DESIGN OF PERFECT PHASE-QUANTIZED SEQUENCES WITH LOW PEAK-TO-AVERAGE-POWER RATIO

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ABSTRACT

Perfect phase-quantized unimodular sequences with entries in $\{x \in \mathbb{C} \mid x^m = 1\}$ have optimal peak-to-average-power ratio (PAR); however, they are extremely rare. For active sensing or communication systems which are able to tolerate sub-optimal PAR values, we show how to construct phase-quantized sequences possessing both virtually perfect periodic autocorrelation and low PAR. Numerical examples are provided to illustrate the performance of the proposed methods.

Index Terms— Periodic autocorrelation, Perfect sequences, Phase-quantized sequences, Peak-to-average-power ratio (PAR).

1. INTRODUCTION

Let $\boldsymbol{x} = \{x_l\}_{l=0}^{n-1}$ be a sequence in the complex field. The periodic autocorrelation of $\boldsymbol{x}$ is defined as

$$R_u = \sum_{l=0}^{n-1} x_l x_{l+u}^*, \quad 0 \le u < n \tag{1}$$

where the indices are used in a periodic manner (i.e. $\bmod\ n$). The design of sequences with good correlation properties is usually considered when small out-of-phase (i.e. $u \neq 0$) autocorrelation lags are required. Several metrics are defined to measure the goodness of such sequences, namely the integrated sidelobe level ISL $\triangleq \sum_{u=1}^{n-1} |R_u|^2$and the peak sidelobe level PSL $\triangleq \max\{|R_u|\}_{u=1}^{n-1}$.

The sequence $\boldsymbol{x}$ is called *perfect* iff

$$R_u = \begin{cases} E & u \equiv 0 \pmod n, \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

where E represents the energy of the sequence. Perfect sequences are of interest in active sensing and communication applications [1][2]. In particular, perfect unimodular sequences (where $x_l = e^{j\phi_l}$ for all l) are of special interest because of their optimal (i.e. unity) peak-to-average-power ratio (PAR). The PAR metric is defined as

$$\text{PAR} \triangleq \frac{\max_l |x_l|^2}{\frac{1}{n}\sum_{l=0}^{n-1} |x_l|^2}. \tag{3}$$

It is known (e.g., see [3]) that there exists an infinite number of independent perfect unimodular sequences for lengths n containing a squared factor (i.e. there is an integer $q > 1$ such that $q^2|n$) and finitely many such sequences for square-free lengths n.

Due to implementation issues, it is usually desirable that the entries of the sequence are from a finite alphabet (and particularly small alphabet sizes). As a result, phase-quantized unimodular sequences (where $x_l = e^{j\frac{2\pi}{m}k_l}$ for $0 \le k_l \le m-1$) have been considered and studied in the literature. Perfect phase-quantized unimodular sequences are typically obtained by analytical construction methods such as Zadoff, Chu, Golomb, P1-4 and Px [3][4]. However, in contrast to unimodular sequences, such sequences are rare; moreover, there exists a very restricted set of alphabet sizes (m) for which it is possible to construct a perfect phase-quantized unimodular sequence of given length [4][5]. On the other hand, we note that in many practical applications some sub-optimality in PAR is tolerable. This practical convenience can be exploited to trade-off the optimal PAR for improved correlation properties. In particular, it could be interesting if one can design virtually perfect phase-quantized sequences with tolerable PAR values. This goal can be achieved by: (i) choosing a suitable set of quantized-phase values, and (ii) letting the absolute value of the entries of the sequence to slightly vary from one (in the sequel, the sequence energy is assumed to be equal to its length n).

In this paper, two different approaches are proposed that can be used to design perfect phase-quantized sequences with low PAR. The key contributions of the proposed methods are: (i) they allow small alphabet sizes $2 \le m \ll n$, (ii) they do not restrict the length or alphabet sizes (in contrast to almost all perfect sequence construction methods), and (iii) they provide many phase-quantized sequences therefore circumventing the rareness dilemma of perfect phase-quantized unimodular sequences. Note that the availability of many perfect

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phase-quantized sequences is useful for communication or sensing systems working in hostile environments which need hard-to-infer sequences in order to avoid detection or jamming.

