# About permutation algebras, (pre)sheaves and named sets *

Fabio Gadducci (gadducci@di.unipi.it)
*Department of Computer Science, University of Pisa. via F. Buonarroti 2, 56127 Pisa, Italy.*

Marino Miculan (miculan@dimi.uniud.it)
*Department of Mathematics and Computer Science, University of Udine. via delle Scienze 206, 33100 Udine, Italy.*

Ugo Montanari (ugo@di.unipi.it)
*Department of Computer Science, University of Pisa. via F. Buonarroti 2, 56127 Pisa, Italy.*

**Abstract.** In this paper, we survey some well-known approaches proposed as general models for calculi dealing with names (like e.g. process calculi with name-passing). We focus on *(pre)sheaf categories*, *nominal sets*, *permutation algebras* and *named sets*. We study the relationships among these models, which allow for transferring techniques and constructions from one model to the other.

**Keywords:** Nominal calculi, permutation algebras, presheaf categories, named sets.

## 1. Introduction

The main aim of this paper is to survey some of the most widely used models for nominal calculi, and to clarify their relationships.

Since the introduction of $\pi$-calculus, the notion of *name* has been recognized as central in models for concurrency, mobility, staged computation, metaprogramming, memory region allocation, etc. In recent years, several approaches have been proposed as general frameworks for streamlining the development of these models featuring name passing and/or allocation. These approaches are based on category theory, non-standard set theory, automata theory, algebraic specifications, etc. It comes as no surprise that there are so many approaches: despite all ultimately cope with the same issues, they are inspired by different aims and perspective, leading to different solutions and choices. It is important to investigate the relationships between these models for many reasons. First of all, this will point out similarities and differences between them. Often these models appear under different names, and with subtle differences, so that it is not always easy to understand whether, and how, they are related. Moreover, apparently peculiar id-

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iosyncrasies are either justified, or revealed to be inessential. Finally, these interconnections allow for transferring properties, techniques and constructions among metamodels, thus cross-fertilizing each other. In fact, this formal comparison allows for highlighting weak points of some metamodel, and possibly for suggesting improvements.

In this paper, we focus on the most widely used and successful models, namely *(pre)sheaf categories*, the various solutions related to Fraenkel-Mostowski set theory, *permutation algebras*, and *named sets*.

Categories of functors over the category $\mathbb{I}$ of finite sets and injective functions, such as $\mathbf{Set}^{\mathbb{I}}$, have been widely used for modeling "staged computations", indexed by the (finite) sets of names currently allocated; see e.g. (Moggi, 1993; Stark, 1994; Hofmann, 1999; Fiore and Turi, 2001). A variation considers only the subcategory of *sheaves* with respect to the atomic topology (the so-called *Schanuel topos*) (Stark, 1994; Hofmann, 1999; Bucalo et al., 2001), leading to models supporting classical logic. These categories allow for extending the standard results about the existence of initial algebras/final coalgebras of polynomial functors also to functors dealing with names.

A well-known alternative (and maybe less technically demanding) approach stems from Fraenkel-Mostowski permutation model of set theory with atoms. Several variants of this theory have been presented as *perm*($\mathbb{A}$)*-sets*, FM-sets, *nominal sets*, etc. (Gabbay and Pitts, 1999; Gabbay and Pitts, 2002; Pitts, 2003). Nominal sets and alike are strictly related to *permutation algebras*, which have been considered for the development of a theory of *structured coalgebras* in the line of algebraic specifications (Ferrari et al., 2002). Permutation algebras are algebras over signatures containing a group of permutation of an enumerable set. A problem with these signatures is that the group of *all* permutations yields a non-countable signature; for this, one can restrict the attention to countable subgroups, such as that of *finite* permutations. Moreover, in many cases we are interested to restrict our attention to permutation algebras whose elements are *finitely supported*—e.g., processes and terms with infinite free names are ruled out. Therefore, there are four possible theories of permutation algebras to consider.

Finally, a different theory of sets with permutations is that of *named sets* (Ferrari et al., 2002). A *named set* is a set in which each element is equipped with a finite set of names and name bijections. Named sets are supposed to be an implementation of permutation algebras, to some extent; indeed, they are the basic building block of the operational model of History Dependent automata (Montanari and Pistore, 2004).

In this paper, we describe precisely the connections among these approaches. The four categories of permutation algebras subsume the several variants of FM-sets which have been introduced in literature.

Then, it is proved that finitely supported permutation algebras, either on signatures with all permutations or on those with only finite ones, are equivalent to the Schanuel topos. This will allow to transfer the known constructions of polynomial functors from the Schanuel topos to the realm of permutation algebras.

Finally, also the category of named sets turns out to be equivalent to the category of algebras with finite support, and hence to the Schanuel topos again. Therefore, named sets can be seen as an "implementation" of sheaves of the Schanuel topos, thus giving a sound base for realizing operational models of nominal calculi whose semantics can be given in these sheaf categories, like e.g. the $\pi$-calculus.

Admittedly, some of these results have been known in the community for a while; however, they are often just cited without proofs, or using different variants of the categories, with different names. One of the aims of the present paper is to clean up and complete this picture, and to fit as much as possible these approaches in a uniform framework.

*Synopsis.* In Section 2 we recall the basic definitions about (finite) permutations, permutation algebras, and finite support, and relate them to nominal sets. In Section 3 we prove that permutation algebras with finite support, over either signatures, ultimately correspond to the Schanuel topos. In Section 4 we consider named sets, and we show that they also form a category which is equivalent to the category of finite permutation algebras with finite support. In Section 5 we cast permutation algebras in the general theory of *continuous G-sets.* This will allow to fit also permutation algebras with arbitrary support in a uniform framework, and to have a different proof of the equivalence between finite support permutation algebras and sheaves of the Schanuel topos. Some conclusions are finally drawn in Section 6.

## 2. Permutation algebras

This section recalls the main definitions on *permutation algebras.* They are mostly drawn from (Montanari and Pistore, 2004), with some additional references to the literature.

### 2.1. Permutation algebras

DEFINITION 1. [permutation group] Let $\mathcal{N}$ be a set (called set of *names*). A *permutation* on $\mathcal{N}$ is a bijective endofunction on $\mathcal{N}$. The set of all such permutations on a given set $\mathcal{N}$ is denoted by $\mathrm{Aut}(\mathcal{N})$, and it forms the *permutation group* of $\mathcal{N}$, where the operation is function composition: For all $\pi_1, \pi_2 \in \mathrm{Aut}(\mathcal{N})$, $\pi_1\pi_2 \triangleq \pi_1 \circ \pi_2$ (that is, for all $x \in \mathcal{N} : (\pi_1\pi_2)(x) = \pi_1(\pi_2(x))$).

Permutations on sets coincide with *automorphisms* (because there is no structure to preserve), hence the notation denoting the permutation group. We stick however to permutations since now this is almost the standard usage in theoretical computer science, and it is the term used in our main references: see (Montanari and Pistore, 2004, Section 2.1) and the initial paragraphs of (Gabbay and Pitts, 2002, Section 3).

DEFINITION 2. [permutation signature and algebras] Let $\mathcal{N}$ be a countable set. The *permutation signature* $\Sigma_\pi$ on $\mathcal{N}$ is given by the set of unary operators $\{\widehat{\pi} \mid \pi \in \mathrm{Aut}(\mathcal{N})\}$, together with the pair of axioms schemata $\widehat{id}(x) = x$ and $\widehat{\pi}_1(\widehat{\pi}_2(x)) = \widehat{\pi_1\pi_2}(x)$.

A *permutation algebra* $\mathcal{A} = (A, \{\widehat{\pi}_A\})$ is an algebra for $\Sigma_\pi$. A *permutation morphism* $\sigma : \mathcal{A} \to \mathcal{B}$ is an algebra morphism, i.e., a function $\sigma : A \to B$ such that $\sigma(\widehat{\pi}_A(x)) = \widehat{\pi}_B(\sigma(x))$. Finally, $Alg(\Sigma_\pi)$ (often shortened as $Alg_\pi$) denotes the category of permutation algebras and their morphisms.

