DESIGN OF PIECEWISE LINEAR POLYPHASE SEQUENCES WITH GOOD CORRELATION PROPERTIES

Mojtaba Soltanalian+, Petre Stoica+, Mohammad Mahdi Naghsh× and Antonio De Maio†*

+Dept. of Information Technology, Uppsala University, Uppsala, Sweden
×ECE Dept., Isfahan University of Technology, Isfahan, Iran
†DIETI, Università degli Studi di Napoli "Federico II", Napoli, Italy

ABSTRACT

In this paper, we devise a computational approach for designing polyphase sequences with two key properties; (i) a phase argument which is piecewise linear, and (ii) an impulse-like autocorrelation. The proposed approach relies on fast Fourier transform (FFT) operations and thus can be used efficiently to design sequences with a large length or alphabet size. Moreover, using the suggested method, one can construct many new such polyphase sequences which were not known and/or could not be constructed by the previous formulations in the literature. Several numerical examples are provided to show the performance of the proposed design framework in different scenarios.

Index Terms— Autocorrelation, peak-to-average-power ratio (PAR), polyphase sequences, radar codes, waveform design

1. INTRODUCTION

A judicious approach to sequence design for active sensing and communication systems is to seek for sequences with small *out-of-phase* autocorrelations, also referred to as good correlation properties [1]-[10]. The periodic (c_k) and aperiodic (r_k) autocorrelations of a sequence $\boldsymbol{x} \in \mathbb{C}^N$ are defined as

$$c_k \triangleq \sum_{l=1}^{N} \boldsymbol{x}(l)\boldsymbol{x}^*(l+k)_{mod\ N}, \quad 0 \leq k \leq (N-1) \quad (1)$$

$$r_k \triangleq \sum_{l=1}^{N-k} \boldsymbol{x}(l)\boldsymbol{x}^*(l+k) = r_{-k}^*, \ 0 \leq k \leq (N-1) \quad (2)$$

where in both cases, the lag $k = 0$ represents the energy of $\boldsymbol{x}$, and the out-of-phase lags are those with $k \neq 0$.

We note that, in many applications, the sequences $\boldsymbol{x}$ with good correlation properties are not only constrained to have low *peak-to-average power ratio* (PAR),

$$\text{PAR} \triangleq \frac{\|\boldsymbol{x}\|_\infty^2}{\frac{1}{N}\|\boldsymbol{x}\|_2^2}, \quad (3)$$

but are also assumed to be *finite-alphabet*. In terms of PAR, the best squences are those with unimodular entries (i.e. $|\boldsymbol{x}(l)| = 1, \forall\, l$). As a result, the construction of finite-alphabet unimodular sequences has been studied widely in the literature. In particular, several (analytical) constructions are available in this case: such sequences for any given length can be constructed for example by Zadoff, Chu, Golomb polyphase, P3 and P4 methods [3]. Other constructions include Frank, P1, P2, Px, and PAT methods that work only when the length is a perfect square ($N = M^2$) [3],[9]. Note that the latter constructions present a unique property in their phase values, namely that their phase argument is *piecewise linear*. A generic piecewise linear polyphase sequence of length $N = MK$ can be formulated as follows [9]. Let the matrix

$$\boldsymbol{\Phi} = \begin{pmatrix} \varphi_{1,1} & \varphi_{1,2} & \cdots & \varphi_{1,K} \\ \varphi_{2,1} & \varphi_{2,2} & \cdots & \varphi_{2,K} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{M,1} & \varphi_{M,2} & \cdots & \varphi_{M,K} \end{pmatrix} \quad (4)$$

include the phase values of a unimodular sequence $\boldsymbol{x}$ via the identity

$$\boldsymbol{x} = e^{j\,(vec(\boldsymbol{\Phi}^T))}. \quad (5)$$

Then $\boldsymbol{x}$ is a piecewise linear polyphase sequence (with parameters M and K) iff

$$\eta_m \triangleq \eta_{m,k} = \varphi_{m,k+1} - \varphi_{m,k} \quad (6)$$

is a fixed constant for $1 \leq k \leq K-1$. Note that piecewise linear polyphase sequences are beneficial to practical implementations owing to the smaller number of variables involved in their construction, as well as their simple structure. As an example, Fig. 1 illustrates the phase values of the Frank sequence of length $N = 25$, with $M = K = 5$.

