Joint Design of the Receive Filter and Transmit Sequence for Active Sensing

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***Abstract*—Due to its long-standing importance, the problem of designing the receive filter and transmit sequence for clutter/interference rejection in active sensing has been studied widely in the last decades. In this letter, we propose a cyclic optimization of the transmit sequence and the receive filter. The proposed approach can handle arbitrary peak-to-average-power ratio (PAR) constraints on the transmit sequence, and can be used for large dimension designs (with $\sim 10^3$ variables) even on an ordinary PC.**

***Index Terms*—Clutter rejection, peak-to-average-power ratio (PAR), probing signal, receive filter.**

I. Introduction and Problem Formulation

A KEY design problem in cognitive active sensing is to jointly optimize the probing sequence and the receive filter (using *apriori* knowledge on clutter/interference) in order to minimize the estimation error of the target parameters. Let $\boldsymbol{s} = (s_1\, s_2 \cdots s_N)^T$ denote the transmit sequence which is used to modulate the train of pulses [1]. In the following, we adopt the discrete model in [2] to formulate the problem. Particularly, we assume that the received baseband signal satisfies the following equation:

$$\boldsymbol{y} = \boldsymbol{A}^H \boldsymbol{\alpha} + \boldsymbol{\epsilon} \tag{1}$$

with

$$\mathbf{A}^H = \begin{pmatrix} s_1 & 0 & \cdots & 0 & s_N & s_{N-1} & \cdots & s_2 \\ s_2 & s_1 & & \vdots & 0 & s_N & & \vdots \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \ddots & s_N \\ s_N & s_{N-1} & \cdots & s_1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{2}$$

$$\boldsymbol{\alpha} = (\alpha_0 \alpha_1 \cdots \alpha_{N-1} \alpha_{-(N-1)} \cdots \alpha_{-1})^T \tag{3}$$

where $\{\alpha_k\}$ are the scattering coefficients of different range cells, and $\boldsymbol{\epsilon}$ denotes the signal independent interference. We also assume that

$$\begin{aligned} \mathbb{E}\{\boldsymbol{\epsilon}\boldsymbol{\epsilon}^H\} &= \boldsymbol{\Gamma}, \\ \mathbb{E}\{|\alpha_k|^2\} &= \beta, \quad k \neq 0 \end{aligned} \tag{4}$$

where the interference covariance matrix $\boldsymbol{\Gamma}$, and the average clutter power β are given (e.g., they are obtained by some prescan procedures [3]), and that $\boldsymbol{\epsilon}$ and $\{\alpha_k\}$ have zero mean and are independent of each other. The estimation of the scattering coefficient of current interest α_0 can be accomplished using a matched filter. However, an estimate of α_0 with generally smaller mean square error (MSE) can be obtained via a suitable mismatched filtering (MMF) of the received data. The MMF estimate of α_0 is given by $\hat{\alpha}_0 = (\boldsymbol{w}^H\boldsymbol{y})/(\boldsymbol{w}^H\boldsymbol{s})$ where $\boldsymbol{w} \in \mathbb{C}^N$ is the MMF vector. The MSE of the above estimate of α_0 can be expressed as

$$\mathrm{MSE}(\hat{\alpha}_0) = \mathbb{E}\left\{\left|\frac{\boldsymbol{w}^H\boldsymbol{y}}{\boldsymbol{w}^H\boldsymbol{s}} - \alpha_0\right|^2\right\} = \frac{\boldsymbol{w}^H\boldsymbol{R}\boldsymbol{w}}{|\boldsymbol{w}^H\boldsymbol{s}|^2} \tag{5}$$

where

$$\boldsymbol{R} = \beta \sum_{0<|k|\le(N-1)} \boldsymbol{J}_k \boldsymbol{s}\boldsymbol{s}^H \boldsymbol{J}_k^H + \boldsymbol{\Gamma} \tag{6}$$

and $\{\boldsymbol{J}_k\}$ are the shifting matrices defined by

$$[\boldsymbol{J}_k]_{l,m} = [\boldsymbol{J}_{-k}^H]_{l,m} \triangleq \delta_{m-l-k} \tag{7}$$

where $\delta_{(\,.\,)}$ denotes the Kronecker delta function. The principal objective of the **c**ognitive **re**ceiver and **w**aveform (CREW) design of $\boldsymbol{w}$ and $\boldsymbol{s}$ is to minimize the MSE of $\hat{\alpha}_0$ (see e.g. [2] for a review of the relevant literature of this design). In the following section, a new approach to CREW is presented.