Permutation algebras and their morphisms correspond trivially to Gabbay and Pitts' *perm($\mathbb{A}$)-sets* and *equivariant functions* (Gabbay and Pitts, 2002). An interesting (and recurring) example of permutation algebra is that for the $\pi$-calculus: the carrier contains all the processes, up-to structural congruence, and the interpretation of a permutation is the associated name substitution (see also (Montanari and Pistore, 2004, Definition 15 and Section 3)).

An unpleasant fact about $Alg_\pi$ is that it has a non-countable set of operators and axioms. In order to have a simpler and more tractable signature, following (Montanari and Pistore, 2004, Section 2.1) we restrict our attention to *finite* permutations.

DEFINITION 3. [finite permutations] Let $\mathcal{N}$ be a countable set, and let $\pi \in \mathrm{Aut}(\mathcal{N})$ be a permutation on $\mathcal{N}$. The *kernel* of $\pi$ is defined as $\ker(\pi) \triangleq \{x \in S \mid \pi(x) \neq x\}$.

A permutation $\pi$ is *finite* if its kernel is finite. The set of all finite permutations is denoted by $\mathrm{Aut}^f(\mathcal{N})$ and it is a subgroup of $\mathrm{Aut}(\mathcal{N})$.

It is well known from group theory that $\mathrm{Aut}^f(\mathcal{N})$ is characterized as the subgroup generated by all *transpositions*, which are permutations whose kernel has exactly 2 elements. Therefore, each finite permutation can be defined as the composition of a finite sequence of transpositions.

DEFINITION 4. [finite permutation signature and algebras] The finite permutation signature $\Sigma_\pi^f$ is obtained as the subsignature of $\Sigma_\pi$ restricted to the unary operators induced by finite permutations.

The associated category of algebras is $Alg(\Sigma_\pi^f)$, shortened as $Alg_\pi^f$.

Each algebra in $Alg^f_\pi$ has a *countable set* of operators and axioms, and thus it is more amenable to the standard results out of the algebraic specification mold. Each algebra in $Alg_\pi$ can be casted trivially to an algebra in $Alg^f_\pi$ (by forgetting the interpretation of non-finite permutations), but as we will see in Section 3.1, this inclusion is strict.

### 2.2. Finitely supported algebras

We provide now a final list of definitions, concerning the *finite support* property. They rephrase definitions in (Montanari and Pistore, 2004, Section 2.1), according to (Gabbay and Pitts, 2002, Definition 3.3), and to our needs in the following sections.

Let us fix in this and the following sections a countable set $\mathcal{N}$, shortening $\mathrm{Aut}(\mathcal{N})$ and $\mathrm{Aut}^f(\mathcal{N})$ with Aut and $\mathrm{Aut}^f$, respectively, usually putting a superscript $\_^f$ for definitions and notations concerning finite permutations. Moreover, subsets of $\mathcal{N}$ will be ranged over by $X, Y$.

DEFINITION 5. [support] Let $\mathcal{A} \in Alg_\pi$ be a permutation algebra. For $a \in A$, the *isotropy group* of $a$ is the set $\mathrm{fix}_A(a)$ of permutations fixing $a$ in $\mathcal{A}$, i.e., $\mathrm{fix}_A(a) \triangleq \{\pi \in \mathrm{Aut} \mid \widehat{\pi}_A(a) = a\}$.

For a subset $X$, we denote by $\mathrm{fix}(X)$ the set of permutations fixing $X$ (i.e., those permutations $\pi \in \mathrm{Aut}$ such that $\pi(k) = k$ for all $k \in X$).

We say that the subset $X$ *supports* the element $a \in A$ if all permutations fixing $X$ also fix $a$ in $\mathcal{A}$ (i.e., if $\mathrm{fix}(X) \subseteq \mathrm{fix}_A(a)$).

The definition can be readily adapted to finite permutation algebras, by replacing Aut by $\mathrm{Aut}^f$ throughout.

The notion of support is a suitable generalization of that of "free variables" of terms, and of "free names" of processes: if $X$ supports $a$, then $a$ is affected only by the action of permutations over the set $X$.

DEFINITION 6. [finitely supported algebras] A permutation algebra $\mathcal{A}$ is *finitely supported* if for each element of its carrier there exists a finite set supporting it.

The full subcategory of $Alg_\pi$ of all finitely supported permutation algebras is denoted by $FSAlg(\Sigma_\pi)$, shortened as $FSAlg_\pi$.

$FSAlg^f_\pi$, the category of finitely supported finite permutation algebras, is defined similarly.

Finitely supported algebras over all permutations correspond trivially to the *perm(𝔸)-sets with finite support* of (Gabbay and Pitts, 2002). On the other hand, it is easy to show that the category of finitely supported algebras over finite permutations corresponds to the category of *nominal sets* as defined in (Pitts, 2003).

In general, an element of the carrier of an algebra may have different sets supporting it. The following proposition, mirroring (Gabbay and Pitts, 2002, Proposition 3.4), ensures that a minimal support does exist.

PROPOSITION 7. *Let $\mathcal{A}$ be a (finite) permutation algebra. If $a \in A$ is finitely supported, then there exists a least finite set supporting $a$, called the* support *of $a$ and denoted by $supp_A(a)$.*

*Remark 1.* Not all algebras in $Alg_\pi$ are finitely supported (hence, neither those in $Alg^f_\pi$). For example, let us consider the set $\mathcal{N} = \{0, 1, \ldots\}$, and the algebra $(\wp(\mathcal{N}), \{\widehat{\pi} \mid \pi \in \mathrm{Aut}^f\})$, where for all $X \in \wp(\mathcal{N})$, $\widehat{\pi}(X) = \{\pi(x) \mid x \in X\}$. The sets $\mathcal{N}_{even} = \{2i \mid i \geq 0\}$ and $\mathcal{N}_{odd} = \{2i + 1 \mid i \geq 0\}$, both elements of $\wp(\mathcal{N})$, are not finitely supported: for all $X$ finite, we can always pick a (finite) permutation $\pi$ fixing $X$ but exchanging $\max(X) + 1$ and $\max(X) + 2$; then $\hat{\pi}(\mathcal{N}_{even}) \neq \mathcal{N}_{even}$.

## 3. Finitely supported algebras and sheaves

In this section we show that the two categories of algebras with finite support with either signatures are equivalent—that is, we can restrict to the countable signature of finite permutations without changing the resulting category. Moreover, these categories are equivalent to the Schanuel topos. Both results have been mentioned (without proof) in the setting of FM-techniques, see (Gabbay and Pitts, 2002, Section 7) and (Pitts, 2003, Section 3).

Then, we take advantage of this correspondence for transferring the constructions of polynomial "behavioural" functors from the Schanuel topos to the categories of permutation algebras.

### 3.1. Some properties of permutation algebras

Recall the existence of the obvious forgetful functor $U : Alg_\pi \to Alg^f_\pi$, which simply drops the interpretation of non-finite permutations, and that can be extended also to the finitely supported counterparts. Actually, all these categories are much more strictly related: in this section we prove this statement, presenting first some (likely folklore) results on permutation algebras.

LEMMA 8 (preserving supports). *Let $\mathcal{A}$ be a permutation algebra, let $a \in A$, and let $X$ be a subset supporting $a$ in $\mathcal{A}$. Then*

*(i) $\pi(X)$ supports $\widehat{\pi}_A(a)$, for all permutations $\pi \in \mathrm{Aut}$;*

*(ii) $X$ supports $\sigma(a)$ in $\mathcal{B}$, for all homomorphisms $\sigma : \mathcal{A} \to \mathcal{B}$.*

Both proofs are easy, and they are skipped. More interesting are their consequences on finitely supported elements.

COROLLARY 9. *Let $\mathcal{A}$ be a permutation algebra, and let $a \in A$ be finitely supported. Then*

*(i)* $supp_A(\widehat{\pi}_A(a)) = \pi(supp_A(a))$*, for all permutations* $\pi \in \mathrm{Aut}$*;*

*(ii)* $supp_B(\sigma(a)) \subseteq supp_A(a)$*, for all homomorphisms* $\sigma : \mathcal{A} \to \mathcal{B}$*;*

*(iii)* $fix_A(a) \subseteq sp_A(a)$*, for* $sp_A(a) \triangleq \{\pi \mid \pi(supp_A(a)) = supp_A(a)\}$*.*

Note also that $sp_A(a)$ is clearly a group.