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* Please address all the correspondence to Mojtaba Soltanalian, Phone: (+46) 18-471-3168; Fax: (+46) 18-511925; Email: mojtaba.soltanalian@it.uu.se

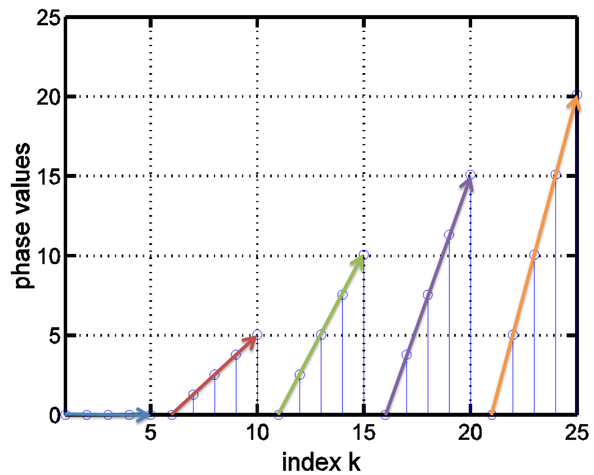


Fig. 1. The phase values of the Frank sequence $\{\boldsymbol{x}(k)\}_{k=1}^{N}$ of length $N = 25$ [3], illustrated based on the formulation in (4)-(6) with $\phi_{m,1} = 0$ and $\eta_m = 2\pi(m-1)/5, 1 \le m \le 5$.

In this paper, a fast computational method for designing piecewise linear polyphase sequences with good correlation is proposed. Particularly, we discuss in detail the sequence design for desirable aperiodic correlation. The reasons for choosing aperiodic correlation (and not its periodic counterpart) are the following; (i) piecewise linear polyphase sequences with optimal periodic correlation are already known in the literature (for instance the Frank sequence), and (ii) the aperiodic autocorrelations are of specific interest due to the higher difficulty of the associated design problem, see e.g. [6]. We also note that a modification of the proposed formulations to tackle the periodic case is straightforward. The contributions of this work can be summarized as follows:

- The analytical construction methods yield polyphase sequences with limited alphabet sizes. As a result, the proposed method can lead to considerable improvements upon the currently known piecewise linear polyphase sequences by alphabet size enlargement.

- The suggested formulation provides the possibility of designing piecewise linear polyphase sequences for lengths $N = MK$ which are not *perfect square*, i.e. for scenarios in which $(M, K) \neq (\sqrt{N}, \sqrt{N})$.

Consequently, using the proposed method, one can construct a new set of piecewise linear polyphase sequences with good correlation properties; a set with large cardinality whose majority of elements are not known and/or cannot be constructed with currently known formulations. See Section 3 for some numerical examples.

Notation: We use bold lowercase letters for vectors and bold uppercase letters for matrices. $(.)^T$, $(.)^*$ and $(.)^H$ denote the vector/matrix transpose, the complex conjugate, and the Hermitian transpose, respectively. $\mathbf{1}$ and $\mathbf{0}$ are the all-one and all-zero vectors/matrices. $\|\boldsymbol{x}\|_n$ or the l_n-norm of the vector $\boldsymbol{x}$ is defined as $(\sum_k |\boldsymbol{x}(k)|^n)^{\frac{1}{n}}$ where $\{\boldsymbol{x}(k)\}$ are the entries of $\boldsymbol{x}$. The symbol $\odot$ stands for the Hadamard element-wise product of matrices. $vec(\boldsymbol{X})$ is a vector obtained by stacking the columns of $\boldsymbol{X}$ successively. Finally, $\mathbb{Z}_Q$ denotes the set $\{0, 1, \cdots, Q-1\}$.

