic out-of-phase correlations of $\{\boldsymbol{x}_u\}_{u=1}^m$. As a result,

$$\begin{aligned}\mathrm{PSL}^{\mathcal{AP}}(X) &\geq \mathcal{W}^{\mathcal{P}}_{m,2n-1} \\ &= \mathcal{W}_{m(2n-1),2n-1} \\ &= \mathcal{W}^{\mathcal{AP}}_{m,n}.\end{aligned} \tag{26}$$

A consequence of the above formulation is the fact that, similar to the case of inner products, the Welch peak correlation bounds can be met if and only if all out-of-phase correlation terms possess the same value. As a side consequence, the above formulation implies that the correlation lags compose the set of inner-products associated with circulant measurement matrices (or frames). Therefore, any of the obtained correlation bounds can be useful when designing measurement matrices (or frames) with circulant structure. In light of the above usage of the Welch peak inner-product bound for deriving peak correlation bounds, we can exploit the tighter peak inner-product bound $\mathcal{I}_{m,n}$ to obtain the following *compound peak correlation bounds*:

$$\begin{aligned}\mathcal{I}^{\mathcal{P}}_{m,n} &\triangleq \mathcal{I}_{mn,n} \\ \mathcal{I}^{\mathcal{AP}}_{m,n} &\triangleq \mathcal{I}^{\mathcal{P}}_{m,2n-1} = \mathcal{I}_{m(2n-1),2n-1}.\end{aligned} \tag{27}$$

Note that achieving the above PSL bounds is harder (both analytically and computationally) not only than meeting the ISL bounds in (20) but also than achieving the aforementioned peak inner-product bounds. It is worth pointing out that for designing sequence sets with constrained alphabet or with other practical limitations, the above bounds can be modified accordingly. For instance, when employing p^{th} root-of-unity (i.e., p-ary) sequences with prime p to design sequence sets with low periodic out-of-phase correlations, one can use the *Sidelnikov bound* [18] which is usually tighter (although not always) than the Welch bound. For the binary alphabet, improved lower bounds on periodic and aperiodic ISL metrics are proposed in [19] and [20], respectively.

The long-standing research problem of finding sequence sets with small out-of-phase correlations has resulted in several analytical constructions for specific values of (m, n) (see e.g., [21]–[23]). However, the analytical constructions are usually proposed for the periodic correlation case and not for the aperiodic case which is deemed to be more difficult [17]. As an example, Kasami family includes sets of binary sequences of length $n = 2^N - 1$ and cardinality $m = 2^{N/2}$ where N is an even natural number [21]. The $\mathrm{PSL}^{\mathcal{P}}$ value of a Kasami set is given by $1+2^{N/2}$. In addition, for odd N, Gold binary sequence sets can be constructed for $(m, n) = (2^N+1, 2^N-1)$ that have a $\mathrm{PSL}^{\mathcal{P}}$ value of $1+\sqrt{2^{N+1}-2}$ [22]. The Weil family consists of sequence sets with $n = N$ and $m = (N-1)/2$, where N is prime, that possess a $\mathrm{PSL}^{\mathcal{P}}$ value of $5 + 2\sqrt{N}$ [23]. Such sets are usually referred to as *asymptotically optimal* owing to the fact that their PSL values behave like $\mathcal{O}(\sqrt{n})$ as $n \to \infty$ similar to the behavior of Welch peak correlation bounds for $s = 1$. We refer the interested reader to [24] for further details on this aspect.

III. Correlation Bounds: Tightness and Improvement

By using the analytical tools provided earlier, we provide a tightness assessment of the compound correlation bounds. In order to improve the tightness condition of the bounds in the aperiodic case, a new improvement of the aperiodic bound is discussed and the obtained improvement is evaluated.

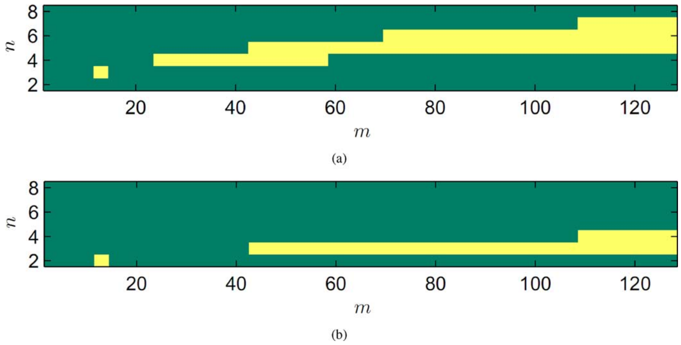


Fig. 1. The values of $(m,n) \in [2,128] \times [2,8]$ (depicted in yellow) for which the peak correlation bounds $\mathcal{I}^{\mathcal{P}}_{m,n}$ and $\mathcal{I}^{\mathcal{AP}}_{m,n}$ were found to be loose (by satisfying both conditions (28) and (29)): (a) periodic correlation, and (b) aperiodic correlation.

A. Tightness of $\mathcal{I}^{\mathcal{P}}_{m,n}$ and $\mathcal{I}^{\mathcal{AP}}_{m,n}$

Our main results regarding the tightness of $\mathcal{I}^{\mathcal{P}}_{m,n}$ and $\mathcal{I}^{\mathcal{AP}}_{m,n}$ can be briefly stated as follows. Examples of (m,n) can be provided for which the tightness of $\mathcal{I}^{\mathcal{P}}_{m,n}$ or $\mathcal{I}^{\mathcal{AP}}_{m,n}$ is straightforward to show. However, there exist (m,n) for which these bounds are not tight. Overall, the theoretical (as well as the computational) evidence suggests that *the tightness of the above bounds may be rather an exception than the rule*. The next two propositions (whose proofs are given in the Appendix) provide examples of cases in which $\mathcal{I}^{\mathcal{P}}_{m,n}$ and $\mathcal{I}^{\mathcal{AP}}_{m,n}$ are tight.

Proposition 1: The peak periodic correlation bound $\mathcal{I}^{\mathcal{P}}_{m,n}$ is tight for $(m,n) = (2,2)$.

Proposition 2: The peak aperiodic correlation bound $\mathcal{I}^{\mathcal{AP}}_{m,n}$ is tight for $(m,n) = (2,2)$.

Next we present a simple computational approach to find cases in which the compound peak correlation bounds are not tight. Specifically, the correlation bounds $\mathcal{I}^{\mathcal{P}}_{m,n}$ and $\mathcal{I}^{\mathcal{AP}}_{m,n}$ are not tight if both conditions below hold:

1) The corresponding Welch bound is not tight, viz.

$$\begin{cases} \mathrm{PSL}^{\mathcal{P}}(X) > \mathcal{W}^{\mathcal{P}}_{m,n} & \text{periodic case,} \\ \mathrm{PSL}^{\mathcal{AP}}(X) > \mathcal{W}^{\mathcal{AP}}_{m,n} & \text{aperiodic case} \end{cases} \tag{28}$$

for all sets X including m sequences of length n, and energy σ.