PROPOSITION 10 (removing infinite supports). *The inclusion functor* $FSAlg_\pi \to Alg_\pi$ *admits a right adjoint.*

*Proof.* Given a permutation algebra $\mathcal{A}$, simply consider the subalgebra obtained by dropping all the elements with infinite support: it is well-defined, thanks to *(i)* of Lemma 8, and it extends to a functor, thanks to *(ii)* of that same lemma. □

Let $X$ be a subset, and let $\pi \in \mathrm{Aut}$: in the following, we denote by $\pi_{|X} : X \to \pi(X)$ the obvious bijection obtained as a restriction of $\pi$. Conversely, given a subset $X$ and a bijection $\pi : X \to Y$, we denote by $\pi^c \in \mathrm{Aut}^f$ any completion of $\pi$, i.e., any finite permutation such that $\pi^c_{|X} = \pi$; and by $\pi^i$ the completion such that moreover $\pi^i_{|\mathcal{N}\setminus(X \cup Y)} = id$.

LEMMA 11 (equating supports). *Let $\mathcal{A}$ be a permutation algebra, let $a \in A$, and let $X$ be a subset supporting $a$ in $\mathcal{A}$. Then, whenever permutations* $\pi, \kappa \in \mathrm{Aut}$ *coincide on $X$ (i.e.,* $\pi_{|X} = \kappa_{|X}$*), their actions coincide on $a$ (i.e.,* $\widehat{\pi}_A(a) = \widehat{\kappa}_A(a)$*).*

Also in this case the proof is easy, simple noting that $\kappa^{-1}\pi$ is the identity on $X$, hence it preserves $a$.

Now, we prove that if we stick to algebras with finite support, the restriction to the countable signature does not change the models.

PROPOSITION 12. *Categories* $FSAlg_\pi$ *and* $FSAlg^f_\pi$ *are isomorphic.*

*Proof.* Note that $U : Alg_\pi \to Alg^f_\pi$ restricts to $U : FSAlg_\pi \to FSAlg^f_\pi$; indeed, for any algebra $\mathcal{A} = (A, \{\hat{\pi} \mid \pi \in \mathrm{Aut}\}) \in Alg_\pi$, if $a \in A$ is supported by a finite subset $X$, then $X$ supports $a$ also in $U(\mathcal{A})$.

It is then enough to show that each finitely supported algebra over finite permutations can be extended to obtain an object of $FSAlg_\pi$. That is, given $\mathcal{A} \in FSAlg^f_\pi$ and $a \in A$, we must define the value of $\widehat{\pi}_A(a)$ for all permutations $\pi$, also infinite ones.

To this end, let us choose a completion $\kappa = \pi^i_{|supp_A(a)} \in \mathrm{Aut}^f$ for any $\pi \in \mathrm{Aut}$ and $a \in A$. Now, the interpretation of $\widehat{\pi}_A(a)$ for any permutation $\pi$ is $\widehat{\kappa}_A(a)$: it is well-given, since thanks to Lemma 11 the choice of the actual completion is irrelevant; and thanks to $(i)$ of Lemma 8 also the axioms of permutation signatures are satisfied. □

For nominal sets, this result has been mentioned (without proof) in (Pitts, 2003, Section 3).

We now conclude with a remark on the categories of all algebras.

PROPOSITION 13. *The forgetful functor* $Alg_\pi \rightarrow Alg^f_\pi$ *is not lluf.*

*Proof.* We show a finite permutation algebra $\mathcal{A}$ which cannot be extended to all permutations, that is, such that it does not exist a $\mathcal{B} = (A, \{\hat{\pi} \mid \pi \in \mathrm{Aut}\}) \in Alg_\pi$ satisfying $U(\mathcal{B}) = \mathcal{A}$.

Let us fix $\mathcal{N} = \{0, 1, 2, \dots\}$, and let $\mathcal{N}_{even} = \{0, 2, 4 \dots\} \subset \mathcal{N}$. We take $A \triangleq \{X \subseteq \mathcal{N} \mid X \cap \mathcal{N}_{even} \text{ infinite}\}$, and for $\pi \in \mathrm{Aut}^f$, let $\widehat{\pi}_A(X) = \{\pi(x) \mid x \in X\}$. Clearly, if $X$ contains infinite even names, also $\widehat{\pi}_A(X)$ does, because $\pi$ is a finite permutation. Let us consider the infinite permutation $\rho(x_{2i}) = x_{2i+1}$, $\rho(x_{2i+1}) = x_{2i}$ $(i \geq 0)$, swapping all odd and even names at once. By the axioms of permutation signatures, the interpretation of $\rho$ must extend those of all finite permutations contained in it, therefore $\widehat{\rho}_A(X) = \{\rho(x) \mid x \in X\}$. But $\mathcal{N}_{even} \in A$, while $\widehat{\rho}_A(\mathcal{N}_{even}) = \{1, 3, 5, \dots\}$ which is not in $A$—absurd. □

### 3.2. Correspondence with sheaves

Recall that the category of *presheaves* over a small category $\mathbf{C}$ is the category of functors $\mathbf{Set}^{\mathbf{C}^{op}}$ and natural transformations among them. In particular, we are interested in the presheaf category $\mathbf{Set}^{\mathbb{I}}$, where $\mathbb{I}$ is (without loss of generality) the category of finite subsets of $\mathcal{N}$ and *injective* maps. This category has been used by many authors for modeling the computational notion of dynamic allocation of names or locations; see e.g. (Moggi, 1993; Stark, 1994; Hofmann, 1999; Fiore and Turi, 2001). Actually, we have to consider a particular subcategory of $\mathbf{Set}^{\mathbb{I}}$, namely the category $\mathrm{Sh}(\mathbb{I}^{op})$ of *sheaves with respect to the atomic topology.* Sheaf conditions are usually expressed in terms of sieves and amalgamations (see e.g. (Mac Lane and Moerdijk, 1994, Section III.4)), but in the case of the atomic topology there exists a simpler, well-known alternative characterization of this subcategory (Johnstone, 2002, Example 2.1.11(h)), which we provide directly here.

PROPOSITION 14. $\mathrm{Sh}(\mathbb{I}^{op})$ *is the full subcategory of* $\mathbf{Set}^{\mathbb{I}}$ *of pullback preserving functors.*

(Clearly, a pullback-preserving functor is also mono preserving, but the converse is not true; see, e.g., $P_\emptyset = \emptyset$ and $P_X = \mathcal{N}$ if $X \neq \emptyset$, and the pullback given by the inclusion of even and odd names in $\mathcal{N}$).

The category $\mathrm{Sh}(\mathbb{I}^{op})$, often called the *Schanuel topos*, features essentially the same important properties of $\mathbf{Set}^{\mathbb{I}}$ above, and indeed it can be used in place of $\mathbf{Set}^{\mathbb{I}}$ for giving the semantics of languages with dynamic name allocations, as in (Stark, 1994; Stark, 1996; Hofmann, 1999; Bucalo et al., 2001). In fact, also the category of FM-sets with finite support (which correspond to $FSAlg_\pi$, as said before) is essentially equivalent to $\mathrm{Sh}(\mathbb{I}^{op})$, as mentioned briefly in (Gabbay and Pitts, 2002, Section 7). Here we give a direct proof of the equivalence between $FSAlg_\pi^f$ and $\mathrm{Sh}(\mathbb{I}^{op})$. The first step is the definition of a categorical version of the notion of support.

DEFINITION 15. Let $F : \mathbb{I} \to \mathbf{Set}$, let $X \in \mathbb{I}$ and let $a \in F_X$. Then, $Y \subseteq X$ *supports* $a$ if for all $h, k : X \rightarrowtail Z$ such that $h_{|Y} = k_{|Y}$ we have $F_h(a) = F_k(a)$.