2. THE PROPOSED METHOD

Let $N = MK$ represent a twin factorization of N. Based on the formulation in Section 1, a piecewise-linear polyphase sequence $\boldsymbol{x}$ can be written as

$$\boldsymbol{x} = \begin{pmatrix} e^{j\varphi_1}\boldsymbol{x}_1 \\ e^{j\varphi_2}\boldsymbol{x}_2 \\ \vdots \\ e^{j\varphi_M}\boldsymbol{x}_M \end{pmatrix}; \quad \boldsymbol{x}_m = \begin{pmatrix} 1 \\ e^{j\eta_m} \\ \vdots \\ e^{j(K-1)\eta_m} \end{pmatrix}, \forall m, \quad (7)$$

where $\varphi_m = \varphi_{m,1}$. Note that, depending on the factorization of N, the parameter M (which denotes the number of linear segments in the phase argument) can attain different values ranging from 1 to N; particularly, the case of $M = 1$ leads to a structure that resembles the steering vectors associated with uniform linear arrays, while $M = N$ corresponds to a sequence design with no piecewise-linearity constraint at all. We assume that the elements of $\boldsymbol{x}$ belong to the Q-ary alphabet $\Omega_Q = \left\{e^{j2k\pi/Q} : k \in \mathbb{Z}_Q\right\}$. Accordingly, we assume that $\{\varphi_m\}$ and $\{\eta_m\}$ are of the form $2\pi k/Q$, with $k \in \mathbb{Z}_Q$.

In the following, we employ the CAN computational framework introduced in [5]. From an intuitive point of view, a sequence with zero out-of-phase periodic correlation has a flat spectrum in the frequency domain— in particular, the more flat the spectrum, the smaller the out-of-phase periodic correlations. The CAN algorithm in [5] (see also [1]) provides the mathematical formalism that confirms such observations. Namely, the periodic out-of-phase correlations of a sequence $\boldsymbol{x}$ can be minimized conveniently via the optimization problem:

$$\min_{\boldsymbol{x},\boldsymbol{v}} \quad \left\|\boldsymbol{A}^H\boldsymbol{x} - \boldsymbol{v}\right\|_2^2 \quad \text{s.t.} \quad \boldsymbol{v} \text{ is unimodular,} \quad (8)$$

where $\boldsymbol{x}$ is constrained, e.g. as in (7), and $\boldsymbol{A}$ denotes the $N \times N$ (inverse) DFT matrix, whose (l,p)-element is given by

$$[A]_{l,p} = \frac{1}{\sqrt{N}} e^{j2\pi lp/N}, \quad l,p = 1,\ldots,N. \quad (9)$$

Note that the aperiodic correlations of $\boldsymbol{x}$ are given by the periodic correlations of the sequence

$$\widetilde{\boldsymbol{x}} = \begin{pmatrix} \boldsymbol{x} \\ \mathbf{0}_{N-1} \end{pmatrix}. \quad (10)$$

Therefore, CAN considers the following frequency-domain design problem to *minimize* the aperiodic out-of-phase correlations of $\boldsymbol{x}$:

$$\min_{\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{v}}} \quad \left\| \widetilde{\boldsymbol{A}}^H \widetilde{\boldsymbol{x}} - \widetilde{\boldsymbol{v}} \right\|_2^2 \qquad (11)$$
$$\text{s.t.} \quad \widetilde{\boldsymbol{v}} \text{ is unimodular,}$$

in which $\boldsymbol{x}$ is constrained as described earlier, and $\widetilde{\boldsymbol{A}}$ denotes the $(2N-1) \times (2N-1)$ (inverse) DFT matrix. For given $\widetilde{\boldsymbol{x}}$, the minimization of (11) with respect to $\widetilde{\boldsymbol{v}}$ is straightforward, viz.