2) The Welch bound dominates both Levestein and Exponential bounds. Due to the fact that the compound bound is the maximum of Welch, Levestein and Exponential bounds, the latter condition is equivalent to

$$\begin{cases} \mathcal{I}^{\mathcal{P}}_{m,n} = \mathcal{W}^{\mathcal{P}}_{m,n} & \text{Periodic case,} \\ \mathcal{I}^{\mathcal{AP}}_{m,n} = \mathcal{W}^{\mathcal{AP}}_{m,n} & \text{Aperiodic case.} \end{cases} \tag{29}$$

Condition 1) can be verified, for example, by checking the two necessary tightness conditions of the Welch bound given in Introduction, see (14) and the related observations. The second condition makes sure that the compound bounds are identical to the Welch bounds. Fig. 1 depicts the values of $(m,n) \in [2,128] \times [2,8]$ for which the use of the above approach shows that the bounds $\mathcal{I}^{\mathcal{P}}_{m,n}$ and $\mathcal{I}^{\mathcal{AP}}_{m,n}$ are not tight. The next sub-section shows that, in general, the (compound) aperiodic correlation bound is loose even more often than what is suggested by Fig. 1.

B. An Improvement of the Aperiodic Correlation Bound

In this sub-section, we propose an improvement of $\mathcal{I}^{\mathcal{AP}}_{m,n}$. The new bound relies on the specific structure of aperiodic correlations. More concretely, one needs to observe that even though the sequence dimensions are increased by zero-padding (with the goal of deriving the aperiodic bound from the periodic one), the sequences retain their intrinsic low dimensional properties. In particular, for subsets of sequences lying in lower dimensional subspaces the angles among the vectors in the set may be smaller— so the inner product may be larger. In the following, a more precise usage of this observation is proposed.

Let $n \geq 2$, and consider the sequence set

$$\{\boldsymbol{Q}_v \widetilde{\boldsymbol{x}}_u\}_{u \in [m],\, v \in [2n-1]}. \tag{30}$$

Now let k be a fixed integer such that $0 \leq k \leq n-1$. Consider the subset of sequences in (30) whose non-zero entries occur only in their first $n+k$ locations. It is straightforward to verify that such property holds for any $0 \leq v \leq k$. As a result, at least $m(k+1)$ sequences of (30) lie in the $n+k$ dimensional space associated with the first $n+k$ entries of the sequences in (30). This fact implies the following lower bound on the peak aperiodic correlation:

$$\mathrm{PSL}^{\mathcal{AP}}(\boldsymbol{X}) \geq \mathcal{I}_{m(k+1),n+k}. \tag{31}$$

Note that the above observation can be made for any window of length $n+k$ over the entries of the sequences in (30), but does not seem to further improve the bound in (31). However, using (31) for $0 \leq k \leq n-1$ yields

$$\mathrm{PSL}^{\mathcal{AP}}(\boldsymbol{X}) \geq \max_{0 \leq k \leq n-1} \mathcal{I}_{m(k+1),n+k} \triangleq \mathcal{N}^{\mathcal{AP}}_{m,n}. \tag{32}$$

Fig. 2 compares the new aperiodic correlation bound $\mathcal{N}^{\mathcal{AP}}_{m,n}$ with the aperiodic bound $\mathcal{I}^{\mathcal{AP}}_{m,n}$. The comparison is accomplished by computing the ratio $\mathcal{N}^{\mathcal{AP}}_{m,n}/\mathcal{I}^{\mathcal{AP}}_{m,n}$ for $(m,n) \in [2,1024] \times [2,256]$. A considerable improvement (even by more than 35%) can be observed for some (m,n). As a specific example, we consider the case of $(m,n) = (450,250)$. In this case, the maximum of $\{\mathcal{I}_{m(k+1),n+k}\}$ occurs for $k=11$ leading to $\mathcal{N}^{\mathcal{AP}}_{m,n} = 0.0604\sigma$, whereas $\mathcal{I}^{\mathcal{AP}}_{m,n} = 0.0447\sigma$. As a result, we obtain $\mathcal{N}^{\mathcal{AP}}_{m,n}/\mathcal{I}^{\mathcal{AP}}_{m,n} = 1.351$.

Finally, we end this section by noting that similar to $\mathcal{I}^{\mathcal{AP}}_{m,n}$, the formulation of $\mathcal{N}^{\mathcal{AP}}_{m,n}$ relies on the inner-product bounds $\{\mathcal{I}_{m,n}\}$ and hence, its growth rate is determined by $\{\mathcal{I}_{m,n}\}$.

IV. Approaching a Correlation Bound

In this section, the challenging problem of meeting a correlation bound is addressed. Particularly, it is of interest to find out how close one can get to a given periodic or aperiodic bound. In the following, we provide a general computational framework (inspired by the formulation of the twisted product in [31]) that can be used to approach any feasible correlation bound.

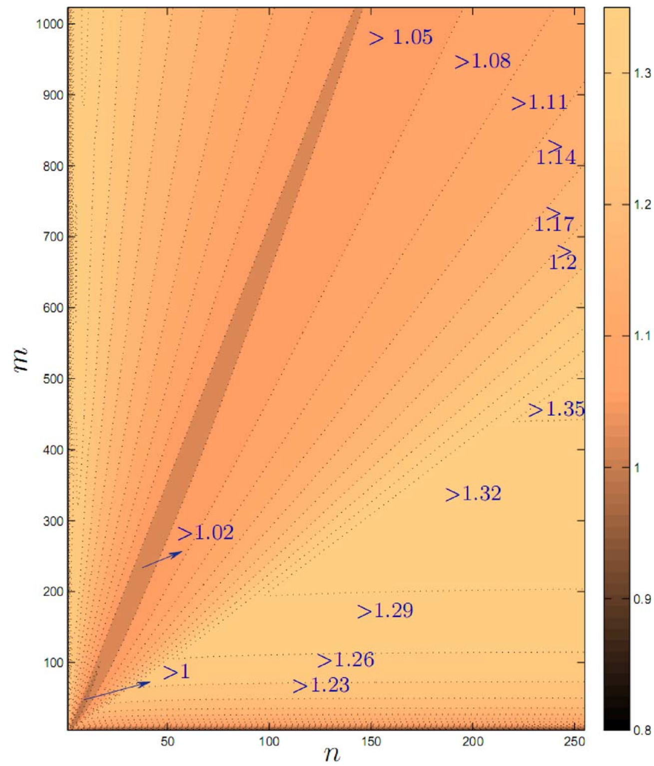


Fig. 2. The improvement of the aperiodic correlation bound. The new bound $\mathcal{N}^{\mathcal{AP}}_{m,n}$ is compared to the bound $\mathcal{I}^{\mathcal{AP}}_{m,n}$ by computing the ratio $\mathcal{N}^{\mathcal{AP}}_{m,n}/\mathcal{I}^{\mathcal{AP}}_{m,n}$. The contours represent the areas with the indicated minimum level of improvement.

