PERFECT ROOT-OF-UNITY CODES WITH PRIME-SIZE ALPHABET

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ABSTRACT

In this paper, Perfect Root-of-Unity Codes (PRUCs) with entries in $\alpha_p = \{x \in \mathbb{C} \mid x^p = 1\}$ where p is a prime are studied. A lower bound on the number of distinct phases in PRUCs over α_p is derived. We show that PRUCs of length $L \geq p(p-1)$ must use all phases in α_p. It is also shown that if there exists a PRUC of length L over α_p then p divides L. We derive equations (which we call principal equations) that give possible lengths of a PRUC over α_p together with their phase distribution. Using these equations, we prove for example that the length of a 3-phase perfect code must be of the form $L = \frac{1}{4}\left(9h_1^2 + 3h_2^2\right)$ for $(h_1, h_2) \in \mathbb{Z}^2$ and we also give the exact number of occurences of each element from α_3 in the code. Finally, all possible lengths (≤ 100) of PRUCs over α_5 and α_7 together with their phase distributions are provided.

Index Terms— Perfect codes, Root-of-unity codes, Periodic autocorrelation, Phase distribution

1. INTRODUCTION

Perfect Root-of-Unity Codes (PRUCs), also known as perfect N-phase [1] or polyphase [2] codes, are unimodular codes with entries in $\alpha_N = \{x \in \mathbb{C} \mid x^N = 1\}$ and the property that all their out-of-phase periodic autocorrelations are equal to zero. These codes are of interest in several applications including communication systems for Frequency-Hopping Spread-Spectrum Multiple-Access (FH/SSMA) and Direct-Sequence Spread-Spectrum Multiple-Access (DS/SSMA) [3]. Other applications include pulse compression for continuous-wave radars [3], fast startup equalization and channel estimation [1].

Due to implementation issues it is usually desirable that the entries of the code are from a small alphabet. With this fact in mind, it is interesting to note that 4, 6 and 11 out of the first 8, 16 and 32 natural numbers, respectively, are prime. The study of PRUCs with prime-size alphabets is important not only because of this relatively high density of prime numbers in small alphabet sizes, but also because of the role of prime numbers as building blocks of natural numbers. A similar building block property can be seen in the PRUC case: let $n = mk$ where m and k are co-prime and assume that there exist PRUCs $\boldsymbol{u} = (u_0, \cdots, u_{m-1})$ and $\boldsymbol{v} = (v_0, \cdots, v_{k-1})$ with alphabet size $m^\dagger$ and $k^\dagger$ respectively; then $\boldsymbol{w} = (w_0, \cdots, w_{n-1})$ where $w_l = u_{(l \bmod m)} v_{(l \bmod k)}$ is a PRUC with alphabet size $n^\dagger = m^\dagger k^\dagger$. This construction is known as Chinese Remainder Theorem (CRT) construction or simply as the direct product [4].

A general computational method to find perfect (or almost-perfect) codes is proposed in [5]. In [3], a unified approach covering all known construction methods of PRUCs was derived. Besides construction methods, some publications (e.g. [1]) have introduced and used the following necessary condition on PRUCs: if $\boldsymbol{x} = \{x_l\}_{l=0}^{L-1}$ is a PRUC of length L then

$$\left| \sum_{l=0}^{L-1} x_l \right| = \sqrt{L} \tag{1}$$

This necessary condition follows directly from the fact that the DFT of a PRUC has a constant magnitude (note that the DFT value at zero frequency is the sum of the code).

In this work, PRUCs with a prime-size alphabet are studied. In Section 2, the phase distribution of PRUCs over α_p is discussed. We show that if there exists such a code over α_p then p must divide the code length. Furthermore, a set of equations (which we call the principal equations) are derived that can be used to show the non-existence of PRUCs for some lengths and to give the exact phase distribution for possible lengths. As an example for the geometrical solution of the principal equations introduced in Section 2, the case of $p = 3$ (leading to 3-phase codes) is discussed in some detail. Finally, section 3 concludes the paper.