$$\widetilde{\boldsymbol{v}} = e^{j \arg(\widetilde{\boldsymbol{A}}^H \widetilde{\boldsymbol{x}})}. \qquad (12)$$

Due to the various constraints on $\boldsymbol{x}$ including the piecewise linearity and a given phase alphabet, the optimization of (11) with respect to $\widetilde{\boldsymbol{x}}$ (or equivalently $\boldsymbol{x}$) appears to be more complicated. However, to achieve a monotonically decreasing objective function, one can simply employ a separate optimization of (11) with respect to the variables $\{\varphi_m\}$ and $\{\eta_m\}$. In order to obtain the minimizer $\{\varphi_m\}$ of (11) for fixed $\widetilde{\boldsymbol{v}}$ and $\{\eta_m\}$, we note that the objective function can be rewritten as

$$\begin{aligned}\left\| \widetilde{\boldsymbol{A}}^H \widetilde{\boldsymbol{x}} - \widetilde{\boldsymbol{v}} \right\|_2^2 &= \left\| \widetilde{\boldsymbol{x}} - \widetilde{\boldsymbol{A}} \widetilde{\boldsymbol{v}} \right\|_2^2 \qquad (13)\\ &= \left\| \begin{pmatrix} e^{j\varphi_1}\boldsymbol{x}_1 \\ e^{j\varphi_2}\boldsymbol{x}_2 \\ \vdots \\ e^{j\varphi_M}\boldsymbol{x}_M \end{pmatrix} - \begin{pmatrix} \widehat{\boldsymbol{v}}_1 \\ \widehat{\boldsymbol{v}}_2 \\ \vdots \\ \widehat{\boldsymbol{v}}_M \end{pmatrix} \right\|_2^2 + \text{const}_1 \\ &= \left\| \begin{pmatrix} e^{j\varphi_1}\boldsymbol{1}_K \\ e^{j\varphi_2}\boldsymbol{1}_K \\ \vdots \\ e^{j\varphi_M}\boldsymbol{1}_K \end{pmatrix} - \begin{pmatrix} \widehat{\boldsymbol{v}}_1 \odot \boldsymbol{x}_1^* \\ \widehat{\boldsymbol{v}}_2 \odot \boldsymbol{x}_2^* \\ \vdots \\ \widehat{\boldsymbol{v}}_M \odot \boldsymbol{x}_M^* \end{pmatrix} \right\|_2^2 + \text{const}_1\end{aligned}$$

where $\widehat{\boldsymbol{v}}_m$ denotes the column vector consisting of the m^{th} K-tuple in the vector $\widetilde{\boldsymbol{A}}\widetilde{\boldsymbol{v}}$. Let $\boldsymbol{u}_m = \widehat{\boldsymbol{v}}_m \odot \boldsymbol{x}_m^*$ for $1 \leq m \leq M$. Then it is easy to verify that the minimization of (13) may be decoupled for different $\{\varphi_m\}$; namely, the minimizer $\varphi_m \triangleq 2\pi g_m/Q$ of (13) is given by the solution to the following optimization problem:

$$\min_{g_m \in \mathbb{Z}_Q} \quad \sum_{k=1}^{K} \left| e^{j\varphi_m} - \boldsymbol{u}_m(k) \right|^2. \qquad (14)$$