The rest of this paper is organized as follows. Section 2 describes the phase selection procedure. Section 3 introduces an extension of the PECAN method in [6] that can be used to construct phase-quantized sequences with low PAR. The design of such sequences is also discussed in Section 4 using a new alternating projection scheme. Numerical examples are provided in Section 5. Finally, Section 6 concludes the paper.

Notation: $\mathbf{1}$ and $\mathbf{0}$ are the all-one and all-zero vectors/matrices. The symbol $\otimes$ stands for the Kronecker product of matrices. $\|\boldsymbol{x}\|_n$ or the l_n-norm of the vector/sequence $\boldsymbol{x}$ is defined as $(\sum_k |x_k|^n)^{\frac{1}{n}}$ where $\{x_k\}$ are the entries of $\boldsymbol{x}$. The Frobenius norm of a matrix $\boldsymbol{X}$ (denoted by $\|\boldsymbol{X}\|_F$) is equal to $\left(\sum_{k,l} |[\boldsymbol{X}]_{k,l}|^2\right)^{\frac{1}{2}}$. For the two matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$, we have $\boldsymbol{X} \geq \boldsymbol{Y}$ if and only if each entry of $\boldsymbol{X}$ is greater than or equal to the corresponding entry in $\boldsymbol{Y}$. $vec(\boldsymbol{X})$ is a vector obtained by stacking the columns of $\boldsymbol{X}$ successively. Finally, for any real number x, the function $[x]$ yields the closest integer to x (the largest is chosen when this integer is not unique).

2. PHASE SELECTION

A phase-quantized sequence is of the form $\boldsymbol{x} = \{\alpha_l\, e^{j\frac{2\pi}{m}k_l}\}_{l=0}^{n-1}$ where $0 \leq k_l \leq m-1$ and $\{\alpha_l\}$ are non-negative real numbers. Without loss of generality, we can assume that the sequence energy is equal to its length:

$$\|\boldsymbol{x}\|_2^2 = \sum_{l=0}^{n-1} \alpha_l^2 = n. \tag{4}$$

In this case, the PAR can be written as

$$\text{PAR} = \max\{\alpha_l^2\}_{l=0}^{n-1}. \tag{5}$$

Clearly, the PAR metric attains lower values when $\boldsymbol{\alpha} = \{\alpha_l\}$ is closer to $\mathbf{1}$ (in the sense of l_∞-norm).

In order to construct $\boldsymbol{x}$, all $\{\alpha_l\}$ and $\{k_l\}$ must be specified. Given the sequence $\{k_l\}$, in the following sections we introduce iterative methods that starting from the sequence $\boldsymbol{x} = \{e^{j\frac{2\pi}{m}k_l}\}_{l=0}^{n-1}$ (which has PAR $= 1$) improve the periodic correlation properties at the expense of an increased PAR. The task of choosing suitable integer phases $\{k_l\}$ is crucial; we need to select $\{k_l\}$ such that they facilitate finding $\boldsymbol{\alpha}$ as close to $\mathbf{1}$ as possible. To this end, let $\boldsymbol{x}^u = \{e^{j\phi_l}\}_{l=0}^{n-1}$ be a perfect unimodular sequence. Such sequences can be easily generated using the PECAN method of [6]. We consider the m-level phase-quantized version of $\boldsymbol{x}^u$ by choosing

$$k_l \equiv \left[m\left(\frac{\phi_l}{2\pi}\right)\right] \pmod m \tag{6}$$

In the following sections, we discuss the design of $\boldsymbol{\alpha}$ for the integer phases $\{k_l\}$ given by (6).