Notation: We use bold lowercase letters for vectors/codes and bold uppercase letters for matrices. $(.)^T$ and $(.)^*$ denote the vector/matrix transpose and complex conjugate respectively. $\mathbf{1}_n$ is the all one vector of length n. $\boldsymbol{e}_l^{(n)}$ is the l^{th} standard basis vector in $\mathbb{R}^n$. $\|\boldsymbol{x}\|_n$ or the l_n-norm of the vector $\boldsymbol{x}$ is defined as $(\sum_k |x(k)|^n)^{\frac{1}{n}}$ where $\{x(k)\}$ are the entries of $\boldsymbol{x}$. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ represent the set of natural, integer, real and complex numbers respectively. $\mathbb{Z}_n$ represents the set $\{0, 1, \cdots, n-1\}$. Finally, p denotes a prime number throughout the paper.

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2. PHASE STUDY

In this section, we study the phase distribution of PRUCs over the alphabet α_p of prime size. Let $\boldsymbol{x} = \{x_l\}_{l=0}^{L-1} = \left\{e^{j\frac{2\pi}{p}k_l}\right\}_{l=0}^{L-1}$ be an L-length PRUC over α_p. All k_l are in $\mathbb{Z}_p$ and we call them the integer phases of the code. The periodic autocorrelation of $\boldsymbol{x}$ at lag $u \in \mathbb{Z}_L$ is defined as

$$R_u = \sum_{l=0}^{L-1} e^{j\frac{2\pi}{p}(k_l - k_{l+u})} = \begin{cases} L & u = 0 \\ 0 & u \in \mathbb{Z}_L - \{0\} \end{cases} \tag{2}$$

where the indices of $\{k_l\}$ are used in a periodic manner (i.e. $mod\ L$). It is interesting to note that R_u is a summation of terms which are also in α_p. Theorem 1 paves the way for using this observation:

Theorem 1. *If $\sum_{k=0}^{p-1} a_k e^{j\frac{2\pi}{p}k} = 0$ for some $a_k \in \mathbb{Z}$ and prime p, then all a_k must be identical.*

Proof: We prove a more general form of the theorem by using some results from the theory of algebraic numbers and minimal polynomials: a number is called algebraic iff it is a root of a polynomial with rational coefficients. The minimal polynomial $P_{min}(x)$ of an algebraic number x_0 is the polynomial with rational coefficients, minimum degree and the leading coefficient equal to 1 which satisfies $P_{min}(x_0) = 0$. It is known [6] that the minimal polynomial of the primitive p^{th} root of unity, p prime, is of the unique form $\sum_{k=0}^{p-1} x^k$. We conclude that if $P(e^{j\frac{2\pi}{p}}) = 0$ for a polynomial $P(x)$ with rational coefficients and degree $d = p-1$ then $P(x)$ must be equal to $w\sum_{k=0}^{p-1} x^k$ for some rational scalar w. This implies that all coefficients of $P(x)$ must be equal, which completes the proof. ■

The next two corollaries follow directly from the above theorem:

Corollary 1. *If there exists a PRUC of length L over α_p then $p|L$.*

Proof: Let $u \in \mathbb{Z}_L - \{0\}$. Then it follows from (2) and Theorem 1 that $R_u = m\sum_{k=0}^{p-1} e^{j\frac{2\pi}{p}k} = 0$ where $m = L/p$ must be an integer. ■

In the sequel we use the notation $L = mp$, $m \in \mathbb{N}$, for the length of PRUCs over α_p.

Corollary 2. *Let $\boldsymbol{x} = \left\{e^{j\frac{2\pi}{p}k_l}\right\}_{l=0}^{mp-1}$ be a PRUC of length $L = mp$ over α_p. Then for every $s \in \mathbb{Z}_p$ and $u \in \mathbb{Z}_L - \{0\}$, there exist exactly m distinct integers $\{l\}$ such that $k_l \equiv k_{l+u} + s\ (mod\ p)$.*

Proof: We only need to observe that, according to Theorem 1, all sums in (2) for $\{R_u\}_{u\in\mathbb{Z}_L-\{0\}}$ must have exactly m terms equal to $e^{j\frac{2\pi}{p}s}$ for every $s \in \mathbb{Z}_p$. ■