II. CREW(Cyclic)

In this section, we propose a cyclic minimization of the MSE criterion in (5). For fixed $\boldsymbol{s}$, the minimization of (5) with respect to (w.r.t.) $\boldsymbol{w}$ results in the closed-form expression:

$$\boldsymbol{w} = \boldsymbol{R}^{-1}\boldsymbol{s} \tag{8}$$

to within a multiplicative constant. For fixed $\boldsymbol{w}$, the minimizing transmit code $\boldsymbol{s}$ of (5) can be obtained as follows. Note that

$$\begin{aligned} \boldsymbol{w}^H\boldsymbol{R}\boldsymbol{w} &= \boldsymbol{w}^H\left(\beta \sum_{0<|k|\le(N-1)} \boldsymbol{J}_k\boldsymbol{s}\boldsymbol{s}^H\boldsymbol{J}_k^H + \boldsymbol{\Gamma}\right)\boldsymbol{w} \\ &= \boldsymbol{s}^H \underbrace{\left(\beta \sum_{0<|k|\le(N-1)} \boldsymbol{J}_k^H\boldsymbol{w}\boldsymbol{w}^H\boldsymbol{J}_k\right)}_{\boldsymbol{Q}} \boldsymbol{s} + \underbrace{\boldsymbol{w}^H\boldsymbol{\Gamma}\boldsymbol{w}}_{\mu}. \end{aligned} \tag{9}$$

As a result, the design metric in (5) can be rewritten as

$$\mathrm{MSE}(\hat{\alpha}_0) = \frac{\boldsymbol{w}^H\boldsymbol{R}\boldsymbol{w}}{|\boldsymbol{w}^H\boldsymbol{s}|^2} = \frac{\boldsymbol{s}^H\boldsymbol{Q}\boldsymbol{s} + \mu}{\boldsymbol{s}^H\boldsymbol{W}\boldsymbol{s}} \tag{10}$$

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where $\boldsymbol{W} = \boldsymbol{w}\boldsymbol{w}^H$. We observe that both the numerator and denominator of (10) are quadratic in $\boldsymbol{s}$. To deal with the minimization of (10), we exploit the idea of fractional programming [4]. Let $a(\boldsymbol{s}) = \boldsymbol{s}^H\boldsymbol{Q}\boldsymbol{s} + \mu, b(\boldsymbol{s}) = \boldsymbol{s}^H\boldsymbol{W}\boldsymbol{s}$, and note that for MSE to be finite we must have $b(\boldsymbol{s}) > 0$. Moreover, let $f(\boldsymbol{s}) = \mathrm{MSE}(\hat{\alpha}_0) = a(\boldsymbol{s})/b(\boldsymbol{s})$ and suppose that $\boldsymbol{s}_\star$ denotes the current value of $\boldsymbol{s}$. We define $g(\boldsymbol{s}) \triangleq a(\boldsymbol{s}) - f(\boldsymbol{s}_\star)b(\boldsymbol{s})$, and $\boldsymbol{s}_\dagger \triangleq \arg\min_{\boldsymbol{s}} g(\boldsymbol{s})$. It is straightforward to verify that $g(\boldsymbol{s}_\dagger) \leq g(\boldsymbol{s}_\star) = 0$. Consequently, we have that $g(\boldsymbol{s}_\dagger) = a(\boldsymbol{s}_\dagger) - f(\boldsymbol{s}_\star)b(\boldsymbol{s}_\dagger) \leq 0$ which implies

$$f(\boldsymbol{s}_\dagger) \leq f(\boldsymbol{s}_\star) \tag{11}$$

as $b(\boldsymbol{s}_\dagger) > 0$. Therefore, $\boldsymbol{s}_\dagger$ can be considered as a new vector $\boldsymbol{s}$ which decreases $f(\boldsymbol{s})$. Note that for (11) to hold, $\boldsymbol{s}_\dagger$ does not necessarily have to be a minimizer of $g(s)$; indeed, it is enough if $\boldsymbol{s}_\dagger$ is such that $g(\boldsymbol{s}_\dagger) \leq g(\boldsymbol{s}_\star)$.

