Signal Processing 93 (2013) 2096–2102

Contents lists available at SciVerse ScienceDirect

Signal Processing

journal homepage: www.elsevier.com/locate/sigpro

Fast communication

A fast algorithm for designing complementary sets of sequences

Mojtaba Soltanalian [a,*], Mohammad Mahdi Naghsh [b], Petre Stoica [a]

[a] *Department of Information Technology, Uppsala University, Uppsala, Sweden*
[b] *ECE Department, Isfahan University of Technology, Isfahan, Iran*

ARTICLE INFO

Article history:
Received 5 October 2012
Received in revised form
8 January 2013
Accepted 4 February 2013
Available online 13 February 2013

Keywords:
Autocorrelation
Binary sequences
Complementary sets
Cyclic minimization
Peak-to-average-power ratio (PAR)
Sidelobe level

ABSTRACT

In this paper, we introduce a fast computational frequency-domain approach for designing complementary sets of sequences. Following the basic idea of CAN-based algorithms, we propose an extension of the *CAN* algorithm to complement*ary* sets of sequences (which we call CANARY). Moreover, modified versions of the proposed algorithm are derived to tackle the complementary set design problems in which low peak-to-average-power ratio (PAR), unimodular or phase-quantized sequences are of interest. Several numerical examples are provided to show the performance of CANARY.

1. Introduction

An active sensing device such as a radar system, transmits suitable waveforms into its surrounding that enable it to measure useful properties (e.g. location or speed) of peripheral objects. The transmit waveforms are generally formulated by using discrete-time sequences (see e.g. [1]). Let $\boldsymbol{x}=(x(1),\ldots,x(N))^T$ represent such a sequence (to be designed). The aperiodic and, respectively, periodic autocorrelations of $\boldsymbol{x}$ are defined as

$$r(k)=\sum_{l=1}^{N-k} x(l)x^*(l+k)=r^*(-k), \quad 0\le k\le (N-1), \tag{1}$$

$$c(k)=\sum_{l=1}^{N} x(l)x^*(l+k)_{\mathrm{mod}\ N}=c^*(-k), \quad 0\le k\le (N-1). \tag{2}$$

In general, transmit sequences $\boldsymbol{x}$ with small out-of-phase (i.e. $k\neq 0$) autocorrelation lags lead to a better performance of an active sensing system. As a result, there exists a rich literature on designing such sequences (see e.g. [1–22] and the references therein).

In order to avoid non-linear side effects and maximize the efficiency of power consumption at the transmitter, unimodular sequences (with $|x(l)|=1$) are desirable. Moreover, for cases with more strict implementation demands, phase-quantized unimodular sequences must be considered. For unimodular sequences it is not possible to make all $\{|r(k)|\}$ much smaller than $r(0)$ (depending on the application, the needed ratio can be around 10^{-5} or even smaller). For instance, it can be easily observed that $|r(N-1)|=1$, no matter how we design the sequence $\boldsymbol{x}$. In contrast with this, unimodular sequences with zero out-of-phase (i.e. perfect) periodic autocorrelation can be obtained for example via construction algorithms [4]. However, even by considering the periodic correlation, finding phase-quantized unimodular sequences with perfect periodic autocorrelation is a hard task. The difficulties in designing sequences with good autocorrelation encouraged the researchers to consider the idea of complementary sets of sequences (CSS). A set $X=\{\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_M\}$

* Corresponding author. Tel.: +46 18 471 3168; fax: +46 18 511925.
E-mail address: mojtaba.soltanalian@it.uu.se (M. Soltanalian).

0165-1684/$ - see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.sigpro.2013.02.008

containing M sequences of length N is called complementary iff the autocorrelations of $\{\boldsymbol{x}_m\}$ sum up to zero at any out-of-phase lag, i.e.

$$\sum_{m=1}^{M} r_m(k) = 0, \quad 1 \le |k| \le (N-1), \tag{3}$$

where $r_m(k)$ represents the kth autocorrelation lag of $\boldsymbol{x}_m$. Consequently, to measure the complementarity of a sequence set $\{\boldsymbol{x}_m\}$ one can consider the integrated sidelobe level (ISL) or the peak sidelobe level (PSL) metrics defined by

$$\mathrm{ISL} = \sum_{k=1}^{N-1} \left| \sum_{m=1}^{M} r_m(k) \right|^2,$$
$$\mathrm{PSL} = \max_k \left\{ \left| \sum_{m=1}^{M} r_m(k) \right| \right\}, \tag{4}$$

as well as the ISL-related merit factor (MF), i.e.

$$\mathrm{MF} = E^2/(2\mathrm{ISL}), \tag{5}$$

where E denotes the sum of the energy of the sequences. Complementary sets containing $M=2$ sequences, which are known as complementary pairs, form a special case of CSS. Complementary pairs with binary (i.e. ± 1) elements were first studied in [5] and are usually referred to as Golay pairs (GP).

CSS have been applied to radar pulse compression [7], multiple-input–multiple-output (MIMO) radars [8], ultrasonic ranging [9], synthetic aperture imaging [10], and ultrasonography [11]. In addition to active sensing systems, CSS have applications in code-division multiple-access (CDMA) communication schemes [12], ultra wideband (UWB) communications [13], orthogonal frequency-division multiplexing (OFDM) [14,15], channel estimation [16], and data hiding [17]. Due to such a wide range of applications, the construction of CSS has been an active area of research during the last decades. The majority of research results on CSS have been concerned with the analytical construction of GP or CSS for restricted sequence lengths N. For example, it is shown in [18] that GPs exist for lengths of the form $N = 2^{\alpha}10^{\beta}26^{\gamma}$ where α,β and γ are non-negative integers. Some conditions on the existence of CSS can be found in [19] and [20]. Furthermore, Ref. [20] considers the extension of GP to general CSS. A theoretical as well as computational investigation of feasible GPs of lengths $N < 100$ is accomplished in [21].