According to Corollary 2, for every $u \in \mathbb{Z}_L - \{0\}$, there exist exactly m distinct integers $\{l\}$ such that $k_l = k_{l+u}$. Let $\boldsymbol{\Phi}_x$ be the circulant matrix made from integer phases $\{k_l\}$ of the code $\boldsymbol{x}$,

$$\boldsymbol{\Phi}_x = \begin{pmatrix} k_0 & k_1 & \cdots & k_{mp-1} \\ k_{mp-1} & k_0 & \cdots & k_{mp-2} \\ \vdots & \vdots & \ddots & \vdots \\ k_1 & k_2 & \cdots & k_0 \end{pmatrix} \tag{3}$$

For the l^{th} column of $\boldsymbol{\Phi}_x$, consider the location of the entries which are equal to k_l ($l = 0, \cdots, p-1$). Considering these locations for all columns, we build an $mp \times mp$ equivalence matrix $\boldsymbol{\Phi}_e$ whose entries in the mentioned locations are 1; otherwise they are 0. Based on Corollary 2, all rows of $\boldsymbol{\Phi}_e$ have exactly m ones except the first row whose all entries are one. If μ_k represents the number of times that $e^{j\frac{2\pi}{p}k}$ occurs in the sequence then $\sum_{k=0}^{p-1}\mu_k = mp$. As we discussed above, by considering the rows of $\boldsymbol{\Phi}_e$, we conclude that there are $mp + m(mp-1)$ ones in $\boldsymbol{\Phi}_e$. On the other hand, since every integer phase $k \in \mathbb{Z}_p$ gives μ_k columns with μ_k ones in each of them, the number of ones in $\boldsymbol{\Phi}_e$ is equal to $\sum_{k=0}^{p-1}\mu_k^2$ and therefore $\sum_{k=0}^{p-1}\mu_k^2 = mp + m(mp-1)$. Now let us assume that t of $\{\mu_k\}$ are nonzero. From the Cauchy-Schwarz inequality we have $\sum_{k=0}^{p-1}\mu_k^2 \geq \frac{1}{t}\left(\sum_{k=0}^{p-1}\mu_k\right)^2$ which implies

$$m^2 p + m(p-1) \geq \frac{(mp)^2}{t} \tag{4}$$

and as a result

$$t \geq \frac{mp^2}{(m+1)p-1} \tag{5}$$

The above lower bound shows that as m increases, a larger number of phases from α_p are needed to build a PRUC of length $L = mp$. For sufficiently large values of m we need all phases:

Corollary 3. *For $m \geq p-1$, all phase values must be used in a PRUC.*

Proof: This is a direct result of the lower bound in (5). ■

Interestingly, the length condition of Corollary 3 covers codes of length p^2 over α_p obtained from construction methods such as Frank, P1, P2 and Px [7].

Now, for every $s \in \mathbb{Z}_p - \{0\}$, let us build the $\boldsymbol{\Phi}_e$ matrix[1] as follows: by finding the locations of the entries $k_{l'}$ in the l^{th} column of $\boldsymbol{\Phi}_x$ such that $k_{l'} \equiv k_l + s\ (mod\ p)$, we represent these locations in $\boldsymbol{\Phi}_e$ by 1, and by 0 otherwise. Based on Corollary 2, for every $u \in \mathbb{Z}_L - \{0\}$, there exist exactly m

[1] The dependency of $\{\boldsymbol{\Phi}_e\}$ matrices on s is not explicitly shown for notational simplicity.

p	Length (L)	Phase distribution $\{\mu_k\}_{k=0}^{p-1}$
5	5	$(2,1,2,0,0)$ $(2,2,0,1,0)$
	20	$(6,4,6,2,2)$ $(6,6,2,4,2)$
	25	$(6,6,6,6,1)$ $(9,4,4,4,4)$
	45	$(12,9,12,6,6)$ $(12,12,6,9,6)$
	55	$(16,12,10,10,7)$ $(16,10,12,7,10)$ $(15,6,10,12,12)$ $(15,12,6,12,10)$
	80	$(20,16,20,12,12)$ $(20,20,12,16,12)$
	100	$(28,18,18,18,18)$ $(22,22,22,22,12)$
7	7	$(2,2,1,0,0,2,0)$
	28	$(6,6,4,2,2,6,2)$ $(7,5,5,2,5,2,2)$
	49	$(13,6,6,6,6,6,6)$ $(8,8,8,8,8,8,1)$
	56	$(11,10,10,5,10,5,5)$ $(11,11,6,11,6,6,5)$
	63	$(12,12,9,6,6,12,6)$
	98	$(17,17,17,10,17,10,10)$ $(18,18,11,18,11,11,11)$