LEMMA 16. *Let $F : \mathbb{I} \to \mathbf{Set}$ be a sheaf, let $X \in \mathbb{I}$ and let $a \in F_X$. Then, there exists a least $Y \in \mathbb{I}$ supporting $a$.*

*Proof.* If two sets $i_1 : Y_1 \subseteq X$ and $i_2 : Y_2 \subseteq X$ both support $a$, then also their pullback $Y_1 \times_X Y_2 = i_1(Y_1) \cap i_2(Y_2) = Y_1 \cap Y_2$ supports $a$, and the cardinality of the pullback is $\leq \min\{|Y_1|, |Y_2|\}$. □

Therefore, for all $X \in \mathbb{I}$ and $a \in F_X$, we can define $supp_X(a)$ as the least $Y$ supporting $a$. Furthermore, we usually drop the subscript, since it is easy to check that such a least $Y$ supporting $a$ does not depend on the particular $X$ the $a$ comes from; that is, if $a \in F_X \cap F_Z$, then $supp_X(a) = supp_Z(a)$.

PROPOSITION 17. *Categories $FSAlg_\pi^f$ and $\mathrm{Sh}(\mathbb{I}^{op})$ are equivalent.*

*Proof.* Let us define first a functor $F : FSAlg_\pi^f \to \mathrm{Sh}(\mathbb{I}^{op})$. Let $\mathcal{A}$ be a finitely supported algebra over finite permutations. The corresponding functor $F\mathcal{A} : \mathbb{I} \to \mathbf{Set}$ is defined

- on objects as $F\mathcal{A}_X \triangleq \{a \in A \mid supp_A(a) \subseteq X\}$;
- for $k : X \rightarrowtail Y$ in $\mathbb{I}$, $F\mathcal{A}_k : F\mathcal{A}_X \to F\mathcal{A}_Y$ maps $a \in A$ to $\widehat{\kappa}_A(a)$, where $\kappa \in \mathrm{Aut}^f$ is a(ny) finite permutation extending $k$ to the whole $\mathcal{N}$. Since $a$ has finite support, by Lemma 11 this definition is well given.

Thanks to Corollary 9(i), it is easy to check that this $F\mathcal{A}$ is a sheaf, by showing that it preserves pullbacks. Furthermore, let $\sigma : \mathcal{A} \to \mathcal{B}$ be an algebra homomorphism: the associated natural transformation $F\sigma$ :

$F\mathcal{A} \to F\mathcal{B}$ is defined as the obvious restriction $F\sigma_X \triangleq \sigma_{|F\mathcal{A}_X} : F\mathcal{A}_X \to F\mathcal{B}_X$ for all subsets $X$; it is well-given thanks to Corollary 9(ii).

On the other hand, we define a functor $G : \mathrm{Sh}(\mathbb{I}^{op}) \to \mathit{FSAlg}^f_\pi$ as follows. Let $P : \mathbb{I} \to \mathbf{Set}$ be any object of $\mathrm{Sh}(\mathbb{I}^{op})$; the carrier of the corresponding algebra $\mathcal{A} = (A, \{\widehat{\pi}_A \mid \pi \in \mathrm{Aut}^f\})$ is the set

$$A \triangleq \bigcup_{X \in \mathbb{I}} \{a \in P_X \mid \mathit{supp}(a) = X\}$$

For $\pi \in \mathrm{Aut}^f$, the map $\widehat{\pi}_A : A \to A$ is defined as

$$\widehat{\pi}_A(a) \triangleq P_{\pi_{|X}}(a) \quad \text{for } a \in P_X.$$

where $\pi_{|X} : X \rightarrowtail \pi(X)$ is the restriction of $\pi$ to the finite $X$. It is trivial to check that if $a \in P_X$ then $a$ is supported by $X$ according to Definition 5. Finally, any natural transformation $\eta : P \to Q$ induces quite obviously an homomorphism between the corresponding algebras.

It is easy to check that there are two natural isomorphisms

$$\phi : GF \xrightarrow{\sim} Id_{\mathit{FSAlg}^f_\pi} \qquad \psi : FG \xrightarrow{\sim} Id_{\mathrm{Sh}(\mathbb{I}^{op})}.$$

Indeed, for any algebra $\mathcal{A}$ in $\mathit{FSAlg}^f_\pi$, the carrier of $GF\mathcal{A}$ is the set

$$\begin{aligned} GFA &= \{a \mid X \in \mathbb{I}, a \in (FA)_X, \mathit{supp}(a) = X\} \\ &= \{a \mid X \in \mathbb{I}, \mathit{supp}_A(a) \subseteq X, \mathit{supp}(a) = X\} \\ &= \{a \mid X \in \mathbb{I}, \mathit{supp}_A(a) = X\} \cong A \end{aligned}$$

where the last equivalence holds because $\mathcal{A}$ is finitely supported.

On the other hand, for any sheaf $P : \mathbb{I} \to \mathbf{Set}$, the carrier of the algebra $GP$ is the set $\{a \mid Y \in \mathbb{I}, a \in P_Y, \mathit{supp}(a) = Y\}$. Therefore, $FGP$ is the presheaf mapping every $X$ to the set

$$\begin{aligned} (FGP)_X &= \{a \mid Y \in \mathbb{I}, a \in P_Y, \mathit{supp}(a) = Y, \mathit{supp}_{GP}(a) \subseteq X\} \\ &= \{a \mid Y \subseteq X, a \in P_Y, \mathit{supp}(a) = Y\} \\ &= \{a \mid a \in P_X, \mathit{supp}(a) \subseteq X\} = P_X \end{aligned}$$

where the last equivalence holds because by definition and thanks to Lemma 16 the support of $a \in F_X$ is a unique subset of $X$. □

*Remark 2.* Let us consider now the presheaf category $\mathbf{Set}^{\mathbb{B}}$, where $\mathbb{B}$ is the subcategory of $\mathbb{I}$ with only bijective maps. The inclusion functor $\mathbb{B} \hookrightarrow \mathbb{I}$ induces an obvious forgetful functor $|\_| : \mathbf{Set}^{\mathbb{I}} \to \mathbf{Set}^{\mathbb{B}}$, given by composition. As it is well known (Mac Lane and Moerdijk, 1994, Section VII), this functor has a left adjoint $(\_)_! : \mathbf{Set}^{\mathbb{B}} \to \mathbf{Set}^{\mathbb{I}}$, which in this case can be defined on objects as $(P_!)_X \triangleq \sum_{Y \subseteq X} P_X$. In the unpublished work (Fiore, 2001), Fiore proved that the Schanuel topos is equivalent to the Kleisli category of the monad $T : \mathbf{Set}^{\mathbb{B}} \to \mathbf{Set}^{\mathbb{B}}$ arising from this adjunction. More precisely, $T$ is the composition $T_{\_} = |(\_)_!|$, and the Kleisli category $\mathcal{K}(T)$ has as objects the objects of $\mathbf{Set}^{\mathbb{B}}$, and for $P, Q : \mathbb{B} \to \mathbf{Set}$, a morphism $\eta : P \to Q$ in $\mathcal{K}(T)$ is any natural transformation $\eta : P \to |Q_!|$ in $\mathbf{Set}^{\mathbb{B}}$.

In fact, the correspondence in the proof of Proposition 17 can be easily strengthened to work also directly with $\mathcal{K}(T)$. Namely, each finitely supported permutation algebra $\mathcal{A} = (A, \{\widehat{\pi}_A \mid \pi \in \mathrm{Aut}^f\})$ is mapped to a functor $F\mathcal{A}$, object of $\mathbf{Set}^{\mathbb{B}}$, defined as $F\mathcal{A}_X \triangleq \{a \in A \mid supp_A(a) = X\}$, and $F\mathcal{A}_\pi(a) \triangleq \widehat{\pi^c}_A(a)$ for $\pi : X \to X$ in $\mathbb{B}$. For $\sigma : \mathcal{A} \to \mathcal{B}$ in $Alg^f_\pi$, the corresponding morphism $F_\sigma : F\mathcal{A} \to F\mathcal{B}$ in $\mathcal{K}(T)$ is the natural transformation $\eta : F\mathcal{A} \to |(F\mathcal{B})_!|$ in $\mathbf{Set}^{\mathbb{B}}$, defined as $\eta_X \triangleq \sigma_{|F\mathcal{A}_X} : \{a \in A \mid supp_A(a) = X\} \to \{b \in B \mid supp_B(b) \subseteq X\}$. This definition is well given in virtue of Lemma 9.