In contrast to analytical constructions, a computational design of CSS does not impose any restriction on the sequence length N or the set cardinality M. Furthermore, a computational algorithm for designing CSS can provide plenty of CSS without the need for user-tuned parameters of analytical constructions. Such algorithms can also be used to find almost (i.e. sub-optimal) complementary sets of sequences for (N,M) values for which no CSS exists. A computational algorithm (called ITROX) for designing CSS was introduced in [9]. In this paper, we propose an extension of the *CAN* algorithm [23] for designing complement*ary* sets of sequences (which we call CANARY). The proposed algorithm works in the frequency domain, and is generally faster than ITROX. This is due to the fact that ITROX is based on certain eigenvalue decompositions with $\mathcal{O}(MN^2)$ complexity, whereas CANARY relies on fast Fourier transform (FFT) operations with $\mathcal{O}(MN\log(N))$ complexity (the difference in computational burdens between the two algorithms can be clearly observed in practice when N grows large).

The rest of this work is organized as follows. Section 2 presents the CANARY algorithm for CSS design. The extension of the CANARY algorithm to phase-quantized (and other constrained) CSS is studied in Section 3. Section 4 is devoted to numerical examples, whereas Section 5 concludes the paper.

Notation: We use bold lowercase letters for vectors and bold uppercase letters for matrices. $(.)^T$, $(.)^*$ and $(.)^H$ denote the vector/matrix transpose, the complex conjugate, and the Hermitian transpose, respectively. $\mathbf{1}$ and $\mathbf{0}$ are the all-one and all-zero vectors/matrices, respectively. $\|\boldsymbol{x}\|_n$ or the l_n-norm of the vector $\boldsymbol{x}$ is defined as $(\sum_k |\boldsymbol{x}(k)|^n)^{1/n}$ where $\{\boldsymbol{x}(k)\}$ are the entries of $\boldsymbol{x}$. The Frobenius norm of a matrix $\boldsymbol{X}$ (denoted by $\|\boldsymbol{X}\|_F$) with entries $\{\boldsymbol{X}(k,l)\}$ is equal to $(\sum_{k,l} |\boldsymbol{X}(k,l)|^2)^{\frac{1}{2}}$, whereas the l_1-norm of $\boldsymbol{X}$ (denoted as $\|\boldsymbol{X}\|_1$) is given by $\sum_{k,l} |\boldsymbol{X}(k,l)|$. The matrix $e^{j\boldsymbol{X}}$ is defined element-wisely as $[e^{j\boldsymbol{X}}]_{k,l} = e^{j[\boldsymbol{X}]_{k,l}}$. $\arg(.)$ denotes the phase angle (in radians) of the vector/matrix argument. The symbol $\odot$ stands for the Hadamard (element-wise) product of matrices. $\mathbb{C}$ represents the set of complex numbers. Finally, δ_k is the Kronecker delta function which is equal to one when $k=0$ and to zero otherwise.

2. CANARY

It is well-known that for any sequence $\boldsymbol{x}$ of length N with aperiodic autocorrelation lags $\{r(k)\}$ (see e.g. [24])

$$\Phi(\omega) \triangleq \left| \sum_{n=1}^{N} x(n) e^{-j\omega n} \right|^2 = \sum_{k=-(N-1)}^{N-1} r(k) e^{-j\omega k} \tag{6}$$

where $\Phi(\omega)$ is the "spectrum" of $\boldsymbol{x}$. Consider a complementary set $X = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_M\}$ containing M sequences of length N. It follows from the Parseval equality that

$$2\mathrm{ISL} = \sum_{k=-(N-1)}^{N-1} \left| \sum_{m=1}^{M} r_m(k) - MN\delta_k \right|^2$$
$$= \frac{1}{2N} \sum_{p=1}^{2N} \left[\sum_{m=1}^{M} \Phi_m(\omega_p) - MN \right]^2 \tag{7}$$

with $\Phi_m(\omega_p)$ representing the spectrum of the mth sequence at the angular frequency $\omega_p = 2p\pi/(2N)$. Therefore, the minimization of the ISL metric in (4) can be accomplished by minimizing the following frequency-domain metric:

$$\sum_{p=1}^{2N} \left[\sum_{m=1}^{M} \left| \sum_{n=1}^{N} x_m(n) e^{-j\omega_p n} \right|^2 - MN \right]^2. \tag{8}$$

Inspired by the basic idea of the CAN algorithm in [23] that considers (8) with $M=1$, we propose a cyclic algorithm (which we call CANARY) for designing CSS.

Let $\boldsymbol{X} \triangleq (\boldsymbol{x}_1 \boldsymbol{x}_2 \ldots \boldsymbol{x}_M)$ and let $\boldsymbol{A}^H$ represent the $2N \times 2N$ DFT matrix given by

$$[\boldsymbol{A}^H]_{p,n} = \frac{1}{\sqrt{2N}} e^{-jn\omega_p}, (p,n) \in \{1,2,\ldots,2N\}^2. \tag{9}$$

The design problem associated with the frequency-domain metric in (8) can be dealt with conveniently via considering the following minimization problem:

$$\min_{\boldsymbol{Z},\boldsymbol{S}} \quad \|\boldsymbol{A}^H \boldsymbol{Z} - \boldsymbol{S}\|_F \tag{10}$$

$$\text{s.t.} \quad (\boldsymbol{S} \odot \boldsymbol{S}^*) \boldsymbol{1}_M = MN \, \boldsymbol{1}_{2N}, \tag{11}$$

$$\boldsymbol{Z} = \begin{pmatrix} \boldsymbol{X} \\ \boldsymbol{0}_{N \times M} \end{pmatrix}, \tag{12}$$

where $\boldsymbol{S}$ is an auxiliary matrix variable.