3. PECAN FOR LOW-PAR PHASE-QUANTIZED SEQUENCE DESIGN

In [6] an iterative method called PECAN is devised to design perfect unimodular sequences of arbitrary length. In this section, an extension of this algorithm is discussed that can be used for designing perfect low-PAR phase-quantized sequences. For a unimodular sequence $\{x_l\}$ (with $|x_l| = 1$), the PECAN algorithm minimizes the criterion (see [6])

$$C = \sum_{l=0}^{n-1} |X_l - e^{j\psi_l}|^2 \tag{7}$$

where $\{X_l\}$ denotes the discrete Fourier transform (DFT) of $\{x_l\}$:

$$X_l = \sum_{k=0}^{n-1} x_k e^{-j2\pi\frac{kl}{n}} \tag{8}$$

and $\{\psi_l\}$ are arbitrary phase values in $[0, 2\pi)$. In each iteration, the algorithm performs a cyclic minimization as follows: for given $\{x_l\}$ (and hence given $\{X_l\}$), the minimizing $\{\psi_l\}$ are simply given by

$$\psi_l = \arg(X_l), \;\; 0 \leq l < n, \tag{9}$$

whereas for given $\{\psi_l\}$ the minimizing $\{x_l\}$ are given by

$$x_l = e^{j\arg(z_l)}, \;\; 0 \leq l < n \tag{10}$$

where $\{z_l\}$ represents the Inverse DFT of $\{e^{j\psi_l}\}$:

$$z_l = \frac{1}{n}\sum_{k=0}^{n-1} e^{j\psi_k} e^{j2\pi\frac{kl}{n}}. \tag{11}$$

To propose an extension of the PECAN algorithm, we assume that the absolute values of the entries of the sequence belong to the interval $[1-\varepsilon_1, 1+\varepsilon_2]$ where $1 \geq \varepsilon_1 \geq 0$ and $\varepsilon_2 \geq 0$. Note that PAR is a global metric. In some applications a local alternative to the optimization of the PAR metric is more important. The above limitation on the absolute values of the entries of the sequences plays such a role.

Similar to the previous derivations of the PECAN, the minimizing $\{\psi_l\}$ are given by Eq. (9). However, the minimizing $\{x_l\}$ are obtained in a slightly different way. Let $x_l = \alpha_l e^{j\frac{2\pi}{m}k_l}$ be the l^{th} entry of $\boldsymbol{x}$. The minimizing x_l can be obtained independently of the other entries of the sequence by solving the minimization problem

$$\begin{aligned} &\min_{\alpha_l, k_l} |\alpha_l e^{j\frac{2\pi}{m}k_l} - z_l| \\ &\text{s.t. } \alpha_l \in [1-\varepsilon_1, 1+\varepsilon_2], \\ &\quad k_l \in \{0, 1, \cdots, m-1\}. \end{aligned} \tag{12}$$

Table 1. The Extended PECAN for Designing Perfect Low-PAR Phase-Quantized Sequences

Step 0 (Initialization): Generate a perfect unimodular sequence and consider the m-level phase-quantized version of it.
Step 1: Compute $\{\psi_l\}$ using Eq. (9).
Step 2: Compute $\{\alpha_l\}$ using Eq. (14).
Step 3: Repeat the cyclic minimization steps 1 and 2 until a stop criterion is satisfied, e.g. the absolute change of any specific entry of $\{\alpha_l\}$ in successive iterations is less than a given $\epsilon > 0$, or the obtained PAR has reached the maximum tolerable value.

For fixed $\{k_l\}$ obtained by (6), the optimal $\alpha_l \in [1-\varepsilon_1, 1+\varepsilon_2]$ is given by the minimizer of the function:

$$\left|\alpha_l - |z_l| \cos\left(\arg(z_l) - \frac{2\pi}{m} k_l\right)\right| \tag{13}$$

Let $\beta_l = |z_l| \cos\left(\arg(z_l) - \frac{2\pi}{m} k_l\right),\ 0 \le l < n$. Then the minimizing $\{\alpha_l\}$ are given by

$$\alpha_l = \begin{cases} \beta_l, & \beta_l \in [1-\varepsilon_1, 1+\varepsilon_2] \\ 1+\varepsilon_2, & \beta_l > 1+\varepsilon_2 \\ 1-\varepsilon_1, & \beta_l < 1-\varepsilon_1. \end{cases} \tag{14}$$

Remark: It is worth mentioning that the function in (13) can be minimized with respect to $\{k_l\}$ as well. However, due to the multi-modal structure of the loss function in (7), the performance does not necessarily improve. We also note that based on the discussions in Section 2, $\{k_l\}$ is initialized such that it provides a generally good compromise between virtually perfect correlation properties and an increased PAR. ■

The proposed extended PECAN for designing low-PAR phase-quantized sequences is summarized in Table 1. It is interesting to note that while the criterion (7) is quadratic, it effectively minimizes the harder-to-optimize ISL metric which is quartic in $\{x_l\}$ [6]. The extended PECAN algorithm can also be considered for cases in which no entry of the sequence is limited in absolute value. Interestingly, the latter scenario is only a special case of the extension described above corresponding to $\varepsilon_1 = 1$ and $\varepsilon_2 = \infty$.