A. Problem Formulation

The *twisted product* of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ of length n is defined as $\boldsymbol{x} \circlearrowright \boldsymbol{y}^H \triangleq$

$$\begin{pmatrix} \boldsymbol{x}(1)\boldsymbol{y}^*(1) & \boldsymbol{x}(2)\boldsymbol{y}^*(2) & \cdots & \boldsymbol{x}(n)\boldsymbol{y}^*(n) \\ \boldsymbol{x}(1)\boldsymbol{y}^*(2) & \boldsymbol{x}(2)\boldsymbol{y}^*(3) & \cdots & \boldsymbol{x}(n)\boldsymbol{y}^*(1) \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{x}(1)\boldsymbol{y}^*(n) & \boldsymbol{x}(2)\boldsymbol{y}^*(1) & \cdots & \boldsymbol{x}(n)\boldsymbol{y}^*(n-1) \end{pmatrix} \tag{33}$$

where $\boldsymbol{x}(k)$ and $\boldsymbol{y}(k)$ are the k^{th} entries of $\boldsymbol{x}$ and $\boldsymbol{y}$ respectively. In a more general context, we define the twisted product of two matrices $\boldsymbol{X} = (\boldsymbol{x}_1\, \boldsymbol{x}_2 \cdots \boldsymbol{x}_p)$ and $\boldsymbol{Y} = (\boldsymbol{y}_1\, \boldsymbol{y}_2 \cdots \boldsymbol{y}_q)$ as

$$\boldsymbol{X} \circlearrowright \boldsymbol{Y}^H \triangleq \begin{pmatrix} \boldsymbol{x}_1 \circlearrowright \boldsymbol{y}_1^H \\ \vdots \\ \boldsymbol{x}_1 \circlearrowright \boldsymbol{y}_q^H \\ \boldsymbol{x}_2 \circlearrowright \boldsymbol{y}_1^H \\ \vdots \\ \vdots \\ \boldsymbol{x}_p \circlearrowright \boldsymbol{y}_q^H \end{pmatrix} \tag{34}$$

where all $\{\boldsymbol{x}_k\}$ and $\{\boldsymbol{y}_l\}$ are of length n. Interestingly, meeting a PSL bound can be formulated by using the concept of twisted product for both periodic and aperiodic correlations. It should be observed that $\boldsymbol{X}$ meets a peak periodic correlation bound $B^{\mathcal{P}}$ if and only if the entries of

$$\boldsymbol{c} = (\boldsymbol{X} \circlearrowright \boldsymbol{X}^H)\mathbf{1}_{n\times 1} \tag{35}$$

satisfy

$$\begin{cases} \boldsymbol{c}(t) = \sigma & t = l(m+1)n + 1, 0 \le l \le m-1, \\ |\boldsymbol{c}(t)| \le \mathcal{B}^{\mathcal{P}} & \text{otherwise} \end{cases} \tag{36}$$

where the first condition corresponds to the energy constraints on $\{\boldsymbol{x}_k\}$.

Next note that for any two sequences $\boldsymbol{x}_u, \boldsymbol{x}_v \in \mathbb{C}^n$ the periodic cross-correlations $\{\widetilde{c}_{u,v}(k)\}$ of $\widetilde{\boldsymbol{x}}_u = (\boldsymbol{x}_u^T\, \mathbf{0}_{1\times(n-1)})^T$ and $\widetilde{\boldsymbol{x}}_v = (\boldsymbol{x}_v^T\, \mathbf{0}_{1\times(n-1)})^T$ are given by

$$\widetilde{c}_{u,v}(k) = \begin{cases} r_{u,v}(k) & 0 \le k \le n-1, \\ r^*_{u,v}(2n-k-1) & n \le k \le 2n-2. \end{cases} \tag{37}$$

Consequently, a similar approach as in the case of the periodic correlation can be used to characterize the sequence sets meeting a peak aperiodic correlation bound $\mathcal{B}^{\mathcal{AP}}$. Let

$$\widetilde{\boldsymbol{X}} = \begin{pmatrix} \boldsymbol{X} \\ \mathbf{0}_{(n-1)\times m} \end{pmatrix}. \tag{38}$$

Now note that $\boldsymbol{X}$ meets $\mathcal{B}^{\mathcal{AP}}$ if and only if the entries of

$$\widetilde{\boldsymbol{c}} = (\widetilde{\boldsymbol{X}} \circlearrowright \widetilde{\boldsymbol{X}}^H)\mathbf{1}_{(2n-1)\times 1} \tag{39}$$

satisfy

$$\begin{cases} \widetilde{\boldsymbol{c}}(t) = \sigma & t = l(m+1)(2n-1)+1, 0 \le l \le m-1, \\ |\widetilde{\boldsymbol{c}}(t)| \le \mathcal{B}^{\mathcal{AP}} & \text{otherwise.} \end{cases} \tag{40}$$

B. Computational Framework

In the sequel, we devise a computational framework based on alternating projections to approach the given bounds $\mathcal{B}^{\mathcal{P}}$ and $\mathcal{B}^{\mathcal{AP}}$.

1) The Periodic Case: Consider the convex set $\Gamma^{\mathcal{P}}_{n,m}$ of all matrices $\boldsymbol{Z}$ for which the entries of $\boldsymbol{c} = \boldsymbol{Z}\mathbf{1}_{n\times 1}$ satisfy the conditions in (36). Furthermore, consider the set

$$\Lambda^{\mathcal{P}}_{n,m} = \left\{\boldsymbol{Z} \mid \boldsymbol{Z} = \boldsymbol{X} \circlearrowright \boldsymbol{X}^H,\ \boldsymbol{X} \in \mathbb{C}^{n\times m}\right\}. \tag{41}$$

Let $\Psi^{\mathcal{P}}_{m,n}(\mathcal{B}^{\mathcal{P}})$ denote the sequence sets with a peak periodic correlation equal to $\mathcal{B}^{\mathcal{P}}$. As there exists a one-to-one mapping between the two sets $\Psi^{\mathcal{P}}_{m,n}(\mathcal{B}^{\mathcal{P}})$ and $\Gamma^{\mathcal{P}}_{n,m} \cap \Lambda^{\mathcal{P}}_{n,m}$, a natural approach to find the elements of $\Psi^{\mathcal{P}}_{m,n}(\mathcal{B}^{\mathcal{P}})$ is to employ alternating projections onto the two sets $\Gamma^{\mathcal{P}}_{n,m}$ and $\Lambda^{\mathcal{P}}_{n,m}$.

Let $vec(\boldsymbol{X}) = (\boldsymbol{x}_1^T\, \boldsymbol{x}_2^T \cdots \boldsymbol{x}_m^T)^T$. It can be seen that all the entries of $\boldsymbol{X} \circlearrowright \boldsymbol{X}^H$ occur in $vec(\boldsymbol{X})\, vec^H(\boldsymbol{X})$ exactly one time. Therefore, there exists a unique re-ordering function that maps the two matrices to one another. We denote this function by $\mathcal{G} : \mathbb{C}^{m^2 n\times n} \to \mathbb{C}^{mn\times mn}$ which is such that

$$\mathcal{G}(\boldsymbol{X} \circlearrowright \boldsymbol{X}^H) = vec(\boldsymbol{X})\, vec^H(\boldsymbol{X}). \tag{42}$$

In words, this mapping defines the (k,l) element of the right-hand side as the corresponding let us say $(\overline{k}, \overline{l})$ element of the matrix argument. As such, it can be easily generalized to any arbitrary matrix. The Frobenius norm projection $\boldsymbol{Z}^{\Lambda}_{\perp}$ of any $\boldsymbol{Z} \in \mathbb{C}^{m^2 n\times n}$ on $\Lambda^{\mathcal{P}}_{n,m}$ can be obtained as the solution to the optimization problem

$$\begin{aligned} \min_{\boldsymbol{X}_\perp, \boldsymbol{Z}^{\Lambda}_\perp} \quad & \|\boldsymbol{Z} - \boldsymbol{Z}^{\Lambda}_\perp\|_F \\ \text{s.t.} \quad & \boldsymbol{Z}^{\Lambda}_\perp = \boldsymbol{X}_\perp \circlearrowright \boldsymbol{X}^H_\perp \end{aligned} \tag{43}$$

whose objective function may be recast as:

$$\begin{aligned}\|\boldsymbol{Z}-\boldsymbol{Z}_{\perp}^{\Lambda}\|_F &= \|\boldsymbol{Z}-\boldsymbol{X}_{\perp} \circlearrowleft \boldsymbol{X}_{\perp}^H\|_F \\ &= \|\mathcal{G}(\boldsymbol{Z}) - vec(\boldsymbol{X}_{\perp})\, vec^H(\boldsymbol{X}_{\perp})\|_F. \quad (44)\end{aligned}$$

By using (44), the minimizer $\boldsymbol{X}_{\perp}$ of (43) can be obtained as $vec(\boldsymbol{X}_{\perp}) = \sqrt{\eta_1(\mathcal{G}(\boldsymbol{Z}))}\, \boldsymbol{v}_1(\mathcal{G}(\boldsymbol{Z}))$, which yields

$$\boldsymbol{Z}_{\perp}^{\Lambda} = \boldsymbol{X}_{\perp} \circlearrowleft \boldsymbol{X}_{\perp}^H \quad (45)$$

as the optimal projection on $\Lambda_{n,m}^{\mathcal{P}}$.