For fixed $\boldsymbol{Z}$ (equivalently fixed $\boldsymbol{X}$), the minimizer $\boldsymbol{S}$ of (10) can be obtained as follows. Since the constraint (11) is imposed row-wise, we can consider the optimization of the entries in each row of $\boldsymbol{S}$ independently. Suppose that $\overline{\boldsymbol{s}}^T$ represents a generic row of $\boldsymbol{A}^H \boldsymbol{Z}$. Then the goal is to find a vector $\boldsymbol{s}$ that solves the optimization problem

$$\min_{\boldsymbol{s}} \quad \|\overline{\boldsymbol{s}} - \boldsymbol{s}\|_2^2$$
$$\text{s.t.} \quad \|\boldsymbol{s}\|_2^2 = MN. \tag{13}$$

The solution to (13) is simply given by

$$\boldsymbol{s} = \sqrt{MN} \, \frac{\overline{\boldsymbol{s}}}{\|\overline{\boldsymbol{s}}\|_2}. \tag{14}$$

In sum, let $\overline{\boldsymbol{s}}_k^T$ $(k = 1, \ldots, 2N)$ denote the kth row of $\overline{\boldsymbol{S}} = \boldsymbol{A}^H \boldsymbol{Z}$. Then the minimizer $\boldsymbol{S}$ of (10) can be obtained as

$$\boldsymbol{S} = \sqrt{MN} \begin{pmatrix} \overline{\boldsymbol{s}}_1^T / \|\overline{\boldsymbol{s}}_1\|_2 \\ \overline{\boldsymbol{s}}_2^T / \|\overline{\boldsymbol{s}}_2\|_2 \\ \vdots \\ \overline{\boldsymbol{s}}_{2N}^T / \|\overline{\boldsymbol{s}}_{2N}\|_2 \end{pmatrix}. \tag{15}$$

Next we study the optimization of (10) with respect to $\boldsymbol{Z}$. For cases in which the sequences $\{\boldsymbol{x}_m\}$ are not constrained, the minimizer $\boldsymbol{Z}$ of (10) is given by

$$[\boldsymbol{Z}]_{n,m} = \begin{cases} [\boldsymbol{A}\boldsymbol{S}]_{n,m}, & 1 \le n \le N, \\ 0, & n > N. \end{cases} \tag{16}$$

However in many practical applications, the sequences are constrained (see the discussion on this aspect in the Introduction). Particularly, we will consider unimodularity constraints as well as more general peak-to-average-power ratio (PAR) constraints. For unimodular $\boldsymbol{X}$, the minimizer $\boldsymbol{Z}$ of (10) can be expressed as

$$[\boldsymbol{Z}]_{n,m} = \begin{cases} e^{j \arg[\boldsymbol{A}\boldsymbol{S}]_{n,m}}, & 1 \le n \le N, \\ 0, & n > N. \end{cases} \tag{17}$$

On the other hand, the minimizer $\boldsymbol{Z}$ of (10) for PAR constraint set, viz.

$$\text{PAR}(\boldsymbol{x}_m) = \frac{\|\boldsymbol{x}_m\|_\infty^2}{\frac{1}{N}\|\boldsymbol{x}_m\|_2^2} \le \gamma, \quad 1 \le m \le M \tag{18}$$

can be obtained by solving the optimization problem

$$\min_{\boldsymbol{Z}} \quad \|\boldsymbol{Z} - \boldsymbol{A}\boldsymbol{S}\|_F$$
$$\text{s.t.} \quad \|\boldsymbol{x}_m\|_\infty^2 \le \gamma, \quad 1 \le m \le M,$$
$$\|\boldsymbol{x}_m\|_2^2 = N, \quad 1 \le m \le M,$$
$$\boldsymbol{Z} = \begin{pmatrix} \boldsymbol{X} \\ \boldsymbol{0}_{N \times M} \end{pmatrix}. \tag{19}$$

Interestingly, the problem (19) can be tackled using an efficient recursive algorithm suggested in [25]. Briefly, first we note that (19) can be solved via a separate optimization with respect to the sequences $\{\boldsymbol{x}_m\}$ (i.e. the columns of $\boldsymbol{X}$), and that for each sequence $\boldsymbol{x}_m$ (19) boils down to a "nearest-vector" problem with PAR constraint. Let $\overline{\boldsymbol{x}}_m$ denote the vector containing the first N entries of the mth column of $\boldsymbol{A}\boldsymbol{S}$. If the magnitudes of the entries of $\overline{\boldsymbol{x}}_m$ are below $\sqrt{\gamma}$ then $\boldsymbol{x}_m = \sqrt{N}\overline{\boldsymbol{x}}_m / \|\overline{\boldsymbol{x}}_m\|_2$ is the solution. Otherwise, the entry of $\boldsymbol{x}_m$ corresponding to the entry of $\overline{\boldsymbol{x}}_m$ (say x_{max}) with maximal magnitude is given by $\sqrt{\gamma} e^{j \arg(x_{max})}$; and the other entries of $\boldsymbol{x}_m$ are obtained solving the same type of "nearest-vector" problem but with the remaining energy, i.e. $N - \gamma$.

Based on the previous analysis, the CANARY algorithm for designing CSS is summarized in Table 1. Note that each iteration of CANARY is computationally efficient as it is based solely on FFT operations. As a result, the CANARY algorithm can be used for large values of N and M (e.g. $MN \sim 10^6$ or even larger).

We conclude this section with two remarks.