For a given MMF vector $\boldsymbol{w}$, and any $\boldsymbol{s}_\star$ of the minimizer $\boldsymbol{s}$ of (10) we have (assuming $\|\boldsymbol{s}\|_2^2 = N$):

$$g(\boldsymbol{s}) = \boldsymbol{s}^H(\boldsymbol{Q} + (\mu/N)\boldsymbol{I} - f(\boldsymbol{s}_\star)\boldsymbol{W})\boldsymbol{s} = \boldsymbol{s}^H\boldsymbol{T}\boldsymbol{s} \tag{12}$$

where $\boldsymbol{T} \triangleq \boldsymbol{Q} + (\mu/N)\boldsymbol{I} - f(\boldsymbol{s}_\star)\boldsymbol{W}$. Now, let λ be a real number larger than the maximum eigenvalue of $\boldsymbol{T}$. Then the minimization of (10) w.r.t. unimodular $\boldsymbol{s}$ can be cast as the following unimodular quadratic program (UQP) [5]:

$$\begin{aligned} \max_{\boldsymbol{s}} \quad & \boldsymbol{s}^H\tilde{\boldsymbol{T}}\boldsymbol{s} \\ \text{s.t.} \quad & |s_k| = 1, \quad 1 \leq k \leq N, \end{aligned} \tag{13}$$

in which $\tilde{\boldsymbol{T}} \triangleq \lambda\boldsymbol{I} - \boldsymbol{T}$ is positive definite. Note that (13) is NP-hard in general (see, e.g., [6]). A possible approach to deal with (13) is to employ the semi-definite relaxation (SDR) method which is widely used in the literature. However, SDR is based on a core semi-definite program (SDP) which makes it computationally expensive as N grows large. To tackle (13) efficiently, in [5] a set of *power method-like* iterations was introduced that can be used to monotonically increase the criterion in (13) (or equivalently decrease $f(\boldsymbol{s})$); namely, the vector $\boldsymbol{s}$ is updated using the nearest-vector problem

$$\begin{aligned} \min_{\boldsymbol{s}^{(t+1)}} \quad & \left\| \boldsymbol{s}^{(t+1)} - \tilde{\boldsymbol{T}}\boldsymbol{s}^{(t)} \right\|_2 \\ \text{s.t.} \quad & \left| s_k^{(t+1)} \right| = 1, \quad 1 \leq k \leq N. \end{aligned} \tag{14}$$

The solution of (14) is simply given by $\boldsymbol{s}^{(t+1)} = e^{j\arg(\tilde{\boldsymbol{T}}\boldsymbol{s}^{(t)})}$. A proof of monotonically increasing behavior of the associated UQP objective function through the above power method-like iterations is presented in Appendix A.

In many applications, unimodularity (i.e., unit PAR) is not required for the transmit sequence $\boldsymbol{s}$. As a result, one can consider a more general PAR constraint, viz. $\mathrm{PAR} = \|\boldsymbol{s}\|_\infty^2/(\frac{1}{N}\|\boldsymbol{s}\|_2^2) \leq \gamma$ for designing $\boldsymbol{s}$. In such a situation, a similar formulation as in the case of unimodular $\boldsymbol{s}$ can be used. More concretely, a decrease of the MSE criterion in (10) for a PAR constrained $\boldsymbol{s}$ can be achieved via increasing the objective function of the following optimization problem:

$$\begin{aligned} \max_{\boldsymbol{s}} \quad & \boldsymbol{s}^H\tilde{\boldsymbol{T}}\boldsymbol{s} \\ \text{s.t.} \quad & |s_k| \leq \sqrt{\gamma}, \quad 1 \leq k \leq N, \\ & \|\boldsymbol{s}\|_2^2 = N \end{aligned} \tag{15}$$

TABLE I
CREW (CYCLIC)