4. ALTERNATING PROJECTIONS

The periodic autocorrelation lags of $\boldsymbol{x} = \{\alpha_l\, e^{j\frac{2\pi}{m}k_l}\}_{l=0}^{n-1}$ are given by

$$R_u = \sum_{l=0}^{n-1} \alpha_l \alpha_{l+u}\, e^{j\frac{2\pi}{m}(k_l - k_{l+u})},\ \ 0 \le u < n \tag{15}$$

where the indices are used in a periodic manner. As discussed in the Introduction, $\boldsymbol{x}$ is perfect iff all its out-of-phase autocorrelation lags are equal to zero. Note that because the sequence $\{k_l - k_{l+u}\}_{l=0}^{n-1}$ is given, Eq. (15) can be written as $R_u = \boldsymbol{e}^T \boldsymbol{a}_u$ in which $\boldsymbol{e} = \left(1\ e^{j\frac{2\pi}{m}(1)} \cdots e^{j\frac{2\pi}{m}(m-1)}\right)^T$,

$$\boldsymbol{a}_u = \boldsymbol{P}_u \begin{pmatrix} \alpha_0\alpha_{u \bmod n} \\ \alpha_1\alpha_{(1+u) \bmod n} \\ \vdots \\ \alpha_{n-1}\alpha_{(n-1+u) \bmod n} \end{pmatrix} \tag{16}$$

and $\boldsymbol{P}_u$ is a binary selection matrix whose $[\boldsymbol{P}_u]_{r+1,s+1}$ entry is equal to one iff $(k_s - k_{(s+u) \bmod n}) \bmod m = r$, and is zero otherwise. By combining (16) for all out-of-phase lags $1 \le u < n$, we obtain

$$\underbrace{\begin{pmatrix} \boldsymbol{P}'_1 \\ \boldsymbol{P}'_2 \\ \vdots \\ \boldsymbol{P}'_{n-1} \end{pmatrix}}_{\boldsymbol{P}} vec(\boldsymbol{\alpha}\boldsymbol{\alpha}^T) = \underbrace{\begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_{n-1} \end{pmatrix}}_{\boldsymbol{a}} \tag{17}$$

where $\{\boldsymbol{P}'_u\}$ is constructed from $\{\boldsymbol{P}_u\}$ such that it selects the correct indices of $\{\alpha_l\alpha_{(l+u) \bmod n}\}$ in $vec(\boldsymbol{\alpha}\boldsymbol{\alpha}^T)$. To give a more precise definition of $\{\boldsymbol{P}'_u\}$, note that $\{\alpha_l\alpha_{(l+u) \bmod n}\}$ appears in the $(n((l+u) \bmod n) + l + 1)^{th}$ position of $vec(\boldsymbol{\alpha}\boldsymbol{\alpha}^T)$. This implies that $[\boldsymbol{P}'_u]_{r+1,(n((s+u) \bmod n)+s+1)}$ is equal to one iff $(k_s - k_{(s+u) \bmod n}) \bmod m = r$ and is zero otherwise.