Table 1. All possible lengths (less than or equal to 100) of PRUCs over α_p for $p = 5$ and 7 together with phase distributions.

distinct integers $\{l\}$ such that $k_{l+u} \equiv k_l + s \ (mod\ p)$. Therefore the $\mathbf{\Phi}_e$ matrix has exactly m ones in each of its rows except for the first row which is all zero. This implies that $\mathbf{\Phi}_e$ has $m(mp-1)$ ones. On the other hand, the number of ones in $\mathbf{\Phi}_e$ is equal to $\sum_{k=0}^{p-1} \mu_k \mu_{k+s}$ as it equals the number of all pairs with the property $k_{l+u} \equiv k_l + s \ (mod\ p)$. Therefore, all out-of-phase correlations $\{\sum_{k=0}^{p-1} \mu_k \mu_{k+s}\}_{s \in \mathbb{Z}_p - \{0\}}$ of the $\{\mu_k\}$ sequence are equal to $m(mp-1)$.

If we define $r_k = \mu_k - m$, we obtain a set of equations which we call *the principal equations*:

$$\begin{cases} \sum_{k=0}^{p-1} r_k = 0 \\ \sum_{k=0}^{p-1} r_k^2 = m(p-1) \\ \sum_{k=0}^{p-1} r_k r_{k+s} = -m, \quad s \in \mathbb{Z}_p - \{0\} \end{cases} \tag{6}$$

It is interesting to note that if $\{r_k\}$ is a solution to (6), then $\{-r_k\}$, $\{r_{-k}\}$ and $\{r_{k+l}\}$ where $l \in \mathbb{Z}_p$ are also valid solutions to (6). In other words, the set of principal equations induces a certain type of equivalence class on its solutions. Next, we note that the unimodular perfect codes enjoy a similar set of equivalence properties: let $\boldsymbol{x}$ be a unimodular perfect code, then $\boldsymbol{x}^*$ and $e^{j\phi}\boldsymbol{x}$ (where ϕ can be chosen arbitrarily) are also unimodular perfect codes. This shows that given a solution $\{r_k\}$ to the principal equations, the solutions $\{r_{-k}\}$ and $\{r_{k+l}\}$ do not lead to new PRUCs. In contrast, the solution $\{-r_k\}$ might lead to new codes. Table 1 shows all feasible lengths (less than or equal to 100) and their corresponding phase distribution obtained from the principal equations for $p = 5$ and 7. Using the equivalence properties of PRUCs, the $\{\mu_k\}$ sequences are circularly shifted such that μ_0 take the maximum value among all $\{\mu_k\}$. The following discussion is devoted to a geometrical study of the problem.

Let $\boldsymbol{r}_0 = (r_0, \cdots, r_{p-1})^T$ and also let $\boldsymbol{r}_k$ represent the circularly shifted version of $\boldsymbol{r}_0$ by $k \in \mathbb{Z}_p$. The principal equations can be rephrased as follows over all vectors $\{\boldsymbol{r}_k\}$:

$$\begin{cases} \mathbf{1}_p^T \boldsymbol{r}_k = 0 \\ \|\boldsymbol{r}_k\|_2 = \sqrt{m(p-1)} \\ \boldsymbol{r}_k^T \boldsymbol{r}_l = -m, \quad k \neq l \end{cases} \tag{7}$$

The angle between each pair of vectors $\{(\boldsymbol{r}_k, \boldsymbol{r}_l)\}_{k \neq l}$ is

$$\theta = \arccos\left(\frac{\boldsymbol{r}_k^T \boldsymbol{r}_l}{\|\boldsymbol{r}_k\|_2 \|\boldsymbol{r}_l\|_2}\right) = \arccos\left(\frac{-1}{p-1}\right) \tag{8}$$