Step 0: Initialize the transmit sequence $\boldsymbol{s}$ with a unimodular (or low PAR) vector in $\mathbb{C}^N$.
Step 1: Compute the matrix $\boldsymbol{R}$, and find the optimal MMF vector $\boldsymbol{w}$ using (8).
Step 2 Compute the scalar μ, and the matrices $\boldsymbol{Q}$ and $\boldsymbol{W}$. Use t as the internal iteration counter of step 2, and while $f(\boldsymbol{s}^{(t)}) - f(\boldsymbol{s}^{(t+1)}) > \delta$ (for some fixed $\delta > 0$) do:
Step 2-1: Form the matrix $\tilde{\boldsymbol{T}}$ (as defined in (13)) using the current vector $\boldsymbol{s}$.
Step 2-2: Employ the power method-like iterations following (14) or (16) (depending on the code constraint) to update $\boldsymbol{s}$; until convergence.
Step 3: Repeat steps 1 and 2 until a stop criterion is satisfied, e.g. $\left\|\mathrm{MSE}^{(v+1)} - \mathrm{MSE}^{(v)}\right\| < \epsilon$ for some given $\epsilon > 0$, where v denotes the outer loop iteration number.

where $\tilde{\boldsymbol{T}}$ is defined as in (13). To this end, we note that the derivation of the power method-like iterations in [5] can be conveniently generalized to the case with a PAR constraint. In particular, one can increase the objective function of (15) by updating $\boldsymbol{s}$ using the nearest-vector problem

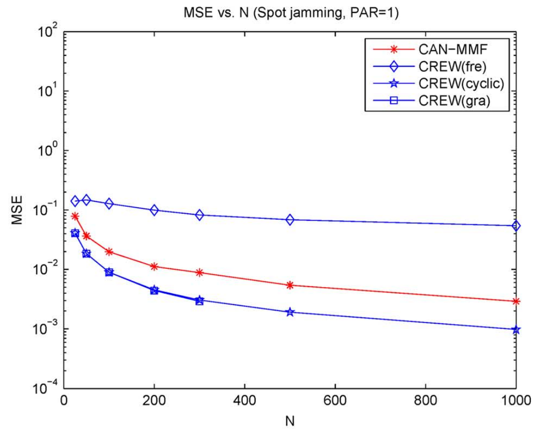
$$\begin{aligned} \min_{\boldsymbol{s}^{(t+1)}} \quad & \left\| \boldsymbol{s}^{(t+1)} - \tilde{\boldsymbol{T}}\boldsymbol{s}^{(t)} \right\|_2 \\ \text{s.t.} \quad & \left| s_k^{(t+1)} \right| \leq \sqrt{\gamma}, \quad 1 \leq k \leq N, \\ & \left\| \boldsymbol{s}^{(t+1)} \right\|_2^2 = N \end{aligned} \tag{16}$$

which can be solved efficiently via a recursive algorithm suggested in [7].

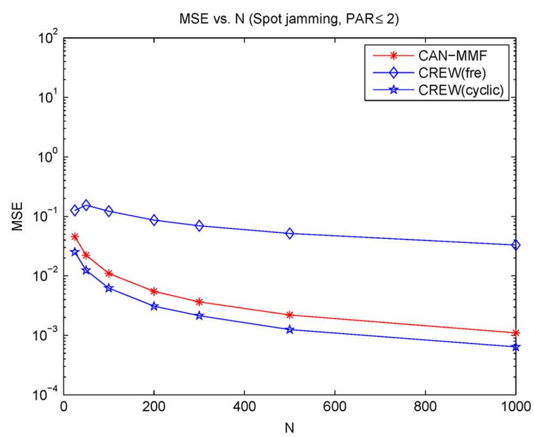
The CREW(cyclic) algorithm derived above is summarized in Table I. Note that the matrices $\boldsymbol{R}$ and $\boldsymbol{Q}$ can be computed efficiently by employing fast Fourier transform (FFT) operations. We refer the interested reader to Appendix B for the derivation of such an efficient computational scheme.