Consequently, the minimizer φ_m of (13) becomes

$$\varphi_m = \Psi_Q \left(\arg \left(\sum_{k=1}^{K} \boldsymbol{u}_m(k) \right) \right), \quad 1 \leq m \leq M, \qquad (15)$$

where $\Psi_Q(.)$ yields the closest phase value to the argument in the Q-ary alphabet. Now suppose $\{\varphi_m\}$ and $\widetilde{\boldsymbol{v}}$ (equivalently $\{\widehat{\boldsymbol{v}}_m\}$) are given. According to (13), the minimization of (11) with respect to $\{\eta_m\}$ may be accomplished using the optimization problems

$$\min_{\boldsymbol{x}_m} \quad \left\| \boldsymbol{x}_m - e^{-j\varphi_m} \widehat{\boldsymbol{v}}_m \right\|_2^2 \qquad (16)$$
$$\text{s.t.} \quad \boldsymbol{x}_m \text{ has the structure in (7),}$$

for $1 \leq m \leq M$. Let $\eta_m \triangleq 2\pi h_m/Q$, and note that one can restate the objective function of (16) as

$$\begin{aligned}&\left\| \boldsymbol{x}_m - e^{-j\varphi_m} \widehat{\boldsymbol{v}}_m \right\|_2^2 \\ &= \sum_{k=1}^{K} \left| e^{j2\pi h_m (k-1)/Q} - e^{-j\varphi_m} \widehat{\boldsymbol{v}}_m(k) \right|^2 \\ &= \text{const}_2 - 2\,\Re \left\{ \sum_{k=1}^{K} (e^{-j\varphi_m} \widehat{\boldsymbol{v}}_m(k)) e^{-j2\pi h_m (k-1)/Q} \right\}.\end{aligned}$$

Hence, the optimization problem in (16) is equivalent to

$$\max_{h_m \in \mathbb{Z}_Q} \quad \Re \left\{ \sum_{k=1}^{K} (e^{-j\varphi_m} \widehat{\boldsymbol{v}}_m(k))\, e^{-j2\pi h_m (k-1)/Q} \right\}. \qquad (17)$$

Interestingly, the solution to (17) can be obtained efficiently using an FFT operation due to the fact that the objective function represents the real-part of the Q-point DFT sequence associated with $\{e^{-j\varphi_m} \widehat{\boldsymbol{v}}_m(k)\}_{k=1}^{K}$.

Finally, the steps of the proposed method are summarized in Table 1. We note that the approach proposed in this work relies on FFT operations and hence can be used efficiently for large lengths N, or phase alphabet sizes Q.

Table 1. The Proposed Algorithm for Designing Piecewise-Linear Polyphase Sequences with Good Correlation

Input parameters: sequence length $= N$, alphabet size $= Q$, twin factorization of N into (M, K).

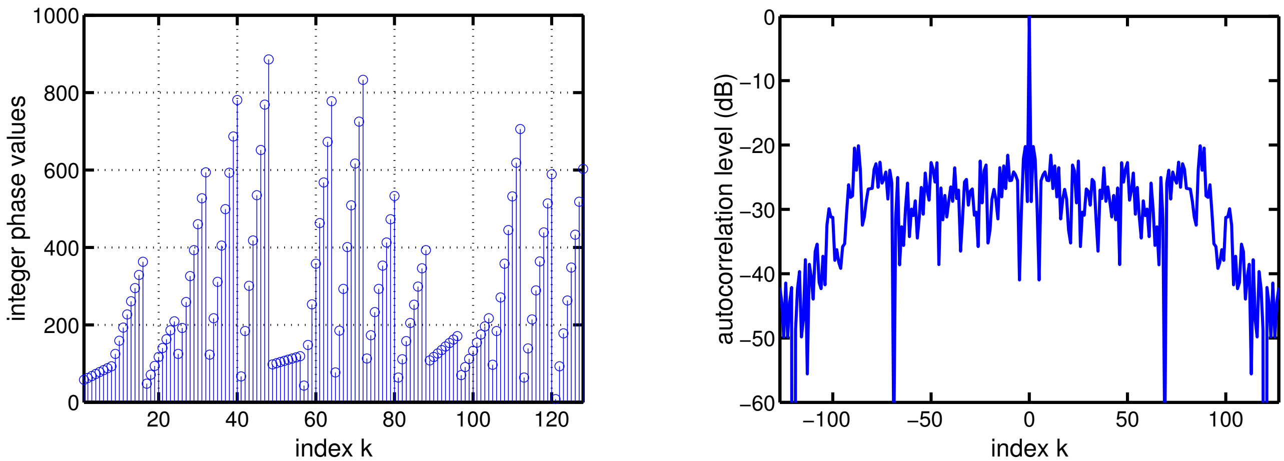
Step 0: Initialize the variables $\{\varphi_m\}$ and $\{\eta_m\}$ of the form $2\pi k/Q$ ($k \in \mathbb{Z}_Q$) randomly (or set the values by a previously known sequence).
Step 1: Form the sequence $\boldsymbol{x}$ using (7), based on the current values of $\{\varphi_m\}$ and $\{\eta_m\}$.
Step 2: Compute $\widetilde{\boldsymbol{v}}$ using (12).
Step 3: Compute $\{\varphi_m\}$ using (15).
Step 4: Compute $\{\eta_m\}$ using (17).
Step 5: Let $\epsilon = \left\| \widetilde{\boldsymbol{A}}^H \widetilde{\boldsymbol{x}} - \widetilde{\boldsymbol{v}} \right\|_2$. Repeat the steps 1-4 until