Remark 1. To make the paper as concise as possible, we only derived the CANARY algorithm for aperiodic auto-correlations. However, the main ideas of CANARY can also be used to design CSS with good periodic correlations. In the latter case, CANARY can be useful when single sequences with perfect periodic correlation do not exist (such as in the certain design example in the next section). Let $\tilde{\boldsymbol{A}}^H$ denote the $N \times N$ DFT matrix. It is straightforward to verify that the design of CSS with good periodic correlations can be formulated as the following optimization problem:

$$\min_{\boldsymbol{X},\, \boldsymbol{S}} \quad \|\tilde{\boldsymbol{A}}^H \boldsymbol{X} - \boldsymbol{S}\|_F$$
$$\text{s.t.} \quad (\boldsymbol{S} \odot \boldsymbol{S}^*) \boldsymbol{1}_M = MN \boldsymbol{1}_N, \tag{20}$$

which can be tackled in the same manner as proposed for (10). $\square$

Table 1
The CANARY algorithm.

Step 0: Initialize $\boldsymbol{Z}$ using a random $\boldsymbol{X} \in \mathbb{C}^{N \times M}$
Step 1: Compute the minimizer $\boldsymbol{S}$ of (10) using (15)
Step 2: Depending on the constraint imposed on the sequences $\{\boldsymbol{x}_m\}$, compute the minimizer $\boldsymbol{Z}$ (equivalently $\boldsymbol{X}$) of (10) using (16), (17), or (19)
Step 3: Repeat steps 1 and 2 until a stop criterion is satisfied, e.g. $\|\boldsymbol{X}^{(l+1)} - \boldsymbol{X}^{(l)}\|_F \le \epsilon$ for some pre-defined $\epsilon > 0$ (where $\boldsymbol{X}^{(l)}$ denotes the matrix $\boldsymbol{X}$ obtained at the lth iteration)

Remark 2. An alternative approach to designing CSS is to use the weighted CAN (WeCAN) algorithm in [23]. To see how this can be done, let $\boldsymbol{y} \triangleq (\boldsymbol{x}_1^T, \boldsymbol{0}_{N-1}^T, \boldsymbol{x}_2^T, \boldsymbol{0}_{N-1}^T, \ldots, \boldsymbol{x}_M^T, \boldsymbol{0}_{N-1}^T)$ be an auxiliary sequence of length $M(2N-1)$. Note that the first N aperiodic autocorrelation lags of $\boldsymbol{y}$ (denoted by $\{R(k)\}$) can be written as

$$R(k) = \sum_{m=1}^{M} r_m(k), \quad 0 \le k \le (N-1). \tag{21}$$

Therefore, the sequence set $\{\boldsymbol{x}_m\}$ is complementary if and only if $\boldsymbol{y}$ has a zero correlation zone (ZCZ) for lags in the interval $1 \le k \le (N-1)$. Such a ZCZ design (with the given sequence structure) can be carried out using the WeCAN algorithm. However, this approach is computationally expensive compared to the CANARY algorithm. □

3. Phase-quantized CSS design

A sequence $\boldsymbol{x}$ of length N is phase-quantized (with phase quantization level L) iff

$$\arg(x(n)) \in \left\{0, \frac{2\pi}{L}(1), \ldots, \frac{2\pi}{L}(L-1)\right\} \tag{22}$$

for all $1 \le n \le N$. In particular, $\boldsymbol{x}$ is a phase-quantized unimodular sequence (with phase quantization level L) iff for any $1 \le n \le N$

$$x(n) \in \{1, e^{j(2\pi/L)(1)}, \ldots, e^{j(2\pi/L)(L-1)}\}. \tag{23}$$

The CANARY algorithm can be used to try to find (unimodular) phase-quantized CSS (or sub-optimal CSS whenever a perfect CSS does not exist) for arbitrary N and M; however, a certain modification is needed. Let $Q_L(\varphi)$ denote the closest element in the set of quantized levels in (22) to a given φ. Also let $v_{n,m} = |v_{n,m}| e^{j\varphi_{n,m}} = [\boldsymbol{AS}]_{n,m}$. For unimodular phase-quantized CSS (with phase quantization level L), the minimizer $\boldsymbol{Z}$ of (10) is given by

$$[\boldsymbol{Z}]_{n,m} = \begin{cases} e^{jQ_L(\varphi_{n,m})}, & 1 \le n \le N, \\ 0, & n > N \end{cases} \tag{24}$$

and for just phase-quantized CSS by

$$[\boldsymbol{Z}]_{n,m} = \begin{cases} |v_{n,m}| \cos(\varphi_{n,m} - Q_L(\varphi_{n,m}))\, e^{jQ_L(\varphi_{n,m})}, & 1 \le n \le N, \\ 0, & n > N. \end{cases} \tag{25}$$

However, for small values of L, unimodular (or low PAR) sequences with practically optimal correlation properties are rare. In addition, we note that the objective function in (10) is highly multi-modal in such cases (i.e. it may have multiple local optima). Therefore, although using (24) (or (25)) monotonically decreases the objective function in (10), the algorithm might end up in a poor local optimum. To tackle this issue (which was noted in many other publications such as [2,9,4]), we consider a penalized version of (10) in the following.