Remark 1: It is worth noting that for any $\boldsymbol{X} \in \Psi_{m,n}^{\mathcal{P}}(\mathcal{B}^{\mathcal{P}})$, the value of $\eta_1(\mathcal{G}(\boldsymbol{Z}))$ for the corresponding $\boldsymbol{Z}$ represents the total energy of the sequences denoted by $\boldsymbol{X}$. Moreover, finding the close points (or the intersection) of the two sets $\Gamma_{n,m}^{\mathcal{P}}$ and $\Lambda_{n,m}^{\mathcal{P}}$ can be roughly interpreted as the maximization of $\eta_1(\mathcal{G}(\boldsymbol{Z}))$ for $\boldsymbol{Z} \in \Gamma_{n,m}^{\mathcal{P}}$. As a result, for a feasible PSL bound it can be practically assumed that $\eta_1(\mathcal{G}(\boldsymbol{Z})) > 0$ throughout the projections. ■

Next, we study the Frobenius norm projection $\boldsymbol{Z}_{\perp}^{\Gamma}$ of any $\boldsymbol{Z} \in \mathbb{C}^{m^2 n \times n}$ on $\Gamma_{n,m}^{\mathcal{P}}$. Such a projection can be obtained by solving the optimization problem

$$\min_{\boldsymbol{Z}_{\perp}^{\Gamma} \in \Gamma_{n,m}^{\mathcal{P}}} \|\boldsymbol{Z} - \boldsymbol{Z}_{\perp}^{\Gamma}\|_F. \quad (46)$$

We note that the conditions (36) on $\boldsymbol{Z}_{\perp}^{\Gamma} \in \Gamma_{n,m}^{\mathcal{P}}$ are row-wise. Let $\boldsymbol{z}^T$ and $\boldsymbol{z}_{\perp}^T$ represent two generic rows of $\boldsymbol{Z}$ and $\boldsymbol{Z}_{\perp}^{\Gamma}$, respectively. Therefore, we consider the nearest-vector problem

$$\min_{\boldsymbol{z}_{\perp}} \|\boldsymbol{z} - \boldsymbol{z}_{\perp}\|_2 \quad (47)$$

in which $\boldsymbol{z}_{\perp}$ is constrained either to have a given sum n, i.e., $\boldsymbol{z}_{\perp}^T\boldsymbol{1} = n$, or the absolute value of its sum is supposed to be upper bounded by $\mathcal{B}^{\mathcal{P}}$, viz. $|\boldsymbol{z}_{\perp}^T\boldsymbol{1}| \leq \mathcal{B}^{\mathcal{P}}$.

To tackle the above nearest-vector problem, assume $\boldsymbol{z}^T\boldsymbol{1} = \alpha_1 e^{j\theta_1}$ and $\boldsymbol{z}_{\perp}^T\boldsymbol{1} = \alpha_2 e^{j\theta_2}$ for some $\alpha_1, \alpha_2 \in \mathbb{R}_+$, $\theta_1, \theta_2 \in [0, 2\pi)$, and let $\boldsymbol{z}_d = \boldsymbol{z} - \boldsymbol{z}_{\perp}$. By using the Cauchy-Schwarz inequality we have that

$$\|\boldsymbol{z}_d\|_2^2 \geq \frac{|\boldsymbol{z}_d^T\boldsymbol{1}|^2}{\|\boldsymbol{1}\|_2^2} = \frac{|\alpha_1 e^{j\theta_1} - \alpha_2 e^{j\theta_2}|^2}{n} \quad (48)$$

where the equality is attained if and only if all the entries of $\boldsymbol{z}_d$ are identical:

$$\boldsymbol{z}_d(k) = \frac{\alpha_1 e^{j\theta_1} - \alpha_2 e^{j\theta_2}}{n}, \quad 1 \leq k \leq n. \quad (49)$$

Moreover, the equality in (48) can be achieved for any given α_2 and θ_2 via (49). As a result, to minimize $\|\boldsymbol{z} - \boldsymbol{z}_{\perp}\|_2 = \|\boldsymbol{z}_d\|_2$, it is sufficient to minimize $|\alpha_1 e^{j\theta_1} - \alpha_2 e^{j\theta_2}|^2$ with respect to α_2 and θ_2. For any fixed α_2, the minimizer θ_2 of the latter criterion is given by $\theta_2 = \theta_1$. On the other hand, the optimal α_2 depends on the constraint imposed on $\boldsymbol{z}_{\perp}$. In particular, for the constraint $\boldsymbol{z}_{\perp}^T\boldsymbol{1} = n$ then we have the optimum $\alpha_2 = n$. In the case of the constraint $|\boldsymbol{z}_{\perp}^T\boldsymbol{1}| \leq \mathcal{B}^{\mathcal{P}}$, the minimizer α_2 is given by

$$\alpha_2 = \begin{cases} \alpha_1 & \alpha_1 \leq B^{\mathcal{P}}, \\ B^{\mathcal{P}} & \alpha_1 > B^{\mathcal{P}}. \end{cases} \quad (50)$$

TABLE II
THE PROPOSED ALGORITHM FOR APPROACHING A GIVEN PERIODIC/APERIODIC PSL BOUND

Step 0: Initialize $\boldsymbol{X}$ with a random matrix in $\mathbb{C}^{n\times m}$;
(i) in the periodic case: set $\boldsymbol{Z}_{\perp}^{\Lambda} = \boldsymbol{X} \circlearrowleft \boldsymbol{X}^H$,
(ii) in the aperiodic case: set $\boldsymbol{Z}_{\perp}^{\Lambda} = \widetilde{\boldsymbol{X}} \circlearrowleft \widetilde{\boldsymbol{X}}^H$.
Step 1: Compute the optimal projection $\boldsymbol{Z}_{\perp}^{\Gamma}$ of $\boldsymbol{Z}_{\perp}^{\Lambda}$,
(i) in the periodic case: find $\boldsymbol{Z}_{\perp}^{\Gamma} \in \Gamma_{n,m}^{\mathcal{P}}$ by using (46)-(50).
(ii) in the aperiodic case: find $\boldsymbol{Z}_{\perp}^{\Gamma} \in \Gamma_{n,m}^{\mathcal{AP}}$ by using (57).
Step 2: Compute the optimal projection $\boldsymbol{Z}_{\perp}^{\Lambda}$ of $\boldsymbol{Z}_{\perp}^{\Gamma}$,
(i) in the periodic case: find $\boldsymbol{Z}_{\perp}^{\Lambda} \in \Lambda_{n,m}^{\mathcal{P}}$ by using (45).
(ii) in the aperiodic case: find $\boldsymbol{Z}_{\perp}^{\Lambda} \in \Lambda_{n,m}^{\mathcal{AP}}$ by using (56).
Step 3: Repeat steps 1 and 2 until a pre-defined stop criterion is satisfied, e.g. $\|\boldsymbol{Z}_{\perp}^{\Lambda} - \boldsymbol{Z}_{\perp}^{\Gamma}\|_F \leq \xi$, or $\|\boldsymbol{X}_{\perp}^{(t+1)} - \boldsymbol{X}_{\perp}^{(t)}\|_F \leq \xi$, for some $\xi > 0$, in which t denotes the iteration number.