We conclude from (17) that to obtain a perfect sequence $\boldsymbol{x}$ we need $\boldsymbol{\alpha} \ge \boldsymbol{0}$ such that

$$\begin{aligned} (\boldsymbol{I}_{n-1} \otimes \boldsymbol{e})^T \boldsymbol{P}\, vec(\boldsymbol{\alpha}\boldsymbol{\alpha}^T) &= (\boldsymbol{I}_{n-1} \otimes \boldsymbol{e})^T \boldsymbol{a} \\ &= \boldsymbol{0}_{(n-1)\times 1}. \end{aligned} \tag{18}$$

To solve this problem, we consider the alternating projections on the two sets

$$\begin{aligned} \Gamma = \big\{ \boldsymbol{A} \in (\mathbb{R}_+ \cup \{0\})^{n\times n} \mid \\ (\boldsymbol{I}_{n-1} \otimes \boldsymbol{e})^T \boldsymbol{P}\, vec(\boldsymbol{A}) = \boldsymbol{0}_{(n-1)\times 1} \big\} \end{aligned} \tag{19}$$

and $\Lambda = \left\{ \boldsymbol{A} \in \mathbb{R}^{n\times n} \mid \boldsymbol{A} = \boldsymbol{\alpha}\boldsymbol{\alpha}^T,\ \|\boldsymbol{\alpha}\|_2^2 = n,\ \boldsymbol{\alpha} \ge \boldsymbol{0} \right\}$. Note that while Γ is convex, Λ is a non-convex set which may lead the alternating projections to different solutions depending on initialization. In order to find the optimal projection of $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ on Γ, we consider the problem:

$$\min_{\boldsymbol{A}_\perp \in \Gamma} \|\boldsymbol{A} - \boldsymbol{A}_\perp\|_F^2 \tag{20}$$

Let us define the difference matrix $\boldsymbol{A}_\delta = \boldsymbol{A} - \boldsymbol{A}_\perp$. As a result,

$$\begin{aligned} (\boldsymbol{I}_{n-1} \otimes \boldsymbol{e})^T \boldsymbol{P}\, vec(\boldsymbol{A}_\delta) &= (\boldsymbol{I}_{n-1} \otimes \boldsymbol{e})^T \boldsymbol{P}\, vec(\boldsymbol{A}) \\ &\triangleq \boldsymbol{c}_{(n-1)\times 1}. \end{aligned} \tag{21}$$

Therefore, (20) is equivalent to

$$\begin{aligned} &\min_{\boldsymbol{A}_\delta \le \boldsymbol{A}} \|vec(\boldsymbol{A}_\delta)\|_2^2 \\ &\text{s.t.}\ \ (\boldsymbol{I}_{n-1} \otimes \boldsymbol{e})^T \boldsymbol{P}\, vec(\boldsymbol{A}_\delta) = \boldsymbol{c}. \end{aligned} \tag{22}$$

Table 2. The Proposed Alternating Projections

Step 0 (Initialization): Generate a perfect unimodular sequence and consider the m-level phase-quantized version of it.
Step 1: Compute the optimal projection on Γ by solving (22).
Step 2: Compute the optimal projection on Λ using (24).
Step 3: Repeat the projections in steps 1 and 2 until a stop criterion is satisfied, e.g. the distance between the two sets is less than a given $\epsilon > 0$ or the obtained PAR has reached the maximum tolerable value.

The latter optimization problem is a linearly constrained least squares problem which can be solved using standard solvers such as CVX or YALMIP [7].

Next, let $\boldsymbol{\alpha}$ (with $\|\boldsymbol{\alpha}\|_2^2 = n$) be the normalized dominant eigenvector of $\boldsymbol{A}$. Interestingly, the matrices in Γ are non-negative and therefore have non-negative dominant eigenvectors according to the Perron-Frobenius theorem. As a result, the optimal solution $\boldsymbol{A}_\perp$ to the problem

$$\min_{\boldsymbol{A}_\perp \in \Lambda} \|\boldsymbol{A} - \boldsymbol{A}_\perp\|_F^2 \tag{23}$$

is given by

$$\boldsymbol{A}_\perp = \boldsymbol{\alpha}\boldsymbol{\alpha}^T. \tag{24}$$

The proposed alternating projection algorithm can be summarized as in Table 2.