We relax the unimodularity constraint to a penalization of the distance between the magnitudes of $\{x_m(n)\}_{m,n}$ and 1. Therefore, consider the optimization problem (for $\lambda > 0$)

$$\begin{aligned} \min_{\boldsymbol{X},\boldsymbol{S}} \quad & \|\boldsymbol{A}^H\boldsymbol{Z} - \boldsymbol{S}\|_F^2 + \lambda \|(\boldsymbol{X} \odot \boldsymbol{X}^*) - \boldsymbol{1}_{N\times M}\|_1 \\ \text{s.t.} \quad & (\boldsymbol{S} \odot \boldsymbol{S}^*)\boldsymbol{1}_M = MN\, \boldsymbol{1}_{2N}, \\ & \boldsymbol{Z} = \begin{pmatrix} \boldsymbol{X} \\ \boldsymbol{0}_{N\times M} \end{pmatrix}, \\ & \text{all } \{\boldsymbol{x}_m\} \text{ are phase-quantized as in (22).} \end{aligned} \tag{26}$$

The solution $\boldsymbol{S}$ of (26) is identical to that of (10). Let v be a generic element in the $N \times M$ upper sub-matrix of $\boldsymbol{AS}$. To obtain the solution $\boldsymbol{X}$ (and $\boldsymbol{Z}$) of (26), we note that solving (26) for $\boldsymbol{X}$ can be dealt with in an element-wise manner, and hence it can be reduced to the optimization problem

$$\begin{aligned} \min_x \quad & |x - v|^2 + \lambda \big| |x|^2 - 1 \big| \\ \text{s.t.} \quad & x \text{ is phase-quantized as in (22),} \end{aligned} \tag{27}$$

where x denotes a generic entry of $\boldsymbol{X}$. Now let $x = |x| e^{j\varphi_x}$, $v = |v| e^{j\varphi_v}$, and note that the minimizer φ_x of (27) is simply given by $\varphi_x = Q_L(\varphi_v)$. Given φ_x, we can rewrite the criterion in (27) as

$$\begin{aligned} & |x - v|^2 + \lambda \big| |x|^2 - 1 \big| \\ & \quad = \big| |x| - |v| e^{j(\varphi_v - \varphi_x)} \big|^2 + \lambda \big| |x|^2 - 1 \big| \\ & \quad = \text{Const}_1 + \underbrace{(|x| - |v| \cos(\varphi_v - \varphi_x))^2 + \lambda \big| |x|^2 - 1 \big|}_{f(|x|)}. \end{aligned} \tag{28}$$

Note that $f(|x|)$ is both continuous and lower bounded (by zero), and thus has at least one global minimum. A global minimum $|x|$ of $f(|x|)$ satisfying $|x| > 1$ should minimize

$$f(|x|) = (1+\lambda)|x|^2 - 2|x|\,|v| \cos(\varphi_v - \varphi_x) + \text{Const}_2, \tag{29}$$

which implies that $|x| = |v| \cos(\varphi_v - \varphi_x)/(1+\lambda)$. Otherwise, a minimizer $|x|$ of $f(|x|)$ satisfying $|x| < 1$ should minimize

$$f(|x|) = (1-\lambda)|x|^2 - 2|x|\,|v| \cos(\varphi_v - \varphi_x) + \text{Const}_3, \tag{30}$$

which implies $|x| = |v| \cos(\varphi_v - \varphi_x)/(1-\lambda)$. In sum, the minimization of (27) with respect to $|x|$ yields the following *soft-thresholding* type of solution (see [26] for a similar result):

$$|x| = \begin{cases} \dfrac{|v| \cos(\varphi_v - \varphi_x)}{1-\lambda} & |v| < \dfrac{1-\lambda}{\cos(\varphi_v - \varphi_x)}, \\ 1 & \dfrac{1-\lambda}{\cos(\varphi_v - \varphi_x)} \le |v| \le \dfrac{1+\lambda}{\cos(\varphi_v - \varphi_x)}, \\ \dfrac{|v| \cos(\varphi_v - \varphi_x)}{1+\lambda} & |v| > \dfrac{1+\lambda}{\cos(\varphi_v - \varphi_x)}. \end{cases} \tag{31}$$

4. Numerical examples

In this section, we provide numerical examples to illustrate the performance of the CANARY algorithm. The required computational times (on a PC with Intel Core i5 2.8 GHz CPU and 8.0 GB memory) are reported. We use CANARY to design unimodular as well as low PAR CSS of length $N=256$ with $M=1$ (in which case the CSS design becomes a single sequence design), $M=2$ (i.e. a complementary pair), and $M=3$. We stopped the algorithm when the stop criterion was satisfied with $\epsilon = 10^{-15}$. The computational times for designing unimodular CSS with $M=1$, 2, and 3 were approximately 3, 175, and 254 s, respectively. The results are shown in Fig. 1(a). The autocorrelation sums are normalized and expressed in dB

$$\text{autocorrelation level (dB)} = 20\log_{10}\frac{\left|\sum_{m=1}^{M} r_m(k)\right|}{\sum_{m=1}^{M} r_m(0)}. \quad (32)$$

To examine CANARY when dealing with more general PAR constraints, Fig. 1(b) depicts the results of a similar design problem but now the constraint PAR ≤ 2. The needed computational times were 6, 143, and 78 s for $M=1$, 2, and 3, respectively. As expected, the CSS designed for $M\in\{1,2\}$ and PAR ≤ 2 have better MF values compared to their corresponding CSS with PAR $=1$ (i.e. unimodular CSS). Note that increasing M provides more degrees of freedom for CSS design. In particular, it can be observed from the figure that for $M=3$ the autocorrelation sums of the sequences achieve values which are virtually zero (i.e. MF approaches $+\infty$).