$$\epsilon^{(s)} = \epsilon^{(s-1)},$$

where s denotes the iteration number.

3. NUMERICAL RESULTS AND DISCUSSIONS

We provide several numerical examples to show the performance of the proposed method. As discussed earlier, the method can be employed to design piecewise linear polyphase



(a) (b)

Fig. 2. Design of a piecewise linear polyphase sequence (of length $N = 128$) with good aperiodic autocorrelation, and parameters $(M, k) = (16, 8)$, $Q = 128$: (a) the integer phases ($\triangleq$ phase values $\times Q/(2\pi)$) of the sequence; (b) the autocorrelation levels of the sequence.

sequences of non-square length. We use the proposed approach to design a piecewise linear polyphase sequence of length $N = 128$ with $(M, K) = (16, 8)$ and $Q = 128$. The obtained sequence along with its normalized autocorrelation level,

$$\text{autocorrelation level (dB)} \triangleq 20 \log_{10} \left| \frac{r_k}{r_0} \right| \tag{18}$$

are presented in Fig. 2. The correlation peak sidelobe level (PSL), viz.

$$\text{PSL} \triangleq \max\{|r_k|\}_{k=1}^{N-1}, \tag{19}$$

of the sequences obtained during the iterations of the proposed algorithm is shown in Fig. 3. A significant reduction in the PSL of the sequences vs. iteration number can be observed. We note that CAN minimizes an upper bound on the PSL metric, and hence, the resultant PSL values in Fig. 3 are not monotonically decreasing; see [5] and [6] for more details related to this observation.

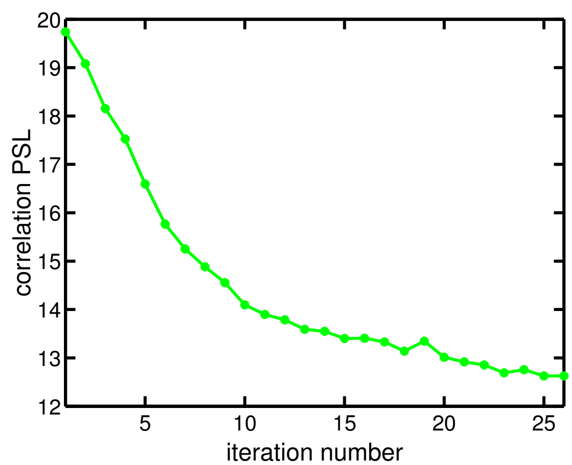


Fig. 3. The PSL values versus iteration number associated with the obtained sequences through the iterations of the proposed algorithm.