Therefore, $\{\boldsymbol{r}_k\}_{k \in \mathbb{Z}_p}$, form a set of p vectors lying in a $(p-1)$-dimensional space which is the hyperplane orthogonal to $\mathbf{1}_p$ and the angle between each pair of them is the value given in (8). We further note that the structure made by connecting all vertices pointed by $\{\boldsymbol{r}_k\}$ is a known multi-dimensional object called a *Regular Simplex* [8]. Such structures are shown in Fig. 1 for one, two and three dimensions. The reference

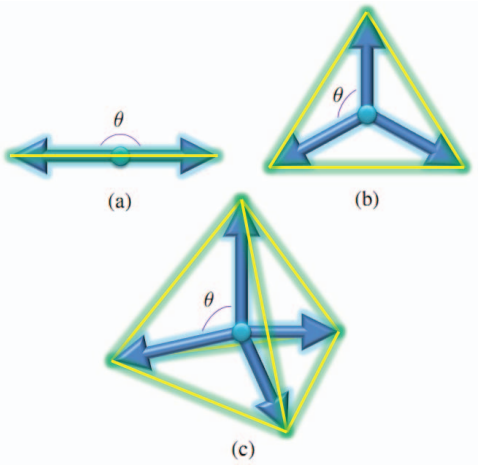

Fig. 1. (a-c) Regular Simplexes in one, two and three dimensional space. In n dimensions they can be characterized with $n+1$ vectors with the same l_2-norm and also the same angle between them as described in Eq. (8).

[8] suggests $\{\boldsymbol{e}_k^{(p)}\}$ (i.e. the standard basis) as vertices of a regular simplex of edge $\sqrt{2}$ lying in the hyperplane $\mathbf{1}_p^T \boldsymbol{x} = 1$. Therefore, a regular simplex characterized by some vectors $\{\widetilde{\boldsymbol{r}}_k\}$ with its center at origin and points with l_2-norm equal to $\sqrt{m(p-1)}$ lying in the hyperplane $\mathbf{1}_p^T \boldsymbol{x} = 0$ can be obtained from $\{\boldsymbol{e}_k^{(p)}\}$ by a translation and scaling. As every two regular simplexes with their center at origin can be obtained from each other by a set of rotations, we can find $\{\boldsymbol{r}_k\}$ vectors by rotations of $\{\widetilde{\boldsymbol{r}}_k\}$ such that their end lie at the integral lattice. Note that as the regular simplex made by $\{\widetilde{\boldsymbol{r}}_k\}$ is in a $(p-1)$-dimensional space, its rotation could be parametrized with $(p-2)$ angles $\psi_0, \cdots, \psi_{p-3}$ and as a result the vectors $\{\boldsymbol{r}_k\}$ could be written as a function of $\sin\psi_k$ and $\cos\psi_k$ for $k \in \mathbb{Z}_{p-2}$. Taking into consideration the fact that $\{\boldsymbol{r}_k\}$ are integral, this gives a closed form solution for m (and as a result a closed form solution for the code length) and also for the phase distribution.

As an example for using the regular simplex to solve the principal equations we study the case of $p = 3$: for 3-phase perfect codes, the $\{\boldsymbol{r}_k\}$ make a two dimensional regular simplex orthogonal to $\mathbf{1}_3$, which has 3 vectors and each two of

them have an angle of $\frac{2\pi}{3}$. The structure of this regular simplex is shown in Fig. 1(b). Let $\boldsymbol{R}_{\mathbf{1}_3}$ be the unitary rotation matrix which maps $\mathbf{1}_3$ to $\sqrt{3}\boldsymbol{e}_3^{(3)}$. Also let

$$\boldsymbol{r}'_k = \sqrt{2m}\left(\cos\left(\frac{2k\pi}{3}+\psi\right), \sin\left(\frac{2k\pi}{3}+\psi\right), 0\right)^T \quad (9)$$

for $k \in \mathbb{Z}_3$ and $\psi \in [0, 2\pi)$. Therefore, $\boldsymbol{r}_k$ is equal to $\boldsymbol{R}_{\mathbf{1}_3}^{-1}\boldsymbol{r}'_k$ for some ψ. This implies that $\boldsymbol{r}_k \in \mathbb{Z}^3$ is of the form