### 3.3. Behavioural functors over permutation algebras

As mentioned before, $\mathbf{Set}^{\mathbb{I}}$ and $\mathrm{Sh}(\mathbb{I}^{op})$ have been widely used in the literature for definining the domain of meaning of name-passing calculi, such as the $\pi$-calculus (Moggi, 1993; Stark, 1994; Hofmann, 1999; Fiore and Turi, 2001). In these approaches, the domain is obtained as the final coalgebra of a "behavioural" endofunctor $B : \mathcal{C} \to \mathcal{C}$, where $\mathcal{C}$ is $\mathbf{Set}^{\mathbb{I}}$ or $\mathrm{Sh}(\mathbb{I}^{op})$ (or a variant of them). The definition of $B$ is usually polynomial, and this ensures the existence of the final coalgebra. Beside the usual constructors of polynomial functor (namely constants, finite sums and products and finite powersets), the categories $\mathrm{Sh}(\mathbb{I}^{op})$ and $\mathbf{Set}^{\mathbb{I}}$ feature the peculiar constructors needed for giving semantics to name-passing calculi. We recall the definition of these constructors on $\mathrm{Sh}(\mathbb{I}^{op})$, which were used in e.g. (Hofmann, 1999; Fiore and Turi, 2001).

1. the *type of names* is the object $N \triangleq \mathbb{I}(1, \_)$ (for all $X \in \mathbb{I}$: $N_X \cong X$);

2. the *shift operator* is the functor $\delta : \mathrm{Sh}(\mathbb{I}^{op}) \longrightarrow \mathrm{Sh}(\mathbb{I}^{op})$ (defined as $\delta(P)_X \triangleq P_{X \uplus 1}$ on objects, and $\delta(P)_f \triangleq P_{f \uplus id}$ on arrows), a type constructor representing the dynamic generation of names;

3. the *finite powerset* $\wp_f : \mathrm{Sh}(\mathbb{I}^{op}) \longrightarrow \mathrm{Sh}(\mathbb{I}^{op})$ is defined pointwise;

4. the *name exponential* $(\_)^N : \mathrm{Sh}(\mathbb{I}^{op}) \longrightarrow \mathrm{Sh}(\mathbb{I}^{op})$ is defined as

$$(P^N)_X = \mathrm{Sh}(\mathbb{I}^{op})(\mathbb{I}(X, \_) \times N, P) \cong (P_X)^X \times P_{X \uplus 1}$$

5. finally, the *partial name exponential* (useful for early semantics) $N \rightharpoonup \_ : \mathrm{Sh}(\mathbb{I}^{op}) \to \mathrm{Sh}(\mathbb{I}^{op})$ is defined as

$$
\begin{aligned}
(N \rightharpoonup P)_X &\triangleq (1 + P_X)^X \\
(N \rightharpoonup P)_f &: (1 + P_X)^X \to (1 + P_Y)^Y \qquad \text{for } f : X \rightarrowtail Y \\
& u \mapsto \lambda y \in Y. \begin{cases} P_f(u(x)) & \text{if } f(x) = y \text{ and } u(x) \in P_X \\ * & \text{otherwise} \end{cases}
\end{aligned}
$$

It is easy to check that any functor defined using these constructors (and finite sums and products) is accessible, and hence admits final coalgebra (Rutten, 2000). For instance, following (Fiore and Turi, 2001) the domain for late semantics of $\pi$-calculus can be defined as the final coalgebra of the functor $B : \mathrm{Sh}(\mathbb{I}^{op}) \to \mathrm{Sh}(\mathbb{I}^{op})$

$$
\begin{aligned}
BP &\triangleq \wp_f(\overbrace{N \times P^N}^{\text{input}} + \overbrace{N \times N \times P}^{\text{output}} + \overbrace{N \times \delta_S P}^{\text{bound output}} + \overbrace{P}^{\tau}) \\
(BP)_X &= \wp_f(X \times (P_X)^X \times P_{X \uplus 1} + X \times X \times P_X + X \times P_{X \uplus 1} + P_X).
\end{aligned}
$$

In virtue of the equivalence between $FSAlg^f_\pi$ and $\mathrm{Sh}(\mathbb{I}^{op})$, it is possible to define these constructors also on $FSAlg^f_\pi$. Moreover, since the equivalence preserves both limits and colimits, we only need to check out the behaviour on the functor for names and on the shift operator.

1. The *algebra of names* is given by $\mathcal{N} = (\mathcal{N}, \mathrm{Aut}^f)$.

2. The *shift* operator $\delta_A : FSAlg^f_\pi \to FSAlg^f_\pi$ is defined as follows. If $\mathcal{A} = (A, \{\widehat{\pi}_A \mid \pi \in \mathrm{Aut}^f\})$ is a permutation algebra, we define

$$\delta(\mathcal{A}) \triangleq (A, \{\widehat{\pi^{+1}}_A \mid \pi \in \mathrm{Aut}^f\})$$

   where for $\pi \in \mathrm{Aut}^f$, $\pi^{+1} \in \mathrm{Aut}^f$ is defined as

$$(\pi^{+1})(s_0) = s_0 \qquad (\pi^{+1})(s_{n+1}) = succ(\pi(s_n)).$$

   for any fixed enumeration $\mathcal{N} = \{s_0, s_1, s_2, \dots\}$.[1]

   For any morphism $\sigma : \mathcal{A} \to \mathcal{B}$, we put $\delta_A \sigma = \sigma$; indeed, for $\pi \in \mathrm{Aut}^f$, we have $\sigma \circ \widehat{\pi^{+1}}_A = \widehat{\pi^{+1}}_B \circ \sigma$ by definition of $\sigma$. It is easy to check that $\delta_A$ is an endofunctor on $FSAlg^f_\pi$.

3. Finite powersets, products and coproducts are defined pointwise.

[1] This is one of the literally infinite possible definitions of $\delta_A$; it corresponds to de Bruijn indexes, where the newly created (i.e., locally bound) name is always $s_0$.

4. By exploiting the equivalence between $\mathrm{Sh}(\mathbb{I}^{op})$ and $FSAlg^f_\pi$, we can derive the definition of $\mathcal{A}^{\mathcal{N}}$ whose carrier is the set

$$\{f : X \to A_X \mid X \subset \mathcal{N} \text{ finite}\} \times A$$

where $A_X \triangleq \{a \in A \mid supp_A(a) \subseteq X\}$. For $\pi \in \mathrm{Aut}^f$, the corresponding operator $\widehat{\pi}_{A^{\mathcal{N}}}$ maps each pair $(f : X \to A_X, a)$ to $(\widehat{\pi}_A \circ f \circ \pi^{-1} : L \to A_L, \widehat{\pi^{+1}}_A)$, where $L = \pi(X)$.

5. Finally, the *partial name exponential* on algebras is defined again by taking advantage of the equivalence with $\mathrm{Sh}(\mathbb{I}^{op})$. For an algebra $\mathcal{A}$, the carrier of the algebra $\mathcal{N} \rightharpoonup \mathcal{A}$ is the set of partial functions

$$B = \{f : X \rightharpoonup A_X \mid X \subset \mathcal{N},\ \text{finite}\}$$

and for $\pi \in \mathrm{Aut}^f$, the operator $\hat{\pi}_B : B \to B$ maps a partial function $u : X \rightharpoonup A_X$ to the partial function $v : Y \rightharpoonup A_Y$ where $Y \triangleq \pi(X)$ and for all $y \in Y$

$$v(y) \triangleq \begin{cases} \hat{\pi}_A(u(\pi^{-1}(y))) & \text{if } u(\pi^{-1}(y)) \text{ is defined} \\ \text{undefined} & \text{otherwise} \end{cases}$$

The coalgebras of the functors over $FSAlg^f_\pi$ are a particular class of the *structured coalgebras* studied for instance in (Corradini et al., 2001; Montanari and Pistore, 2000). Moreover, these functors correspond exactly to the polynomial functors over $\mathrm{Sh}(\mathbb{I}^{op})$ defined using the constructors listed above.

PROPOSITION 18. *Let $B : \mathrm{Sh}(\mathbb{I}^{op}) \to \mathrm{Sh}(\mathbb{I}^{op})$ be a polynomial endofunctor. Then, there exists a functor $\bar{B} : FSAlg^f_\pi \to FSAlg^f_\pi$ such that the category* **Coalg**$(B)$ *is isomorphic to* **Coalg**$(\bar{B})$, *and* vice versa.