Table II summarizes the steps of the proposed algorithm for approaching a given periodic PSL bound. Note that while the projection on the set $\Lambda_{n,m}^{\mathcal{P}}$ is performed by a rank-one approximation, the projection on the set $\Gamma_{n,m}^{\mathcal{P}}$ has a closed-form expression which leads to an even smaller computational burden.

2) The Aperiodic Case: Similar to the derivations in the periodic case, we consider the set

$$\Lambda_{n,m}^{\mathcal{AP}} = \left\{ \boldsymbol{Z} \mid \boldsymbol{Z} = \widetilde{\boldsymbol{X}} \circlearrowleft \widetilde{\boldsymbol{X}}^H, \right.$$
$$\left. \widetilde{\boldsymbol{X}} = \begin{pmatrix} \boldsymbol{X} \\ \boldsymbol{0}_{(n-1)\times m} \end{pmatrix}, \ \boldsymbol{X} \in \mathbb{C}^{n\times m} \right\}. \quad (51)$$

We define the masking matrix $\boldsymbol{M}$ as

$$\boldsymbol{M} = \begin{pmatrix} \boldsymbol{M}' & \cdots & \boldsymbol{M}' \\ \vdots & \ddots & \vdots \\ \boldsymbol{M}' & \cdots & \boldsymbol{M}' \end{pmatrix},$$
$$\boldsymbol{M}'_{(2n-1)\times(2n-1)} = \begin{pmatrix} \boldsymbol{1}_{n\times n} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}, \quad (52)$$

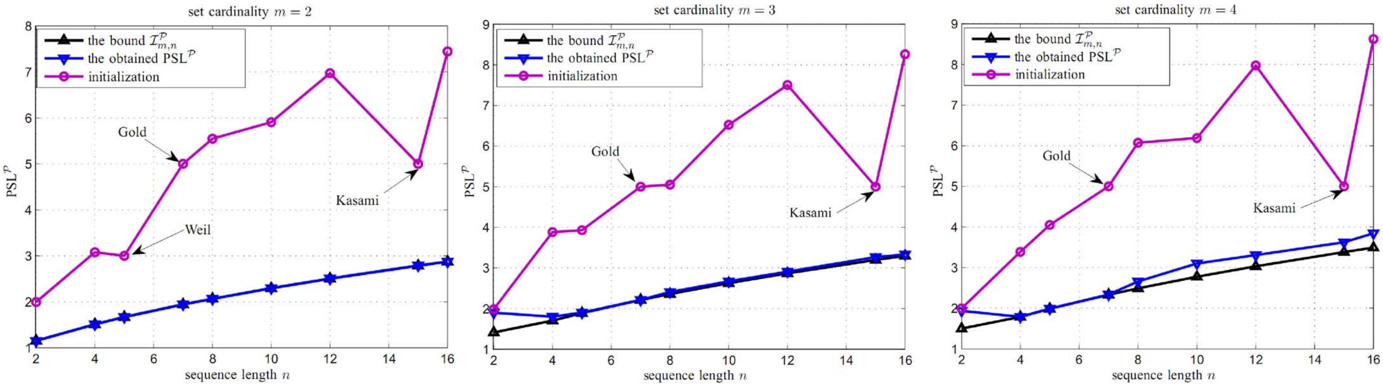
and in addition consider the convex set $\Gamma_{n,m}^{\mathcal{AP}}$ of all matrices $\boldsymbol{Z}$ such that

$$\boldsymbol{Z} \odot \mathcal{G}^{-1}(\boldsymbol{M}) = \boldsymbol{Z}, \quad (53)$$

where $\mathcal{G}$ is as defined in the periodic case but with dimension parameter $2n-1$ in lieu of n, and for which the entries of $\widetilde{\boldsymbol{c}} = \boldsymbol{Z}\boldsymbol{1}_{(2n-1)\times 1}$ satisfy the conditions in (40). Let $\Psi_{m,n}^{\mathcal{AP}}(\mathcal{B}^{\mathcal{AP}})$ denote the sequence sets with a peak aperiodic correlation equal to $\mathcal{B}^{\mathcal{AP}}$. In the following, we propose an alternating projection onto the two sets $\Gamma_{n,m}^{\mathcal{AP}}$ and $\Lambda_{n,m}^{\mathcal{AP}}$ in order to obtain an element (if any) of $\Psi_{m,n}^{\mathcal{AP}}(\mathcal{B}^{\mathcal{AP}}) = \Gamma_{n,m}^{\mathcal{AP}} \cap \Lambda_{n,m}^{\mathcal{AP}}$ associated with the given aperiodic bound $\mathcal{B}^{\mathcal{AP}}$.

Similar to the case of periodic correlation, we use the Frobenius norm as a measure of distance between the two sets. The Frobenius norm projection $\boldsymbol{Z}_{\perp}^{\Lambda}$ of any $\boldsymbol{Z} \in \mathbb{C}^{m^2(2n-1)\times(2n-1)}$ on $\Lambda_{n,m}^{\mathcal{AP}}$ can be obtained as the solution to the optimization problem

$$\begin{aligned} \min_{\boldsymbol{X}_{\perp}, \boldsymbol{Z}_{\perp}^{\Lambda}} \ & \|\boldsymbol{Z} - \boldsymbol{Z}_{\perp}^{\Lambda}\|_F \\ \text{s.t. } & \boldsymbol{Z}_{\perp}^{\Lambda} = \widetilde{\boldsymbol{X}}_{\perp} \circlearrowleft \widetilde{\boldsymbol{X}}_{\perp}^H \\ & \widetilde{\boldsymbol{X}}_{\perp} = \begin{pmatrix} \boldsymbol{X}_{\perp} \\ \boldsymbol{0}_{(n-1)\times m} \end{pmatrix}. \end{aligned} \quad (54)$$



(a) (b) (c)

Fig. 3. $\mathrm{PSL}^{\mathcal{P}}$ of the obtained sequence sets by using the algorithm in Table II (the periodic case), versus sequence length n, and for different set cardinalities m. Gold, Kasami, and Weil sequence sets are used to initialize the algorithm when they exist (pinpointed by arrows).