Next, we consider improving upon a certain piecewise linear polyphase sequence with good correlation. As an example, we use the PAT sequence [9] of length $N = 256$ in order to initialize the algorithm in Table 1. PAT sequences were proposed recently, and have a PSL value which is the minimum of those of Frank, P1, P2, and Px. While improving the correlation properties of a PAT sequence by using numerical methods is not simple, the said properties can be enhanced by considering an alphabet size Q larger than that used by the PAT sequences which is $2\sqrt{N}$. In order to show the potential of such an approach in enhancing the correlation properties, we choose a large alphabet size by setting Q to 2^{16}. Fig. 4 depicts the normalized autocorrelation level of the PAT sequence, as well as the level corresponding to the proposed method. The PSL value corresponding to the initial PAT sequence is equal to 11.3086, while the obtained sequence has a PSL value of 5.6359.

Finally, it can be interesting to examine how the factorization of N into (M, K) affects the correlation properties

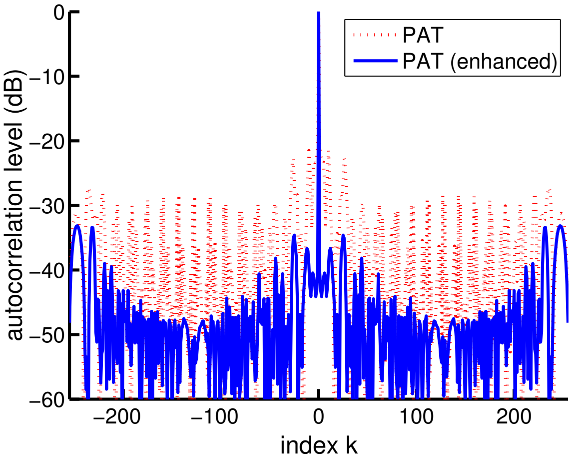


Fig. 4. Enhancement of the aperiodic correlation properties of the PAT sequence of length $N = 256$ via alphabet size enlargement. The figure shows the normalized autocorrelation level of the initial PAT sequence, along with that of the enhanced sequence obtained by the proposed method.

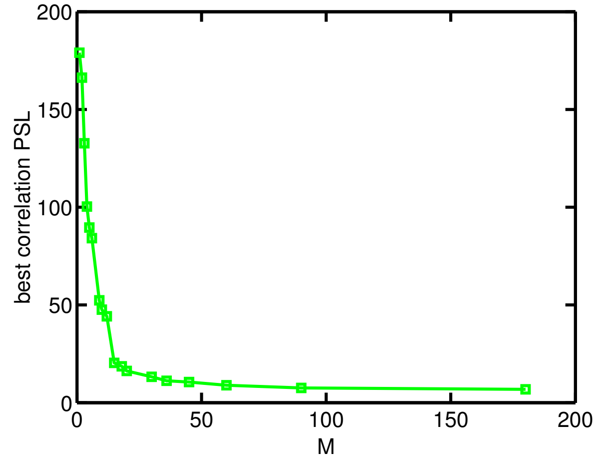


Fig. 5. The best PSL values obtained in 15 trials of the proposed method with different random initializations for $N = 180$, and M set to various divisors of N.

of the obtained sequences. To study this aspect, we consider $N = 2^2 \times 3^2 \times 5 = 180$, and $Q = N$. For all 18 divisors of $N = 180$, we use the proposed algorithm 15 times with different random initializations. Fig. 5 plots the best PSL values for each case, obtained in the 15 trials . A considerable reduction in the obtained PSL values can be seen as M grows large. To explain this behavior, we note that the number of free variables, i.e. degrees of freedom (DOFs) of the problem, is determined by the number of variables $\{\varphi_m\}$ and $\{\eta_m\}$:

$$\#\text{DOFs} = \begin{cases} 2M & M \leq N/2, \\ M & M = N. \end{cases} \tag{20}$$

As a result, the number of DOFs is increasing with M, which lays the ground for a better performance of the method in terms of the correlation PSL. However, increasing M might increase the complexity of implementing the sequences in practice— a trade off which should be dealt with wisely.

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