*Proof.* It is sufficient to check that the functors $F$ and $G$ between $FSAlg^f_\pi$ and $\mathrm{Sh}(\mathbb{I}^{op})$ commute with the constructors of the polynomial functors. This can be proved easily by inspection. □

## 4. Finitely supported algebras and named sets

In this section we compare finitely supported algebras and *named sets*, which were introduced as the building blocks of *HD-automata*.

### 4.1. Named sets

The definitions below are drawn from (Ferrari et al., 2002, Section 3.1), and simplified according to our needs.

DEFINITION 19. [named sets] A *named set* $N$ is a triple

$$N = \langle Q_N, \|\cdot\|_N : Q_N \to \wp_f(\mathcal{N}), G_N : \prod_{q \in Q_N} \wp(\mathrm{Aut}(\|q\|_N))\rangle$$

where $Q_N$ is a set of *states*; $\|\cdot\|_N$ is the *enumerating* function; and for all $q \in Q_N$, the set $G_N(q)$ is a subgroup of $\mathrm{Aut}(\|q\|_N)$, and it is called the *permutation group* of $q$.

Intuitively, a state in $Q_N$ represents a process, and thus the function $\|\cdot\|_N$ assigns to each state the set of variables possibly occurring free in it. Finally, $G_N$ denotes for each state the group of renamings under which it is preserved, i.e., those permutations on names that do not interfere with its possible behavior.

DEFINITION 20. [category of named sets] Let $N$, $M$ be named sets. A *named function* $H : N \to M$ is a pair

$$L = \langle l : Q_N \to Q_M, \Lambda : \prod_{q \in Q_N} \mathbb{I}(\|l(q)\|_M, \|q\|_N)\rangle$$

for $l$ a function and $\Lambda(q)$ an injection from $\|l(q)\|_M$ to $\|q\|_N$, satisfying the additional condition

$$G_N(q) \circ \Lambda(q) \subseteq \Lambda(q) \circ G_M(l(q))$$

Finally, **NSet** denotes the category of named sets and their morphisms.

So, a named function is a state function, equipped with an injective renaming for each $q \in Q_N$, which is somewhat compatible with the permutations in $G_N(q)$ and $G_M(l(q))$. In particular, the identity on $N$ is $\langle id_{Q_N}, id_{\|\cdot\|_N}\rangle$, and composition is defined as expected.

*Remark 3.* We simplified the definition in (Ferrari et al., 2002, Section 3.1) in two ways. First, we did not restrict the enumerating function to taking value in prefixes of $\{0, 1, \ldots\}$: this would correspond to fix a canonical choice of free variables for each state, and albeit important for verification purposes, it does not seem relevant here. Second, on named functions the renaming $\Lambda(q)$ is actually a set of injections, satisfying

$$G_N(q) \circ \lambda \subseteq \Lambda(q) = \lambda \circ G_M(l(q)) \qquad \forall \lambda \in \Lambda(q)$$

In other words, "the whole set of $\Lambda_h(q)$ must be generated by saturating any of its elements by the permutation group of $h(q)$, and the result must be invariant with respect to the permutation group of $q$".

The resulting category has the same cardinality of **NSet**: it is obtained from the latter *via* an obvious restriction on objects, and by imposing an equivalence on hom-sets. We do not further discuss the matter here, referring the reader to (Gadducci et al., 2003) for a detailed correspondence between permutation algebras and that alternative presentation of named sets.

EXAMPLE 21. Let $\mathcal{N} = \{0, 1, 2, \ldots\}$, and let us adopt the usual "set-theoretic" convention of representing finite ordinals by natural numbers, thus $0 = \emptyset$ and $n = \{0, 1, \ldots, n-1\}$.

Now, we consider a few simple examples. Since $1 = \{0\}$ is a singleton, both $N_1 = \langle 1, \|1\| = 1, \mathrm{Aut}(1) = \{id\}\rangle$ and $N_2^p = \langle 1, \|1\| = 2, \mathrm{Aut}(2) = \{id, (1, 0)\}\rangle$ are named sets: same set of states, different enumerating functions. Instead, $N_2^i = \langle 1, \|1\| = 2, \{id\} \subseteq \mathrm{Aut}(2)\rangle$ is a named set with the same set of states and the same enumerating function of $N_2^p$, but with a different permutation group.

Notice that there is no named function from $N_2^p$ to $N_1$, since any injection $\lambda$, when post-composed with $\mathrm{Aut}(2)$, generates the whole $\mathbb{I}(1, 2)$. Instead, denoting by $\lambda_j$, for $j = 0, 1$, the injection mapping 0 to $j$, then $\langle id, \lambda_j\rangle$ is a named function from $N_2^i$ to $N_1$.

Similarly, there is no named function from $N_2^p$ to $N_2^i$, while there exists $\langle id, \lambda\rangle : N_2^i \to N_2^p$, for any $\lambda \in \mathrm{Aut}(2)$. In fact, it is easy to see that, given named sets $\langle Q, \|\cdot\|, G_1\rangle$ and $\langle Q, \|\cdot\|, G_2\rangle$ (i.e., same state set and enumerating function, different permutation groups), with $G_1(q)$ a subgroup of $G_2(q)$ for all $q \in Q$, then $\langle id, \Lambda\rangle$ is a well-defined named function from the former named set to the latter, whenever $\Lambda(q) \in G_2(q)$ for any $q \in Q$.

In the remaining of this section we relate $FSAlg_\pi^f$ and **NSet**, the category of named sets. We plan to sharpen and make more concise some of the results presented in (Montanari and Pistore, 2004, Section 6).

Summarizing, Proposition 22 and Proposition 23 (and the "canonical" version of the latter, Proposition 27: see later) prove the existence of suitable functors between the underlying categories, generalizing the functions on objects presented as Definition 49 and Definition 50, respectively, in (Montanari and Pistore, 2004, Section 6); while Theorem 28 extends to a categorical equivalence the correspondence on objects proved in Theorem 51 of the same paper.

## 4.2. From named sets to permutation algebras

The functor from named sets to (finite) permutation algebras is obtained by a free construction, analogous to the standard correspondence between sets and algebras. First, we need to introduce some notation: for any pair of finite subsets $X$, $Y$ and $\lambda \in \mathbb{I}(X, Y)$, we denote by $\lambda^c \in \mathrm{Aut}^f$ a *completion* of $\lambda$, i.e., a finite permutation such that $\lambda^c_{|X} = \lambda$, i.e., $\lambda^c(x) = \lambda(x)$ for all $x \in X$; and by $\lambda^i$ a completion such that moreover $\lambda^i_{|\mathcal{N}\setminus(X\cup Y)} = id$, i.e., $\lambda^i(z) = z$ for all $z \notin X \cup Y$.

PROPOSITION 22 (from sets to algebras). *Let $F_O$ be the function mapping each named set $N$ to the finite permutation algebra freely generated* *from the elements of $Q_N$, modulo the equivalence $\equiv_N$ induced by the set of axioms $\pi^c(q) =_N q$ for all completions $\pi^c$ of $\pi \in G_N(q)$.*

*Let $L : N \to M$ be a named function, and let $\Xi$ be the function associating to each $q \in Q_N$ the element $\Lambda(q)^i(l(q)) \in T_{\Sigma^f_\pi}(Q_M)$. Hence, let $F_A$ be the function associating to each named function $L$ the free extension of the function $\Xi$.*

*The pair $F = \langle F_O, F_A\rangle$ defines a functor from* **NSet** *to $FSAlg^f_\pi$.*

*Proof.* Since the carrier of $F_O(N)$ is $\{\pi(q) \mid q \in Q_N, \pi \in \mathrm{Aut}^f\}/\equiv_N$, it is easy to see that the resulting algebra has finite support, proving that each element $[\pi(q)]_N$ is supported by the set $\pi(\|q\|_N)$. In order to prove this, we must show that each permutation $\kappa$ fixing $\pi(\|q\|_N)$ also fixes $\widehat{\pi}_{F(N)}(q)$. Then we have that

$$\begin{aligned}
\forall x \in \pi(\|q\|_N) : \kappa(x) = x &\implies \forall k \in \|q\|_N : \kappa(\pi(k)) = \pi(k)\\
&\implies \forall k \in \|q\|_N : \pi^{-1}(\kappa(\pi(k))) = k\\
&\implies \widehat{(\pi^{-1}\kappa\pi)}_{F(N)}(q) \equiv_N q\\
&\implies \widehat{\pi}^{-1}_{F(N)}(\widehat{\kappa}_{F(N)}(\widehat{\pi}_{F(N)}(q))) \equiv_N q\\
&\implies \widehat{\kappa}_{F(N)}(\widehat{\pi}_{F(N)}(q)) \equiv_N \widehat{\pi}_{F(N)}(q)
\end{aligned}$$