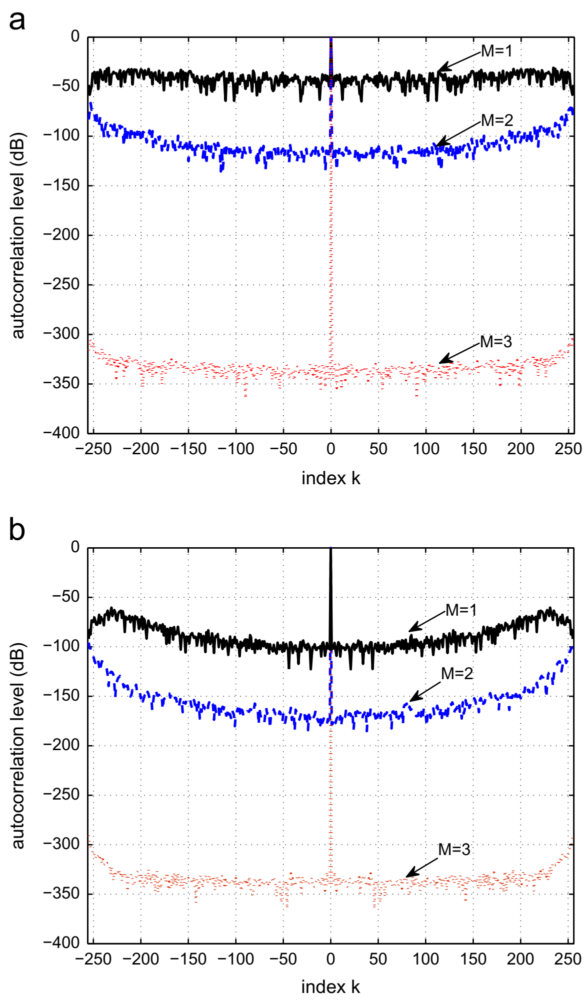

Fig. 1. Unimodular and low PAR CSS design for $N=256$ and $M\in\{1,2,3\}$ with the constraints: (a) PAR $=1$ (i.e. unimodular entries) and (b) PAR ≤ 2. The autocorrelation sums achieve practically zero values as M increases to 3. The MF values corresponding to $M=1,2,3$ in (a) and (b) are given by $(15.9, 1.0\times 10^6, 4.0\times 10^{29})$ and $(6.0\times 10^4, 9.6\times 10^8, 4.1\times 10^{28})$, respectively.

As indicated earlier, CANARY can be used to obtain almost (i.e. sub-optimal) CSS for cases in which no CSS exists. It is known that there is no binary GP of length $N=82$ [20]. With this in mind, we employ the CANARY algorithm to design a sub-optimal GP for $N=82$. Using the relaxed formulation of CANARY in (26) five hundred times (with $\lambda=0.5$), we have designed real-valued complementary pairs with low PAR. Next we clipped the resultant sequences to obtain sub-optimal GP and chose the best sequence pair with respect to the ISL metric. The two sequences obtained in this way are shown in Fig. 2(a). The average autocorrelation of the obtained binary sequences, viz.

$$\frac{1}{2}\left|\sum_{m=1}^{2} r_m(k)\right|, \quad -81\le k\le 81 \quad (33)$$

is presented in Fig. 2(b). The obtained sub-optimal GP achieves a MF value of 19.88. A computational time of 41 s was required on the PC to accomplish the task. As another example, we use the same approach to design a QAM (i.e. with $L=4$) almost complementary pair of length $N=82$. The results are shown in Fig. 2(c)–(d). For the convenience of the reader, the resultant sequences of both binary and QAM examples are provided in Table 2. The obtained QAM CSS has a MF equal to 21.08. As expected the MF corresponding to the obtained QAM CSS is larger than that of the binary example; however, the binary CSS has a smaller PSL. This can be explained by the fact that CANARY is an ISL minimizer (or equivalently a MF maximizer) and not a PSL minimizer. The PAR values of the resultant sequences (before clipping) in the binary and QAM cases are (1.05,1.04) and (1.18,1.15), respectively. As the inner-product (or the distance) of the sequences in the CSS is of interest in some applications, we also report the inner-product values achieved for the above examples. The inner-product metric can be defined as $|\boldsymbol{x}_1^H\boldsymbol{x}_2|/N$, where $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are the sequences in the obtained complementary pairs (both with energy N). The inner-product metric values corresponding to the binary and QAM examples above are 0.024 and 0.039, respectively.

Finally we provide an example to show that CANARY can find known GPs in the search space. Specifically, we consider finding a binary GP of length $N=520$. We perturb the entries of the binary GP by zero-mean i.i.d. Gaussian random variables with a standard deviation σ that take values in the set $\{0.15, 0.25, 0.5, 0.75, 0.85\}$, and then we use the so-obtained perturbed non-binary sequences to initialize CANARY; furthermore we will do this for 1000 times to compute the average statistics. Similar to the previous example, we set $\lambda=0.5$. For each σ, the number of the cases in which CANARY finds the original GP to the total number of tests (i.e. 1000) can be interpreted as the empirical probability p of finding the known GP. The results are reported in Table 3, and the empirical p decreases apparently gracefully and slowly as σ increases.

5. Concluding remarks

The problem of CSS design has been formulated and a fast algorithm (called CANARY) for generating CSS has