We end this section commenting on the integrated sidelobe level (ISL) of the sequences obtained by the proposed alternating projections. At the s^{th} iteration of the method, the ISL metric can be rewritten as

$$\begin{aligned} \text{ISL}^{(s)} &= \sum_{u=1}^{n-1} \left| R_u^{(s)} \right|^2 = \sum_{u=1}^{n-1} \left| \boldsymbol{e}^T \boldsymbol{a}_u^{(s)} \right|^2 \\ &= \sum_{u=1}^{n-1} \left| \boldsymbol{e}^T (\boldsymbol{a}_u^{(s)} - \boldsymbol{a}_u^{\perp\,(s)}) \right|^2 \end{aligned} \tag{25}$$

where $\{\boldsymbol{a}_u^{(s)}\}$ is defined as in (16) and

$$\begin{pmatrix} \boldsymbol{a}_1^{\perp\,(s)} \\ \boldsymbol{a}_2^{\perp\,(s)} \\ \vdots \\ \boldsymbol{a}_{n-1}^{\perp\,(s)} \end{pmatrix} = \boldsymbol{a}^{\perp\,(s)} = \boldsymbol{P}\, vec(\boldsymbol{A}_\perp^{(s)}), \;\; \boldsymbol{A}_\perp^{(s)} \in \Gamma. \tag{26}$$

Using the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} \text{ISL}^{(s)} &\leq m \left(\sum_{u=1}^{n-1} \|\boldsymbol{a}_u^{(s)} - \boldsymbol{a}_u^{\perp\,(s)}\|_2^2 \right) \\ &= m \|\boldsymbol{a}^{(s)} - \boldsymbol{a}^{\perp\,(s)}\|_2^2. \end{aligned} \tag{27}$$

The latter inequality implies that, in each iteration, the ISL of the resultant sequence is bounded by the distance between the obtained points in the two sets, and thus finding the intersection of the two sets leads to an ISL value of zero.

5. NUMERICAL EXAMPLES

In order to illustrate the performance of the newly proposed alternating projection method, we show an example of designing a low-PAR phase-quantized sequence of length $n = 64$ and alphabet size $m = 2$ (i.e. the binary alphabet) in Fig. 1. We used the PECAN method to construct a phase-quantized unimodular sequence for initialization. The autocorrelation levels are normalized and expressed in dB:

$$\text{autocorrelation (dB)} = 20 \log_{10} \left| \frac{R_u}{R_0} \right| \tag{28}$$

Although the binary alphabet is most constrained, the obtained sequence achieves virtually perfect periodic correlations and PAR $= 2.138$. Fig. 1 (c) exhibits an example of the typical increasing/decreasing behavior of PAR/autocorrelation PSL of the obtained sequences vs. iteration number. This behavior makes it possible to stop the algorithms when the desirable PSL or the maximal tolerable PAR is achieved.

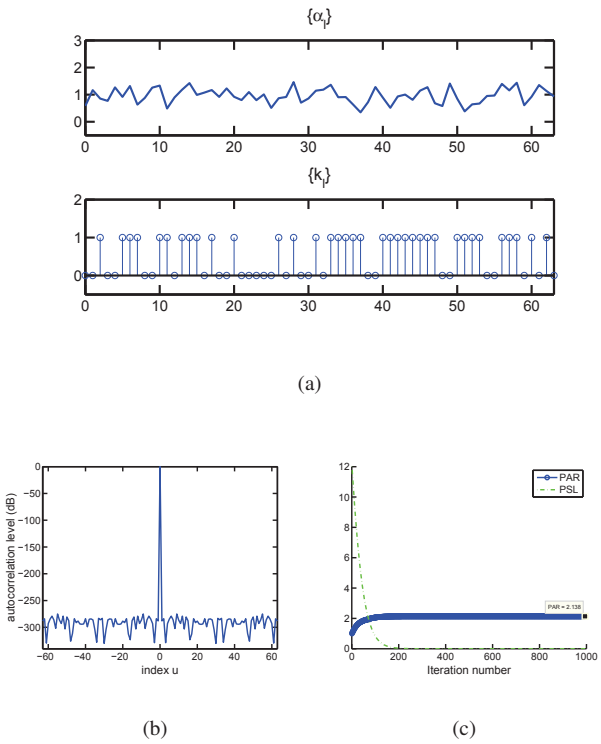

Fig. 1. Design of a phase-quantized sequence of length $n = 64$ with low PAR via the alternating projection method. The phase quantization level is $m = 2$. (a) the sequence amplitudes $\{\alpha_l\}$ and integer phases $\{k_l\}$. (b) the autocorrelation levels (in dB) of the obtained sequence. (c) the PSL and PAR vs. the iteration number for the obtained sequence. The sequence achieves practically perfect periodic correlation properties and PAR $= 2.138$.