III. DISCUSSION AND NUMERICAL EXAMPLES

In this section, we examine the performance of CREW(cyclic) by comparing it with three methods previously devised in [2]; namely CAN-MMF, CREW(gra) and CREW(fre). The CAN-MMF method employs the CAN algorithm in [8] to design a transmit sequence with good correlation properties. As a result, the design of the transmit waveform is independent of the receive filter. The receive filter of CAN-MMF is obtained by (8). Note that no prior knowledge of interference is used in the waveform design of CAN-MMF. CREW(gra) is a gradient based algorithm for minimizing (5) which can only deal with the unimodularity constraint. Moreover, a large number of iterations is needed by CREW(gra) until convergence and, in each iteration, the update of the gradient vector is time consuming. CREW(fre) is a frequency-based approach that yields globally optimal values of the spectrum of the transmit waveform as well as the receive filter for a relaxed version of the original waveform design problem, and hence in general does not provide an optimal solution to the latter problem. Like CAN-MMF, CREW(fre) can handle both unimodularity and PAR constraints. Moreover, it can be used to design relatively long sequences due to the leveraged FFT operations.

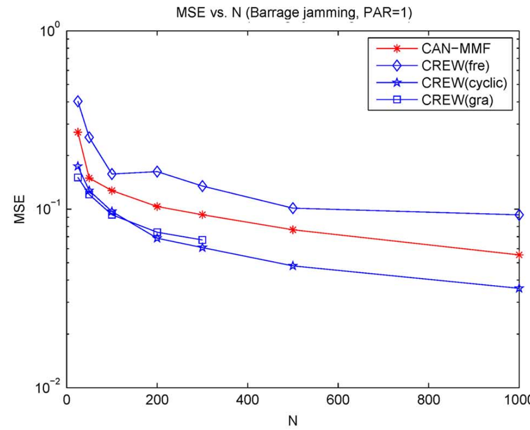


(a)

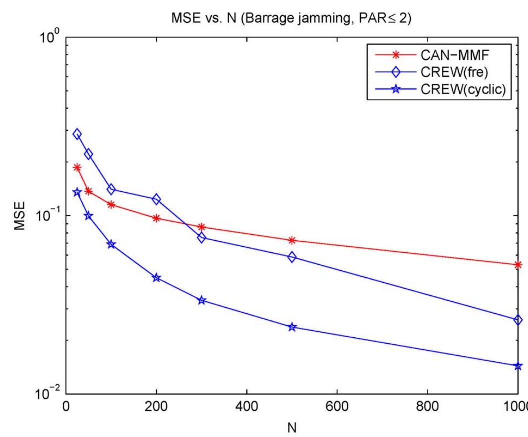


(b)

Fig. 1. MSE values obtained by the different design algorithms for a spot jamming with normalized frequency $f_0 = 0.2$, and the following PAR constraints on the transmit sequence: (a) $\mathrm{PAR} = 1$ (unimodularity constraint), (b) $\mathrm{PAR} \leq 2$.



(a)



(b)

Fig. 2. MSE values obtained by the different design algorithms for a barrage jamming in the normalized frequency interval $[f_1, f_2] = [0.2, 0.3]$ and the following constraints on the transmit sequence: (a) $\mathrm{PAR} = 1$ (unimodularity constraint), (b) $\mathrm{PAR} \leq 2$.

We adopt the same simulation examples as in [2]. Particularly, we consider the following interference covariance matrix:

$$\boldsymbol{\Gamma} = \sigma_J^2 \boldsymbol{\Gamma}_J + \sigma^2 \boldsymbol{I} \tag{17}$$

where $\sigma_J^2 = 100$ and $\sigma^2 = 0.1$ are the jamming and noise powers, respectively, and the jamming covariance matrix $\boldsymbol{\Gamma}_J$ is given by $[\boldsymbol{\Gamma}_J]_{k,l} = q_{k-l}$ where $(q_0\, q_1 \cdots q_{N-1}\, q_{-(N-1)} \cdots q_{-1})$ can be obtained by an inverse FFT (IFFT) of the jamming power spectrum $\{\eta_p\}$ at frequencies $(p-1)/(2N-1), p = 1, \ldots, 2N-1$. We set the average clutter power to $\beta = 1$. Furthermore, the Golomb sequence is used to initialize the transmit code $\boldsymbol{s}$ for all the algorithms.