Note that

$$\begin{aligned}\|\boldsymbol{Z}-\boldsymbol{Z}^{\Lambda}_{\perp}\|_F &= \|\boldsymbol{Z}-\widetilde{\boldsymbol{X}}_{\perp} \circlearrowleft \widetilde{\boldsymbol{X}}^{H}_{\perp}\|_F\\ &= \|\mathcal{G}(\boldsymbol{Z})-vec(\widetilde{\boldsymbol{X}}_{\perp})\, vec^{H}(\widetilde{\boldsymbol{X}}_{\perp})\|_F\\ &= \|\mathcal{M}(\mathcal{G}(\boldsymbol{Z}))-vec(\boldsymbol{X}_{\perp})\, vec^{H}(\boldsymbol{X}_{\perp})\|_F+\text{const.}\end{aligned} \tag{55}$$

where the operator $\mathcal{M}(.)$ collects the entries of the matrix argument corresponding to the non-zero entries of the masking matrix $\boldsymbol{M}$. As a result, the minimizer $\boldsymbol{X}_{\perp}$ of (54) is given by $vec(\boldsymbol{X}_{\perp}) = \sqrt{\eta_1(\mathcal{M}(\mathcal{G}(\boldsymbol{Z})))}\,\boldsymbol{v}_1\,(\mathcal{M}(\mathcal{G}(\boldsymbol{Z})))$, which consequently yields

$$\boldsymbol{Z}^{\Lambda}_{\perp} = \widetilde{\boldsymbol{X}}_{\perp} \circlearrowleft \widetilde{\boldsymbol{X}}^{H}_{\perp} \tag{56}$$

as the optimal projection on $\Lambda^{\mathcal{AP}}_{n,m}$.

The Frobenius norm projection $\boldsymbol{Z}^{\Gamma}_{\perp}$ of any $\boldsymbol{Z} \in \mathbb{C}^{m^2(2n-1)\times(2n-1)}$ on $\Gamma^{\mathcal{AP}}_{n,m}$ can be obtained similarly to that of $\Gamma^{\mathcal{P}}_{n,m}$ with a small modification. Note that the variables α_2 and θ_2 can be calculated by using the same arguments as for $\Gamma^{\mathcal{P}}_{n,m}$. However, the number of non-zero entries in the rows of $\boldsymbol{Z}^{\Gamma}_{\perp}$ is different. Particularly, the exact positions of non-zero entries of $\boldsymbol{Z}^{\Gamma}_{\perp}$ are given by the locations of ones in $\mathcal{G}^{-1}(\boldsymbol{M})$. Therefore, the entries of $\boldsymbol{z}_d$ are given by

$$\boldsymbol{z}_d(k) = \begin{cases} \frac{\alpha_1 e^{j\theta_1}-\alpha_2 e^{j\theta_2}}{|\boldsymbol{\mu}^T\boldsymbol{1}|} & k \in \mathrm{supp}(\boldsymbol{\mu}),\\ 0 & \text{otherwise}\end{cases} \tag{57}$$

where $\boldsymbol{\mu}$ represents the corresponding row in $\mathcal{G}^{-1}(\boldsymbol{M})$, and $\mathrm{supp}(.)$ denotes the set of non-zero locations in the vector argument.

Finally, the steps of the proposed alternating projections, in the periodic and aperiodic cases, are summarized in Table II. Note that in both cases, each iteration of the algorithms has a $\mathcal{O}(m^2n^2)$-complexity. The obtained complexity measure is a direct consequence of the generally large cardinality (i.e., mn) of the data that the algorithms should handle as well as the hardness of the original problem (with m^2n constraints, which should be compared to the fewer constraints (i.e., m^2) for achieving a given peak inner-product level). Due to the practical interest of constrained sequence design, e.g., with finite-alphabet or low-PAR, a modified version of the proposed algorithms that handles such cases is discussed in the Appendix. However, a more extensive discussion of the constrained sequence design is beyond the scope of this paper.

V. Numerical Results

Several numerical examples will be presented to examine the performance of the proposed algorithms for approaching the peak correlation bounds. A main goal of these examples is to determine how close one can get to the peak correlation bounds via the proposed computational tools. The obtained sequence sets are provided online at http://www.anst.uu.se/mojso279/sets.

We employ the suggested algorithm in Table II for different values of (n, m). In the case of periodic correlation, we consider the bound $\mathcal{I}^{\mathcal{P}}_{m,n}$ in (27). Fig. 3 shows the peak periodic correlation ($\mathrm{PSL}^{\mathcal{P}}$) values corresponding to the initializations and the obtained sequence sets along with the bound $\mathcal{I}^{\mathcal{P}}_{m,n}$, for $m \in \{2,3,4\}$ and $n \in \{2,4,5,7,8,10,12,15,16\}$. Note that due to the non-convexity of $\Lambda_{n,m}$, the problem is multi-modal (i.e., it may have many convergence points), and hence, many random initial points might be needed to achieve a certain low peak correlation level. In this example, different random initializations are considered for 40 experiments, and the resultant $\mathrm{PSL}^{\mathcal{P}}$ of the proposed algorithm (in Table II) represents the best outcome of the 40 experiments. To examine the sensitivity to choosing the initial set, well-known sequence sets including Gold, Kasami, and Weil are used as initializing sets for the (m, n) values for which they exist. Such cases are also reported in Fig. 3. It can be observed that $\mathcal{I}^{\mathcal{P}}_{m,n}$ can be practically met in several cases, e.g., for all n with $m = 2$. Furthermore, a considerable decrease of the peak periodic correlation obtained by using the proposed algorithm can be observed in all cases (even for the cases with well-known sets as initialization).

As discussed earlier, if the Welch bound can be met for given (m, n), then the absolute values of all out-of-phase correlations will be identical and have a value equal to the Welch bound. As an example of such behavior, we study the correlation properties of the resultant set for $(m, n) = (2, 32)$. The absolute values of periodic correlations $\{c_{u,v}(k)\}$ are plotted in Fig. 4(a)–(c). In this case, the bound $\mathcal{I}^{\mathcal{P}}_{m,n}$ is nothing but the Welch bound $\mathcal{W}^{\mathcal{P}}_{m,n}$ corresponding to $s = 1$. As expected, all the periodic correlation

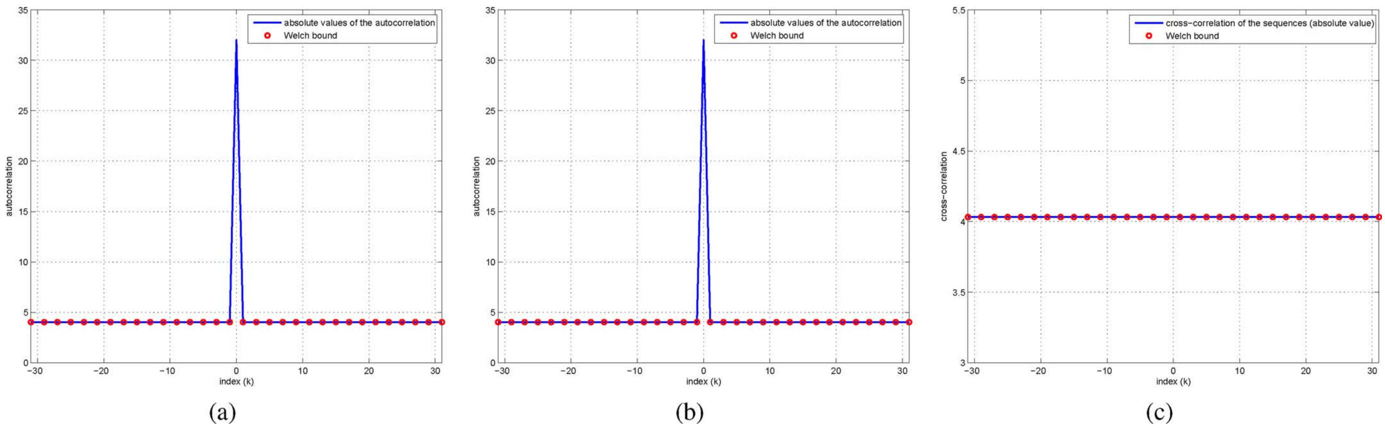


Fig. 4. Correlation levels of the obtained sequence set for $(m, n) = (2, 32)$: (a) autocorrelation of the first sequence, (b) autocorrelation of the second sequence, (c) cross-correlation of the first and the second sequences.