Let us now consider a named set function $L : N \to M$. The function $\Xi$ can be lifted to an algebra homomorphism from the free algebra $T_{\Sigma^f_\pi}(Q_N)$ to the free algebra $T_{\Sigma^f_\pi}(Q_M)$. Moreover, it clearly preserves the axioms on identity and composition: we must then prove that this holds also for the additional axioms arisen from the permutation group. This is equivalent to prove that $\Xi(\widehat{\pi^c}_{F(N)}(q)) \equiv_M \Xi(q)$ for any completion $\pi^c$ of $\pi \in G_N(q)$. By construction, we have that $\Xi(\widehat{\pi^c}_{F(N)}(q)) \triangleq \widehat{\pi^c}_{F(\mathcal{M})}(\widehat{\Lambda(q)^i}_{F(M)}(l(q)))$. Now, remember that there exists a $\kappa \in G_M(l(q))$ such that $\pi \circ \Lambda(q) = \Lambda(q) \circ \kappa$, and then that for a suitable completion $\kappa^c$ we have $\pi^c \circ \Lambda(q)^i = \Lambda(q)^i \circ \kappa^c$: this implies that $\Xi(\pi^c_{F(N)}(q))$ coincides with $\widehat{\Lambda(q)^i}_{F(M)}(\widehat{\kappa^c}_{F(\mathcal{M})}(l(q)))$, which is equivalent to $\widehat{\Lambda(q)^i}_{F(M)}(l(q))$, hence the result.

The identities are clearly preserved. Concerning composition, it is enough to show that the result of the functor is independent with respect to the choice of the completion of the injection, i.e, that given a named function $L : N \to M$, then for any extension $\lambda$ of $\Lambda(q)$ the equality $\widehat{\lambda}(l(q)) \equiv_M \widehat{\Lambda(q)^i}(l(q))$ holds. To prove the latter, note that the conditions on $\Lambda(q)$ ensure on the existence of a permutation $\kappa$ fixing $\|l(q)\|_M$ such that $\lambda = \Lambda(q)^i \circ \kappa$, hence the equality follows. □

Recall now that, according to *(ii)* of Lemma 8, given an algebra homomorphism $\sigma : \mathcal{A} \to \mathcal{B}$, and a finitely supported element $a \in A$, then $supp_B(\sigma(a)) \subseteq supp_A(a)$. So, let $in_\sigma(a)$ be the uniquely associated injection: this remark is sufficient for defining a functor $I$ from finitely supported permutation algebras to named sets.

PROPOSITION 23 (from algebras to sets). *Let $I_O$ be the function mapping each $\mathcal{A} \in FSAlg^f_\pi$ to the named set $\langle A, supp_A(\cdot), G_{I(\mathcal{A})}\rangle$, where*

$$G_{I(\mathcal{A})}(a) \triangleq \{\pi_{|supp_A(a)} \mid \pi \in \mathrm{fix}_A(a)\}.$$

*Let $\sigma : \mathcal{A} \to \mathcal{B}$, and let $in_\sigma(a) : supp_B(\sigma(a)) \to supp_A(a)$ be the uniquely induced arrow. Hence, let $I_A$ be the function associating to $\sigma$ the named function $\langle l_\sigma, \Lambda_\sigma\rangle$ given by the obvious function from $A$ to $B$ and by the set of injections $\Lambda_\sigma(a) = in_\sigma(a)$ for all $a \in A$.*

*The pair $I = \langle I_O, I_A\rangle$ defines a functor from $FSAlg^f_\pi$ to* **NSet**.

*Proof.* It is easy to see that $G_{I(\mathcal{A})}(a)$ is well-defined, since $fix_A(a) \subseteq sp_A(a)$ holds by *(iii)* of Corollary 9, hence $\pi_{|supp_A(a)} \in \mathrm{Aut}(supp_A(a))$; moreover, it is a group, since $\mathrm{fix}_A(a)$ is so.

We must now prove that for all $a \in A$

$$G_{I(\mathcal{A})}(a) \circ \Lambda_\sigma(a) \subseteq \Lambda_\sigma(a) \circ G_{I(\mathcal{B})}(\sigma(a)).$$

This is equivalent to ask that for all $\pi \in G_{I(\mathcal{A})}(a)$ there exists a $\kappa \in G_{I(\mathcal{B})}(\sigma(a))$ such that $\pi \circ in_\sigma(a) = in_\sigma(a) \circ \kappa$. A possible choice is $\pi_{|supp_B(\sigma(a))}$: in fact, since $\mathrm{fix}_A(a) \subseteq \mathrm{fix}_B(\sigma(a))$, any $\pi^c$ fixes $a$ also in $\mathcal{B}$; and since $\mathrm{fix}_B(\sigma(a)) \subseteq sp_B(\sigma(a))$, then $\pi_{|supp_B(\sigma(a))}$ is well-defined and satisfies the requirements.

Identities and composition are preserved, hence the result holds. □

Our next step is a look at the algebras obtained *via* the functor $F$.

LEMMA 24. *Let $\mathcal{N}$ be a named set, and let $q \in Q_N$. Then*

$$\mathrm{fix}_{F(\mathcal{N})}([q]_N) = \{\pi^c \mid \pi \in G_N(q)\}.$$

*Proof.* Let us denote the group $\{\pi^c \mid \pi \in G_N(q)\}$ by $G_N(q)^c$. Clearly, by construction any completion of a permutation $\pi \in G_N(q)$ fixes $q$ in $F(\mathcal{N})$; so, it suffices to prove that if $\widehat{\kappa}_{F(\mathcal{N})}([q]_N) \equiv_N [q]_N$, then $\kappa \in G_N(q)^c$. Now, this implies that there exists a proof for $\kappa(q) =_N q$: then, the result is easily proved by induction on the length of the proof. □

We can now prove the main result of this section.

THEOREM 25 (adjunction). *Let $F$ and $I$ be the functors given in Proposition 22 and Proposition 23, respectively. Then, they form an adjoint pair, for $F$ left-adjoint to $I$.*

*Proof.* Let $\mathcal{N}$ be a named set. By Lemma 24, $supp_{F(N)}([q]_N) = \|q\|_N$ holds and, consequently, $G_{I(F(N))}([q]_N) = G_N(q)$. So, the pair $\eta_N = \langle in_N, id\rangle$ defines a named function from $N$ to $I(F(N))$, for $in_N$ the injection mapping $q$ to $[q]_N$: such a morphism represents the unit.

Let $\mathcal{A} \in FSAlg_\pi$. In order to prove the adjunction $F \dashv I$, it is enough to show that for each named function $H : N \to I(\mathcal{A})$ there exists a unique morphism $\sigma_H : F(N) \to \mathcal{A}$ such that $\eta_N; I(\sigma_H) = H$ (see (Barr and Wells, 1999, Definition 13.2.1)). Such a morphism must behave as $h$ on $Q_N$, meaning that (the equivalence class) $[q]_{\equiv_N}$ has to be mapped into $h(q)$: so, this fact does constrain the choice of $\sigma_H$ to be the free extension of $h$, which indeed satisfies the requirements. □

## 4.4. Strengthening the adjunction

The adjunction proved in the previous section can actually be strengthened. The reason is the peculiar structure of permutations algebras, where each operator is unary and invertible.

Thus, we introduce a last concept, the *orbit* of an element, consisting of the family of all the elements of the carrier of an algebra which can be reached from the given element *via* the application of any operator.

DEFINITION 26. [orbits] Let $\mathcal{A} \in Alg_\pi$ and let $a \in A$. The *orbit* of $a$ is the set $Orb_A(a) \triangleq \{\widehat{\pi}_A(a) \mid \pi \in \mathrm{Aut}\}$.