As the first example, we consider a spot jamming located at a normalized frequency $f_0 = 0.2$, with a power spectrum given by

$$\eta_p = \begin{cases} 1, & p = \lfloor (2N-1) f_0 \rfloor \\ 0, & \text{elsewhere}, \end{cases} \qquad p = 1, \ldots, 2N-1. \tag{18}$$

Fig. 1 shows the MSE values corresponding to CAN-MMF, CREW(fre), CREW(gra), and CREW(cyclic), under the unimodularity constraint, for various sequence lengths. In order to include the CREW(gra) algorithm in the comparison, we show its MSE only for $N \leq 300$ since CREW(gra) is computationally prohibitive for $N > 300$ on an ordinary PC. Fig. 1(b) depicts the MSE values obtained by the different algorithms under the constraint $\mathrm{PAR} \leq 2$ on the transmit sequence. One can observe that CREW(cyclic) provides the smallest MSE values for all sequence lengths. In particular, CREW(cyclic) outperforms CAN-MMF and CREW(fre) under both constraints. Due to the fact that both CREW(gra) and CREW(cyclic) are MSE optimizers, the performances of the two methods are almost identical under the unimodularity constraint for $N \leq 300$. On the other hand, compared to CREW(gra), the CREW(cyclic) algorithm can be used to design longer sequences (even more than $N \sim 1000$) owing to its relatively small computational burden. Furthermore, CREW(cyclic) can handle not only the unimodularity constraint but also more general PAR constraints.

Next we consider a barrage jamming located in the normalized frequency band $[f_1, f_2] = [0.2, 0.3]$, and with a power spectrum given by

$$\eta_p = \begin{cases} 1, & \lfloor (2N-1) f_1 \rfloor \leq p \leq \lfloor (2N-1) f_2 \rfloor \\ 0, & \text{elsewhere}, \end{cases} \qquad p = 1, \cdots, 2N-1. \tag{19}$$

Fig. 2(a) plots the MSE values obtained by CAN-MMF, CREW(fre), CREW(gra) and (CREW)cyclic under the unimodularity constraint. Similar to the previous example, the performances of CREW(gra) and CREW(cyclic) are almost

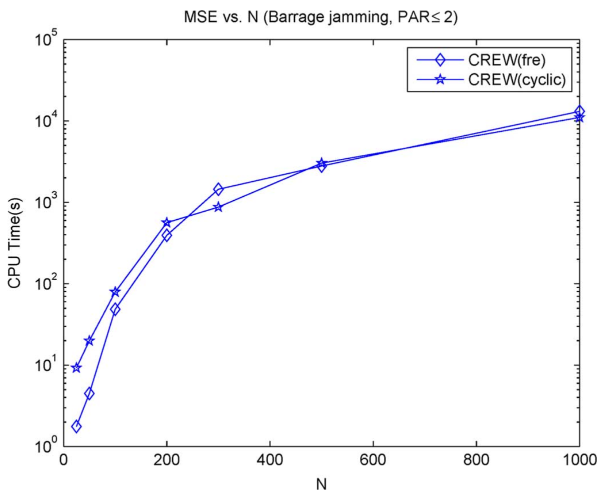


Fig. 3. CPU time of CREW(fre) and CREW(cyclic) for the barrage jamming, and various sequence lengths N, under the constraint of $\mathrm{PAR} \leq 2$.

identical for $N \leq 300$, and the CREW(cyclic) algorithm outperforms the other algorithms for all the sequence lengths. Fig. 2(b) presents the MSE values provided by CAN-MMF, CREW(fre) and CREW(cyclic) for the constraint $\mathrm{PAR} \leq 2$ on the transmit sequence. It can be observed that CREW(cyclic) yields a lower MSE than the other algorithms for all lengths.

According to Fig. 3, although an iteration of CREW(fre) is more computationally efficient than an iteration of CREW(cyclic), the overall CPU time of CREW(fre) until convergence is comparable to that of CREW(cyclic) due to the fact that CREW(fre) generally needs more iterations than CREW(cyclic) until convergence. The results leading to Fig. 3 were obtained using a PC with Intel Core 2 Duo T5250 1.5 GHz CPU, and 1.5 GB memory.