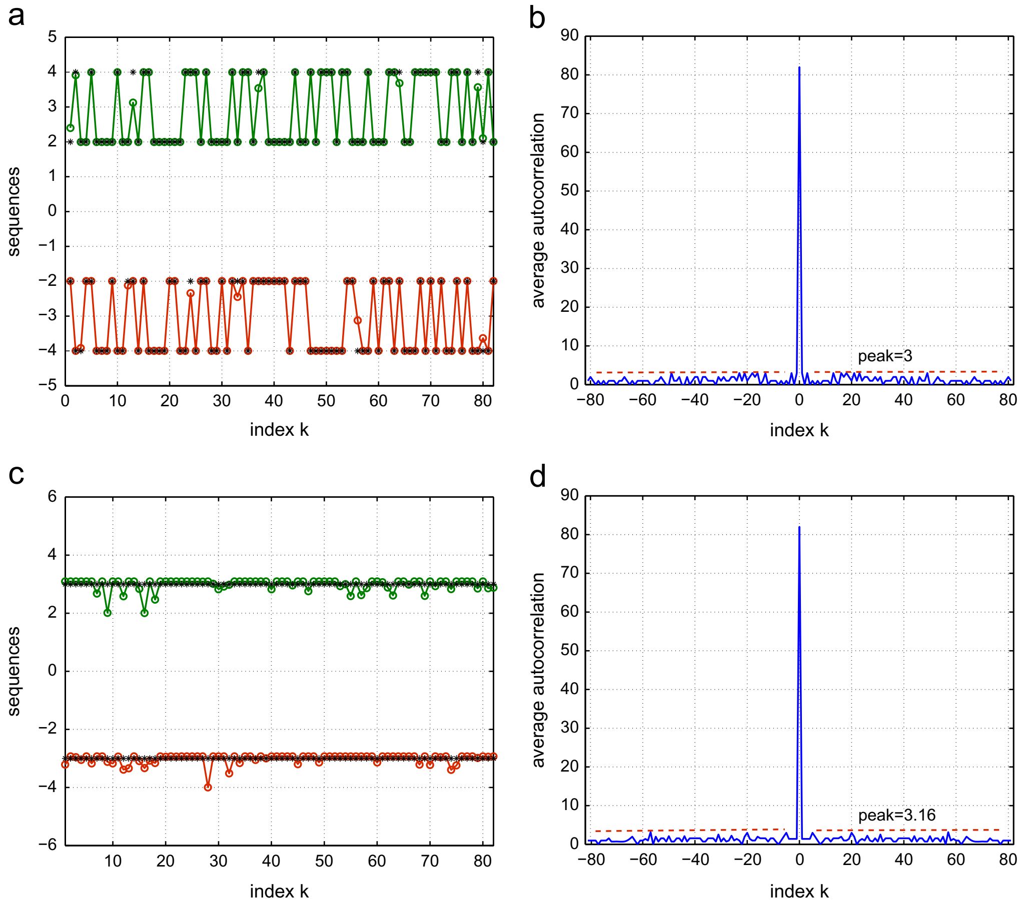

Fig. 2. Design of a sub-optimal GP and an almost CSS (of length 82) using CANARY. (a) Depicts the entries of the binary sequences (shown by $*$) and the resultant sequences of (26). (c) Plots the absolute values of the obtained QAM sequences (shown by $*$) as well as the resultant sequences of (26). In both (a) and (c), a bias of $+3$ and -3 is used to distinguish the sequences. The average autocorrelations of the obtained binary/QAM sequences are shown in (b)/(d).

Table 2
The obtained binary and QAM almost complementary pairs.

Alphabet	Sequence pair
Binary	−1,1,−1,−1,1,−1,−1,−1,−1,1,−1,−1,1,−1,1,1,−1,−1,−1,−1,−1,−1,1,1,1,−1,1,−1,−1,−1,−1,1, → −1,1,1,−1,1,1,−1,−1,−1,−1,−1,1,−1,−1,1,−1,1,1,1,−1,1,1,−1,−1,−1,1,−1,−1,−1,1,1,1,−1, → −1,1,1,1,1,1,−1,−1,1,1,−1,1,−1,1,−1,1,−1
	1,−1,−1,1,1,−1,−1,−1,1,−1,−1,1,1,−1,1,−1,−1,−1,−1,1,1,−1,−1,1,−1,1,1,−1,−1,1,−1,1,1,1, → −1,1,1,1,1,1,1,1,−1,1,1,1,−1,−1,−1,−1,−1,−1,−1,1,1,−1,−1,−1,1,−1,1,1,−1,1,−1,−1,−1,1, → −1,1,−1,1,−1,−1,1,−1,1,−1,−1,−1,−1,1
QAM	j,−1,j,−1,−j,1,−j,1,1,−1,−j,−1,j,1,j,j,j,1,j,−1,j,1,j,−1,j,1,−j,−1,−j,−1,j,1,j,−1,j,1,j,−1, → −j,1,j,1,j,1,j,1,j,−1,j,−1,−j,1,−j,−1,j,−1,−j,−1,−j,1,−j,−1,−j,−1,j,1,−j,−1,j,−1,j,1,j,1, −j, 1, j, 1, −j, 1, j, 1
	j,1,j,1,j,1,j,−1,j,−1,j,−1,−j,−1,j,1,j,−1,j,1,j,−1,−j,1,−j,1,j,−j,−j,−1,j,1,j,1,−j,−1,−j,1,−j, → −1,−j,1,j,−1,j,1,j,1,−j,−1,−j,−1,−j,−1,−j,−1,j,1,j,−1,−j,−1,j,−1,j,1,−j,1,j,−1,−j,−1,−j,1, −j, 1, j, −1, −j, −1, j, 1

been proposed. The main results can be summarized as follows:

- The design of CSS was formulated as a cyclic minimization of (10). Several variations of this minimization problem were proposed for different sequence design constraints (depending on the application). The discussed cases were: (i) a given PAR, (ii) unimodularity of the entries of the sequences, (iii) phase-quantized sequences, and (iv) sequences with unimodular entries and quantized phase values.
- The steps of CANARY are computationally efficient, which enables usage for large (N,M). Moreover, as the sequence design problems are usually solved off-line,

Table 3
The empirical probability p of finding the known GP for various perturbation levels σ.

σ	0.15	0.25	0.5	0.75	0.85
p	1	1	1	0.95	0.70

the computational efficiency of CANARY can be leveraged to perform an efficient search of CSS when only a single set is not enough.

- The formulation in this paper can be exploited to deal with the CSS design for good periodic correlation properties as well. Detailed derivations were not presented for the sake of brevity.
- Numerical examples were provided to examine the performance of CANARY when dealing with different CSS design problems.

We conclude this paper by returning to the fact that CANARY is a scheme that attempts to minimize the ISL. We note that one can generally make the PSL metric "small" by minimizing the ISL. However, a direct minimization of the PSL metric appears to be more complicated and remains a topic for future work.

Acknowledgment

This work was supported in part by the European Research Council (ERC) under Grant #228044 and the Swedish Research Council.