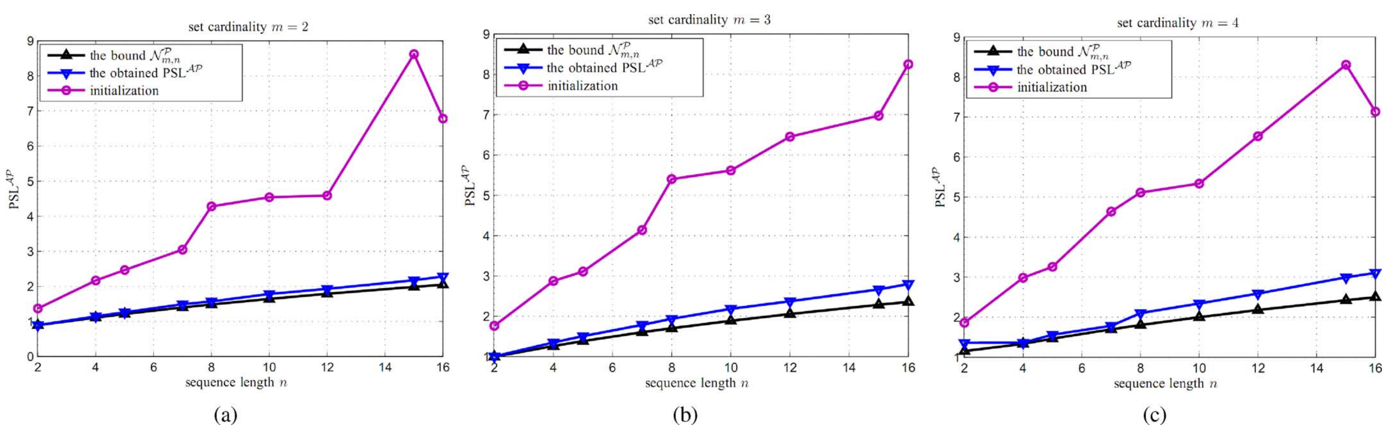


Fig. 5. $\mathrm{PSL}^{\mathcal{AP}}$ of the obtained sequence sets by using the algorithm in Table II (aperiodic case), versus sequence length n, for different set cardinalities m.

levels (excluding the in-phase one) are equal to $\mathcal{W}^{\mathcal{P}}_{m,n} = \frac{32}{\sqrt{63}} \approx 4.03162$.

The mixed bound $\max\{\mathcal{N}^{\mathcal{AP}}_{m,n}, \mathcal{I}^{\mathcal{AP}}_{m,n}\}$ is used to conduct a similar numerical investigation in the aperiodic case. Fig. 5 illustrates the achieved $\mathrm{PSL}^{\mathcal{AP}}$ values by using the proposed alternating projections along with the mixed bound $\max\{\mathcal{N}^{\mathcal{AP}}_{m,n}, \mathcal{I}^{\mathcal{AP}}_{m,n}\}$. As expected, the bound is met for the case $(m, n) = (2, 2)$ (see Proposition 2). For other cases, in which the aperiodic bound cannot be met exactly, significant reductions in the obtained $\mathrm{PSL}^{\mathcal{AP}}$ can be observed compared to the $\mathrm{PSL}^{\mathcal{AP}}$ values corresponding to the initial sets.

VI. Concluding Remarks

Peak correlation bounds have been studied, and the problem of meeting peak periodic and aperiodic correlation bounds has been addressed. The main results can be summarized as follows:

- Welch, Levenstein, and Exponential bounds on peak inner-product level of sequence sets were discussed. Peak correlation bounds were derived based on the peak inner-product bounds.
- An improvement of the peak aperiodic correlation bound was provided.
- Analytical examples of the tightness of the peak correlation bounds were provided in both the periodic and aperiodic cases.
- Two novel algorithms were devised to tackle the problem of approaching a given periodic or aperiodic bound. Numerical examples were provided to show the potential of the proposed methods. In several cases, particularly in the case of periodic correlation, the considered peak correlation bound was met by using the proposed methods. In all examples, a significant decrease in the peak correlations of the designed sets was observed, compared to the PSL of the initial sequence sets (even for initializations by well-known sets such as Gold, Kasami and Weil families).

We believe that more studies are needed to achieve a deeper understanding and formulation of tighter peak correlation bounds. The focus of this work was on studying and on attempting to achieve peak correlation bounds when no extra constraints on the sequences were enforced. However, a modification of the algorithms to handle constrained sequence design was proposed. Extensive studies concerning constrained sequence set design, e.g., sets containing low peak-to-average-power ratio (PAR), unimodular or root-of-unity sequences, can be interesting topics for future research.

Appendix

Proof of Proposition 1: First note that $\mathcal{I}_{2,2}^{\mathcal{P}} = \mathcal{W}_{2,2}^{\mathcal{P}}$ with $s = 1$. We provide the characterization of all sequence sets meeting the peak periodic bound $\mathcal{W}_{2,2}^{\mathcal{P}}$. Let

$$\boldsymbol{X} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \tag{58}$$

and observe that the out-of-phase periodic correlation levels of the two sequences $(a_1 \ a_2)^T$ and $(b_1 \ b_2)^T$ belong to the set $\{|2\Re(a_1 a_2^*)|, |2\Re(b_1 b_2^*)|, |a_1 b_1^* + a_2 b_2^*|, |a_1 b_2^* + a_2 b_1^*|\}$. The necessary and sufficient conditions for meeting the Welch peak correlation bound $\mathcal{W}_{m,n}^{\mathcal{P}}$ (for $s = 1$) imply that all the elements in the latter set should be equal to $\mathcal{W}_{2,2}^{\mathcal{P}} = \frac{2}{\sqrt{3}}$. The structure of $\boldsymbol{X}$ for meeting $\mathcal{I}_{2,2}^{\mathcal{P}}$ can be studied as follows.