Appendix A
Effectiveness of the Power Method-Like Iterations in (14) and (16)

We show that the power method-like iterations in (14) and (16) yield a monotonic increase of the objective functions of the associated UQPs. Let $\boldsymbol{s}^{(t+1)}$ be an update of the vector s obtained by the aforementioned power method-like iterations. Note that for fixed $\boldsymbol{s}^{(t)}$, the update vector $\boldsymbol{s}^{(t+1)}$ is the minimizer of the criterion

$$\left\| \boldsymbol{s}^{(t+1)} - \tilde{\boldsymbol{T}}\boldsymbol{s}^{(t)} \right\|_2^2 = \mathrm{const} - 2\,\Re\left\{ \boldsymbol{s}^{(t+1)\,H}\,\tilde{\boldsymbol{T}}\boldsymbol{s}^{(t)} \right\} \tag{20}$$

or, equivalently, the maximizer of the criterion $\Re\{\boldsymbol{s}^{(t+1)\,H}\,\tilde{\boldsymbol{T}}\boldsymbol{s}^{(t)}\}$ in the search space satisfying the constraints. We have that

$$\left(\boldsymbol{s}^{(t+1)} - \boldsymbol{s}^{(t)}\right)^H \tilde{\boldsymbol{T}} \left(\boldsymbol{s}^{(t+1)} - \boldsymbol{s}^{(t)}\right) \geq 0 \tag{21}$$

which implies

$$\begin{aligned} &\boldsymbol{s}^{(t+1)\,H}\tilde{\boldsymbol{T}}\boldsymbol{s}^{(t+1)} \\ &\quad \geq 2\,\Re\left\{ \boldsymbol{s}^{(t+1)\,H}\,\tilde{\boldsymbol{T}}\boldsymbol{s}^{(t)} \right\} - \boldsymbol{s}^{(t)\,H}\tilde{\boldsymbol{T}}\boldsymbol{s}^{(t)} \\ &\quad \geq \boldsymbol{s}^{(t)\,H}\tilde{\boldsymbol{T}}\boldsymbol{s}^{(t)} \end{aligned} \tag{22}$$

as $\Re\{\boldsymbol{s}^{(t+1)\,H}\,\tilde{\boldsymbol{T}}\boldsymbol{s}\} \geq \boldsymbol{s}^{(t)\,H}\tilde{\boldsymbol{T}}\boldsymbol{s}^{(t)}$.

Appendix B
Efficient Computation of $\boldsymbol{R}$ and $\boldsymbol{Q}$

We have that

$$\boldsymbol{R} + \beta\boldsymbol{s}\boldsymbol{s}^H = \beta \sum_{0 \leq |k| \leq (N-1)} \boldsymbol{J}_k\boldsymbol{s}\boldsymbol{s}^H\boldsymbol{J}_k^H = \beta\boldsymbol{A}^H\boldsymbol{A}. \tag{23}$$

The entries of $\boldsymbol{A}^H\boldsymbol{A}$ are nothing but the aperiodic autocorrelations of $\boldsymbol{s}$:

$$[\mathbf{A}^H\mathbf{A}]_{l,m} = r_{l-m} \tag{24}$$

where

$$r_k = \sum_{l=k+1}^{N} s_l s_{l-k}^* = r_{-k}^* \tag{25}$$

for $0 \leq k \leq N-1$. Note that the aperiodic autocorrelations $\{r_k\}$ of $\boldsymbol{s}$ are identical to the periodic autocorrelations of the sequence $\tilde{\boldsymbol{s}} = (\boldsymbol{s}^T\mathbf{0}_{N-1}^T)^T$ where $\mathbf{0}$ denotes the all-zero vector. As a result, one can obtain $\{r_k\}$ by calculating the Inverse FFT (IFFT) of $\{|v_k|^2\}$, where the sequence $\{v_k\}$ is the FFT of $\tilde{\boldsymbol{s}}$. Once $\{r_k\}$ are calculated, the matrix $\boldsymbol{R}$ can be obtained using (23)–(24).

In order to compute the matrix $\boldsymbol{Q}$, we note that

$$\begin{aligned} \boldsymbol{Q} &= \beta \sum_{0 < |k| \leq (N-1)} \boldsymbol{J}_k^H\boldsymbol{w}\boldsymbol{w}^H\boldsymbol{J}_k \\ &= \beta \sum_{0 < |k| \leq (N-1)} \boldsymbol{J}_k\boldsymbol{w}\boldsymbol{w}^H\boldsymbol{J}_k^H \end{aligned} \tag{26}$$

which implies that by using the variable $\boldsymbol{w}$ in lieu of $\boldsymbol{s}$, the matrix $\boldsymbol{Q}$ can be obtained via the same technique as devised above for the computation of $\boldsymbol{R}$.

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