References

[1] H. He, J. Li, P. Stoica, Waveform Design for Active Sensing Systems: A Computational Approach, Cambridge University Press, Cambridge, UK, 2012.
[2] J. Jedwab, A survey of the merit factor problem for binary sequences, in: T. Helleseth, D. Sarwate, H.-Y. Song, K. Yang (Eds.), Sequences and their Applications—SETA 2004, Lecture Notes in Computer Science, vol. 3486, Springer, Berlin, Heidelberg, 2005, pp. 19–21.
[3] B. Dang, J. Li, G. Liao, Taylor polynomial expansion based waveform correlation cancellation for bistatic MIMO radar localization, Signal Processing 92 (6) (2012) 1404–1410.
[4] P. Stoica, H. He, J. Li, On designing sequences with impulse-like periodic correlation, IEEE Signal Processing Letters 16 (8) (2009) 703–706, http://dx.doi.org/10.1109/LSP.2009.2021378.
[5] M.J.E. Golay, Multi-slit spectrometry, Journal of the Optical Society of America 39 (6) (1949) 437, http://dx.doi.org/10.1364/JOSA.41.000468.
[6] C.D. Marziani, J. Ure na, lvaro. Hernández, J.J. García, F.J. lvarez, A. Jiménez, M.C. Pérez, Recursive algorithm to directly obtain the sum of correlations in a CSS, Signal Processing 91 (5) (2011) 1343–1346.
[7] N. Levanon, Noncoherent radar pulse compression based on complementary sequences, IEEE Transactions on Aerospace and Electronic Systems 45 (2) (2009) 742–747, http://dx.doi.org/10.1109/TAES.2009.5089554.
[8] S. Searle, S. Howard, B. Moran, The use of complementary sets in MIMO radar, in: Asilomar Conference on Signals, Systems and Computers, 2008, pp. 510–514. http://dx.doi.org/10.1109/ACSSC.2008.5074457.
[9] M. Soltanalian, P. Stoica, Computational design of sequences with good correlation properties, IEEE Transactions on Signal Processing 60 (5) (2012) 2180–2193.
[10] P. Cristea, R. Tuduce, J. Cornelis, Complementary sequences for coded aperture imaging, in: 50th International ELMAR Symposium, vol. 1, 2008, pp. 53–56.
[11] I. Trots, A. Nowicki, W. Secomski, J. Litniewski, Golay sequences-sidelobe-cancelling codes for ultrasonography, Archive of Acoustics 29 (1) (2004) 87–97.
[12] S.M. Tseng, M. Bell, Asynchronous multicarrier DS-CDMA using mutually orthogonal complementary sets of sequences, IEEE Transactions on Communications 48 (1) (2000) 53–59, http://dx.doi.org/10.1109/26.818873.
[13] E. Garcia, J. Garcia, J. Urena, M. Perez, D. Ruiz, Multilevel complementary sets of sequences and their application in UWB, in: International Conference on Indoor Positioning and Indoor Navigation (IPIN), 2010, pp. 1–5. http://dx.doi.org/10.1109/IPIN.2010.5647962.
[14] J. Davis, J. Jedwab, Peak-to-mean power control in OFDM, Golay complementary sequences, and Reed–Muller codes, IEEE Transactions on Information Theory 45 (7) (1999) 2397–2417, http://dx.doi.org/10.1109/18.796380.
[15] K.U. Schmidt, Complementary sets, generalized Reed–Muller codes, and power control for OFDM, IEEE Transactions on Information Theory 53 (2) (2007) 808–814, http://dx.doi.org/10.1109/TIT.2006.889723.
[16] P. Spasojevic, C. Georghiades, Complementary sequences for ISI channel estimation, IEEE Transactions on Information Theory 47 (3) (2001) 1145–1152, http://dx.doi.org/10.1109/18.915670.
[17] T. Kojima, N. Ohtani, T. Matsumoto, U. Parampalli, On multiple information embedding by digital watermarking based on complete complementary codes, in: Signal Design and its Applications in Communications (IWSDA), 2011, pp. 100–103. http://dx.doi.org/10.1109/IWSDA.2011.6159397.
[18] R. Turyn, Hadamard matrices, Baumert–Hall, four symbol sequences, pulse compression, and surface wave encodings, Journal of Combinatorial Theory, Series A 16 (1974) 313–333.
[19] L. Bomer, M. Antweiler, Periodic complementary binary sequences, IEEE Transactions on Information Theory 36 (6) (1990) 1487–1494, http://dx.doi.org/10.1109/18.59954.
[20] M. Parker, K.G. Paterson, Tellambura, Golay Complementray Sequences, 1st ed. Wiley, 2003.
[21] P. Borwein, R. Ferguson, A complete description of Golay pairs for lengths up to 100, Mathematics of Computation 73 (246) (2004) 967–986.
[22] A.T. Fam, I. Sarkar, A new class of interlaced complementary codes based on components with unity peak sidelobes, Signal Processing 88 (2) (2008) 307–314.
[23] P. Stoica, H. He, J. Li, New algorithms for designing unimodular sequences with good correlation properties, IEEE Transactions on Signal Processing 57 (4) (2009) 1415–1425, http://dx.doi.org/10.1109/TSP.2009.2012562.
[24] P. Stoica, R.L. Moses, Spectral Analysis of Signals, Prentice Hall, Upper Saddle River, NJ, 2005.
[25] J. Tropp, I. Dhillon, R. Heath, T. Strohmer, Designing structured tight frames via an alternating projection method, IEEE Transactions on Information Theory 51 (1) (2005) 188–209, http://dx.doi.org/10.1109/TIT.2004.839492.
[26] J. Loubes, S. Van De Geer, Adaptive estimation with soft thresholding penalties, Statistica Neerlandica 56 (4) (2002) 453–478, http://dx.doi.org/10.1111/1467-9574.t01-1-00212.