By considering the energy constraint as well as the constraint $|2\Re(a_1 a_2^*)| = \frac{2}{\sqrt{3}}$, we obtain that

$$\begin{cases} a_1 = \frac{1}{2}\left(e^{j\phi_1}\sqrt{2 \pm \frac{2}{\sqrt{3}}} + e^{j\phi_0}\sqrt{2 \mp \frac{2}{\sqrt{3}}}\right) \\ a_2 = \frac{1}{2}\left(e^{j\phi_1}\sqrt{2 \pm \frac{2}{\sqrt{3}}} - e^{j\phi_0}\sqrt{2 \mp \frac{2}{\sqrt{3}}}\right) \end{cases} \tag{59}$$

for some phase angles ϕ_0 and ϕ_1. In a similar manner, for $(b_1 \ b_2)^T$ we have that

$$\begin{cases} b_1 = \frac{1}{2}\left(e^{j\theta_1}\sqrt{2 \mp \frac{2}{\sqrt{3}}} + e^{j\theta_0}\sqrt{2 \pm \frac{2}{\sqrt{3}}}\right) \\ b_2 = \frac{1}{2}\left(e^{j\theta_1}\sqrt{2 \mp \frac{2}{\sqrt{3}}} - e^{j\theta_0}\sqrt{2 \pm \frac{2}{\sqrt{3}}}\right) \end{cases} \tag{60}$$

with θ_0 and θ_1 being auxiliary phase angles. The result in (60) is also based on verifying that the assumption of the same order in "$\pm$" signs of a_1 and b_1 (as well as a_2 and b_2) in (59) and (60) violates the satisfaction of the following constraints

$$\begin{cases} |a_1 b_1^* + a_2 b_2^*| = \frac{2}{\sqrt{3}} \\ |a_1 b_2^* + a_2 b_1^*| = \frac{2}{\sqrt{3}} \end{cases} \tag{61}$$

On the other hand, for a_1, a_2 as given in (59) and b_1, b_2 as given in (60), the constraints in (61) lead to the following equation

$$\left| e^{j(\phi_1 - \theta_1)} \sqrt{4 - 4/3} \pm e^{j(\phi_0 - \theta_0)} \sqrt{4 - 4/3} \right| = \frac{4}{\sqrt{3}} \tag{62}$$

which implies

$$\cos\left((\phi_1 - \theta_1) - (\phi_0 - \theta_0)\right) = 0. \tag{63}$$

Therefore, $\Psi_{2,2}^{\mathcal{P}}(\mathcal{S}_{2,2})$ includes $\boldsymbol{X}$ characterized by (59) and (60), and with the auxiliary phase angles such that

$$(\phi_1 - \theta_1) = (\phi_0 - \theta_0) + (2k+1)\frac{\pi}{2}, \tag{64}$$

for some $k \in \mathbb{Z}$.

Proof of Proposition 2: Observe that $\mathcal{I}_{2,2}^{\mathcal{AP}} = \mathcal{W}_{2,2}^{\mathcal{AP}} = \frac{2}{\sqrt{5}}$ (with $s = 1$) and let

$$\boldsymbol{X} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}. \tag{65}$$

In what follows, a characterization of $\boldsymbol{X}$ that meets the aperiodic bound $\mathcal{W}_{2,2}^{\mathcal{AP}}$ is derived. The set of aperiodic out-of-phase correlation levels $\{|r_{u,v}(k)|\}$ of the columns of $\boldsymbol{X}$ is given by $\{|a_1 a_2^*|, |b_1 b_2^*|, |a_1 b_1^* + a_2 b_2^*|, |a_2 b_1^*|, |a_1 b_2^*|\}$. By using the necessary and sufficient conditions for meeting the Welch peak correlation bound, we conclude that

$$\begin{cases} |a_1| = |b_1| \\ |a_2| = |b_2| \end{cases} \tag{66}$$

and that $|a_1||a_2| = \frac{2}{\sqrt{5}}$. By applying the energy constraint we obtain

$$|a_2|^4 - 2|a_2|^2 + \frac{4}{5} = 0. \tag{67}$$

The solutions to (67) are given by $|a_2| = \sqrt{1 \pm \frac{\sqrt{5}}{5}}$, which also yields $|a_1| = \frac{2}{\sqrt{5 \pm \sqrt{5}}}$.

To determine the phase angles of the sequences in $\boldsymbol{X}$, we employ the equation $|a_1 b_1^* + a_2 b_2^*| = \frac{2}{\sqrt{5}}$ which results in

$$\left| |a_1|^2 \, e^{j(\theta_1 - \phi_1)} + |a_2|^2 \, e^{j(\theta_2 - \phi_2)} \right| = \frac{2}{\sqrt{5}} \tag{68}$$

with $\theta_1, \theta_2, \phi_1$, and ϕ_2 being the phase angles of a_1, a_2, b_1, and b_2, respectively. Equation (68) can be simplified to obtain

$$\cos\left((\theta_1 - \phi_1) - (\theta_2 - \phi_2)\right) = -1. \tag{69}$$

Consequently, $\boldsymbol{X}$ meeting $\mathcal{W}_{2,2}^{\mathcal{AP}}$ have the structure

$$\boldsymbol{X} = \begin{pmatrix} \frac{2}{\sqrt{5 \pm \sqrt{5}}} e^{j\theta_1} & \frac{2}{\sqrt{5 \pm \sqrt{5}}} e^{j\phi_1} \\ \sqrt{1 \pm \frac{\sqrt{5}}{5}} e^{j\theta_2} & \sqrt{1 \pm \frac{\sqrt{5}}{5}} e^{j\phi_2} \end{pmatrix} \tag{70}$$

such that $(\theta_1 - \phi_1) = (\theta_2 - \phi_2) + 2(k+1)\pi$ where $k \in \mathbb{Z}$.

Modified Projections for Constrained Sequence Design: With some modifications, the alternating projections proposed in Section IV can be used for designing constrained sequences such as cases with finite-alphabet or low PAR. Note that in constrained cases, finding the optimal projection on the two sets $\Gamma_{n,m}$ and $\Lambda_{n,m}$ could be more complicated. However, the convergence of the projections is guaranteed if the distance between the latest projection points on the two sets is decreasing. In the following, we discuss a set of modifications that can enable the proposed approaches in Section IV to tackle the constrained case.

Let $\boldsymbol{X} \in \Omega_{n,m}$ represent the required sequence structure. We revise the definition of $\Lambda_{n,m}$ by replacing the constraint $\boldsymbol{X} \in \mathbb{C}^{n \times m}$ with $\boldsymbol{X} \in \Omega_{n,m}$. Therefore, finding the projection on $\Lambda_{n,m}$ becomes equivalent to minimizing (44) and (55), in the periodic and aperiodic cases, respectively, but for $\boldsymbol{X} \in \Omega_{n,m}$. Hereafter, we study the periodic case as the extension to the aperiodic case is straightforward. Due to the fact that $\boldsymbol{X}$ has a fixed power (or Frobenius norm), minimizing (44) is equivalent to:

$$\begin{aligned} \max_{\boldsymbol{X}_\perp} \quad & vec^H(\boldsymbol{X}_\perp)\, \boldsymbol{Z}'\, vec(\boldsymbol{X}_\perp) \\ \text{s.t.} \quad & \boldsymbol{X}_\perp \in \Omega_{n,m} \end{aligned} \tag{71}$$

where $\boldsymbol{Z}' = \lambda \boldsymbol{I} + \mathcal{G}(\boldsymbol{Z})$ is positive-definite. More important, any increase in the objective function of (71) leads to a decrease of (44). Interestingly, increasing quadratic functions such as the one in (71), over constrained vector sets can be dealt with conveniently via the power-method like iterations proposed in [32] and [33]. Namely, considering the previously known projection on $\Lambda_{n,m}^{\mathcal{P}}$ as initialization ($\boldsymbol{X}_{\perp}^{(0)}$), the quadratic function in (71) can be increased monotonically by using the iterations:

$$\min_{\boldsymbol{X}_{\perp}^{(t+1)} \in \Omega_{n,m}} \left\| vec\left(\boldsymbol{X}_{\perp}^{(t+1)}\right) - \boldsymbol{Z}'\, vec\left(\boldsymbol{X}_{\perp}^{(t)}\right) \right\|_2 \qquad (72)$$

The solution to (72) for unimodular or p-ary vector sets can be obtained analytically. In low-PAR scenarios, (72) can be solved using an efficient recursive algorithm developed in [34].

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