$$\sqrt{2m}\begin{pmatrix} \frac{\sqrt{2}}{2}\cos\left(\frac{2k\pi}{3}+\psi\right) - \frac{\sqrt{6}}{6}\sin\left(\frac{2k\pi}{3}+\psi\right) \\ \frac{\sqrt{6}}{3}\sin\left(\frac{2k\pi}{3}+\psi\right) \\ -\frac{\sqrt{2}}{2}\cos\left(\frac{2k\pi}{3}+\psi\right) - \frac{\sqrt{6}}{6}\sin\left(\frac{2k\pi}{3}+\psi\right) \end{pmatrix}$$

for $k \in \mathbb{Z}_3$. As $\{\boldsymbol{r}_k\}$ are the circularly shifted versions of each other, it is sufficient to study one of them. For $k = 0$, we infer that both $h_1 = 2\sqrt{\frac{m}{3}}\sin\psi$ (which is the second entry of $\boldsymbol{r}_0$) and $h_2 = 2\sqrt{m}\cos\psi$ (which is the difference between the first and the third entry of $\boldsymbol{r}_0$) must be integers. We conclude that $3h_1^2 + h_2^2 = 4m$ and $\boldsymbol{r}_0 = \frac{1}{2}\left((h_2-h_1), 2h_1, -(h_2+h_1)\right)^T$. Therefore, the code length must be of the form $L = \frac{1}{4}\left(9h_1^2 + 3h_2^2\right)$ and the phase distribution is given by

$$\frac{1}{4}\left(3h_1^2+h_2^2\right)\mathbf{1}_3 + \frac{1}{2}\begin{pmatrix} (h_2-h_1) \\ 2h_1 \\ -(h_2+h_1) \end{pmatrix} \quad (10)$$

for integers h_1 and h_2.

Finally, let S be the sum of entries of a PRUC, i.e. $S = \sum_{l=0}^{mp-1} e^{j\frac{2\pi}{p}k_l}$. For $\{r_k\}$ satisfying the principal equations:

$$\sum_{k=0}^{p-1} r_k r_{k+s} = \left(\sum_{k=0}^{p-1} r_k^2\right) - mp = -m, \quad s \in \mathbb{Z}_p - \{0\} \quad (11)$$

We have $|S|^2 = \left|\sum_{k=0}^{p-1}\mu_k e^{j\frac{2\pi}{p}k}\right|^2 = \left|\sum_{k=0}^{p-1} r_k e^{j\frac{2\pi}{p}k}\right|^2$ and as a result of (11),

$$|S|^2 = \sum_{s=0}^{p-1}\left(\sum_{k=0}^{p-1} r_k r_{k+s}\right)e^{j\frac{2\pi}{p}s} = mp \quad (12)$$

which shows that satisfaction of the principal equations implies the necessary condition in (1).

3. CONCLUDING REMARKS

Perfect root-of-unity codes with prime-size alphabets have been studied. A lower bound on the number of distinct phases which must be used in a PRUC over α_p was derived. The lower bound was used to show that for PRUCs of length $L \geq p(p-1)$ over α_p, the code must use all phase values. Guidelines to find possible lengths (L) of PRUCs over α_p were given. It was shown that p must divide L and there should exist a phase distribution $\{\mu_k\}$ (introduced in Section 2) satisfying the principal equations for which $\sum_{k=0}^{p-1}\mu_k = L$. A geometrical analytical method to solve the principal equations was introduced for a specific p using the regular simplex. It was shown that satisfaction of the principal equations implies satisfaction of the previously known necessary condition (1), and therefore that the latter condition is weaker than the principal equations.

We conclude the paper with two remarks. First of all, satisfaction of the principal equations is necessary but not sufficient for a PRUC. The necessity induced by the principal equations guarantees that if a code exists over α_p then it will have a specific length and phase distribution. To the best of our knowledge, these new results can be used to show the impossibility of some lengths and to significantly reduce the size of search space of PRUCs in general. Finally, although Theorem 1 shows that a uniform distribution of phases leads to perfect codes, for *almost-perfect* codes we can focus on *almost-uniform* distributions. A clear possibility here can be outlined as follows: for lengths for which a perfect code does not exist, one can try to build codes with a phase distribution which approximately satisfies the principal equations.

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