n \ m	2	4	8	16	32	64
10	**1.842**	1.386	**1.085**	1.085	**1.022**	**1.022**
	1.842	**1.250**	1.250	**1.022**	**1.022**	**1.022**
	1.842	1.842	**1.085**	**1.022**	**1.022**	**1.022**
25	**1.904**	**1.904**	**1.904**	1.895	1.290	**1.104**
	1.992	1.992	1.992	**1.610**	**1.289**	1.245
	1.960	1.960	1.960	1.717	1.447	1.142
50	**1.999**	**1.999**	**1.999**	**1.999**	**1.507**	1.382
	1.999	**1.999**	**1.999**	**1.999**	1.655	1.350
	2.250	2.250	2.250	2.250	1.600	**1.290**
100	2.907	2.748	2.739	2.739	2.324	1.878
	3.095	3.095	3.095	**2.097**	2.027	1.515
	2.250	**2.250**	**2.250**	2.250	**1.844**	**1.444**

Table 3. The lowest PAR values obtained for different lengths (n) and alphabet sizes (m) by running the two proposed methods 50 times; alternating projections (see the first row for each length) and the extended PECAN method with $\varepsilon_1 = 1$ and $\varepsilon_2 = \infty$ (see the second row for each length). The third row for each length presents the results of running the extended PECAN method 10 times for the same values of (n, m) and $\varepsilon_1 = \varepsilon_2 = \varepsilon \in \{0, 0.05, 0.1, 0.15, 0.3, 0.4, 0.5, 0.7, 0.9\}$. For each length/alphabet size, the bold font is used to indicate the lowest obtained PAR using the proposed methods.

Next, we consider the lengths $n \in \{10, 25, 50, 100\}$ and alphabet sizes $m \in \{2, 4, 8, 16, 32, 64\}$. As the sequence design problems are usually performed off-line, we have run the proposed alternating projections and the extended PECAN method in Section 3 (with $\varepsilon_1 = 1$ and $\varepsilon_2 = \infty$) 50 times and have chosen the sequence with the best PAR. We also present results obtained by the extended PECAN when the absolute values of the entries of the sequences are limited. We have run the extension of the PECAN method 10 times for the same set of lengths and alphabet sizes as above, $\varepsilon_1 = \varepsilon_2 = \varepsilon \in \{0, 0.05, 0.1, 0.15, 0.3, 0.4, 0.5, 0.7, 0.9\}$ and have selected the sequence with the best PAR. In all cases, we stopped the algorithms when PSL values less than 10^{-10} were achieved. The results are shown in Table 3. Note that the difficult optimization problems considered in this paper have many local optima and as a result, different optimization paths can be chosen in the search space. On the other hand, higher values of ε enlarge the constraint set and thus it is possible that we achieve lower PAR values for higher ε.

6. CONCLUDING REMARKS

Computational methods have been proposed to tackle the problem of designing perfect phase-quantized sequences with low PAR. It is important to note that:

- The PECAN algorithm in [6] can be used to design many perfect unimodular sequences for any length n. However, these sequences have infinite-alphabet. In some applications, finite-alphabet or phase-quantized perfect sequences are required.
- Perfect phase-quantized unimodular sequences can be obtained using analytical construction methods. However, these sequences are rare and do not exist for arbitrary alphabet sizes.
- Quantized PECAN sequences (as described in Eq. (6)) are phase-quantized and unimodular but not perfect.

The key idea proposed in this paper is as follows. We obtain a quantized PECAN sequence and fix its phase values. We propose polynomial-time algorithms (namely an extension of the PECAN method and an alternating projection scheme) to trade-off PAR for perfect periodic correlation properties. With such a compromise, we can obtain many perfect phase-quantized sequences for any arbitrary length and alphabet size; yet the PAR larger than 1. These sequences are of interest for applications in which sub-optimal PAR values are tolerable. Numerical examples are provided to examine the performance of the proposed methods.

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