Orbits obviously partition a permutation algebra. So, let us assume the existence for each orbit $Orb_A(a)$ of a canonical representative $a_O$ (we come back on this later on, in Remark 4), and let $A_O \triangleq \{a_O \mid a \in A\}$.

PROPOSITION 27 (from algebras to sets, II). *Let $\widehat{I}_O$ be the function mapping each $\mathcal{A} \in FSAlg^f_\pi$ to the named set $\langle A_O, supp_A(\cdot), G_{\widehat{I}(\mathcal{A})}\rangle$, for $G_{\widehat{I}(\mathcal{A})}(a_O)$ the set of permutations given by*

$$\{\pi_{|supp_A(a_O)} \mid \pi \in \mathrm{fix}_A(a_O)\}.$$

*Let $\sigma : \mathcal{A} \to \mathcal{B}$, let $in_\sigma(a_O) : supp_B(\sigma(a_O)) \to supp_A(a_O)$ be the uniquely induced arrow, let $\lambda$ be a chosen finite permutation such that $\widehat{\lambda}_A(\sigma(a_O)_O) = \sigma(a_O)$, and let $\lambda_{a_O} : supp_A(\sigma(a_O)_O) \to supp_A(\sigma(a_O))$ be the associated restriction. Hence, let $\widehat{I}_A$ be the function associating to $\sigma$ the named function $\langle h_\sigma, \Lambda_\sigma\rangle$ such that $h_\sigma(a_O) = \sigma(a_O)_O$ and $\Lambda_\sigma(a_O) = in_\sigma(a_O) \circ \lambda_{a_O}$ for all $a_O \in A$.*

*The pair $\widehat{I} = \langle \widehat{I}_O, \widehat{I}_A\rangle$ defines a functor from $FSAlg^f_\pi$ to* **NSet**.

*Proof.* The key remark for the correctness of $\Lambda_\sigma$ is the obvious coincidence between $\lambda \circ \mathrm{fix}_B(\sigma(a_O)_O)$ and $\mathrm{fix}_B(\sigma(a_O)) \circ \lambda$, for any $\lambda \in Hom_B[\sigma(a_O)_O, \sigma(a_O)]$; so that the equality $\Lambda_\sigma(a_O) \circ G_{\widehat{I}(\mathcal{B})}(\sigma(a_O)_O) = in_\sigma(a_O) \circ G_{I(\mathcal{B})}(\sigma(a_O)) \circ \lambda_{a_O}$ holds. Then, it is enough to mimic the proof for Proposition 23. □

Using the previously defined functor, it is easy to realize that named sets are just a different presentation for finite permutation algebras.

THEOREM 28. *Categories* **NSet** *and* $FSAlg^f_\pi$ *are isomorphic.*

*Proof.* Let $N$ be a named set: It is easy to prove that it is isomorphic to $\widehat{I}(F(N))$. First of all, the set of states of $\widehat{I}(F(N))$ is $\bigcup_{q \in Q_N}(([q]_N)_O)$; then, its set of permutations $G_{\widehat{I}(F(\mathcal{N}))}(([q]_N)_O) \subseteq \mathrm{Aut}(\|q\|_N)$ satisfies

$$\lambda_{([q]_N)_O} \circ G_{\widehat{I}(F(\mathcal{N}))}(([q]_N)_O) = G_{I(F(\mathcal{N}))}([q]_N) \circ \lambda_{([q]_N)_O};$$

finally, remember that by Lemma 24 the equality $supp_{F(N)}([q]_N) = \|q\|_N$ holds, and it implies $G_{I(F(\mathcal{N}))}([q]_N) = G_N(q)$. So, the corresponding (natural) isomorphism is given by $\langle([-]_N)_O, \lambda_{([-]_N)_O}\rangle$.

Analogous considerations hold for the isomorphism $F(\widehat{I}(\mathcal{A})) \to \mathcal{A}$ on algebras, which is obtained as the free extension of the function mapping $[a_O]_{\widehat{I}(\mathcal{A})}$ into $a_O$. □

*Remark 4.* The canonical representative $a_O$ of each orbit can be constructively defined, as long as the underlying set $\mathcal{N}$ is totally ordered. In fact, this property allows for both $\wp_{fin}$ and $\mathrm{Aut}^f$ also being naturally equipped with a total order, and the latter is then lifted to sets of permutations. Hence, for each orbit an element $a_c$ can be chosen, such that $supp_A(a_c)$ is minimal, and which has the minimal permutation group associated to it. The definition is well-given, since it is easy to prove that $\mathrm{fix}_A(a) = \mathrm{fix}_A(b)$ implies $a = b$ for all finitely supported $a, b \in A$ such that $b \in Orb_A(a)$.

*Remark 5.* A different notion of "named set" is considered in (Fiore and Staton, 2004), namely

> a named set is a pair $(A, f)$ where $A$ is a set and for all $a \in A$, $f(a)$ is a subgroup of Aut.

This definition is simpler than our Definition 19, because it basically lacks the enumerating function for each element; nevertheless, the notion of "supporting set" can be recovered by stating that $X \subseteq \mathcal{N}$ supports $a \in A$ if and only if $\mathrm{fix}(X) \subseteq f(a)$.

According to this alternative definition, a named set is not finitely supported *a priori*, but the property must be required explicitly; on the other hand, all the named sets of Definition 19 are finitely supported. In fact, in (Fiore and Staton, 2004) the subcategory of finitely supported named sets is proved equivalent to the Schanuel topos, and hence, by the results above, to the category **NSet** of Definition 20. Hence, we can see the more explicit Definition 20 as the "implementation-oriented" notion of named sets, while the more compact definition used in (Fiore and Staton, 2004) appears to be more "theoretical-oriented".

## 5. Permutation algebras and continuous $G$-sets

In the previous section we have proved the equivalences

$$FSAlg_\pi \cong FSAlg^f_\pi \cong \mathrm{Sh}(\mathbb{I}^{op})(\cong \mathbf{NSet})$$

by providing directly the corresponding equivalence functors. In this section we re-analyze these correspondence in the light of a well-known theory from algebraic topology, namely that of *(continuous) G-sets*. This allows for accommodating in a single framework also the categories $Alg_\pi$ and $Alg^f_\pi$, which were omitted in the previous analysis.

### 5.1. Continuous $G$-sets

In this subsection we recall some standard definitions and results about continuous $G$-sets; see e.g. (Kelley, 1975) for a presentation in the context of general topology, and (Mac Lane, 1971, Section V.9) and (Mac Lane and Moerdijk, 1994, Chapter II) in the context of category and topos theory.

DEFINITION 29. [$G$-sets] Let $G$ be a group. A *$G$-set* is a pair $(X, \cdot_X)$ where $X$ is a set and $\cdot_X : X \times G \to X$ is a *right $G$-action*, that is

$$x \cdot_X id = x \qquad (x \cdot_X g_1) \cdot_X g_2 = x \cdot_X (g_1 g_2)$$

A morphism $f : (X, \cdot_X) \to (Y, \cdot_Y)$ between $G$-sets is a function $f : X \to Y$ such that $f(x \cdot_X g) = f(x) \cdot_Y g$ for all $x \in X$.

The $G$-sets and their morphisms form a category denoted by $\mathbf{B}G^\delta$.

For instance, the $perm(\mathbb{A})$-sets and equivariant functions used in (Gabbay and Pitts, 2002) form the category $\mathbf{B}perm(\mathbb{A})^\delta$.

More generally, we are interested in $G$-sets where $G$ is a *topological group*, i.e., its carrier is equipped with a topology and multiplication and inverse are continuous. We recall some basic definitions of topology.

DEFINITION 30. A *topological space* is a pair $(X, \mathcal{O}(X))$ for $X$ a set and $\mathcal{O}(X) \subseteq \wp(X)$ (the *topology* over $X$) is closed with respect to arbitrary union and finite intersection, and $\emptyset, X \in \mathcal{O}(X)$.

A function $f : X \to Y$ is a *continuous map* $f : (X, \mathcal{O}(X)) \to (Y, \mathcal{O}(Y))$ if $f^{-1}(U) \in \mathcal{O}(X)$ for all $U \in \mathcal{O}(Y)$.

We denote by **top** the category of topological spaces and continuous maps form.

The elements of $\mathcal{O}(X)$ are referred to as the *open sets* of the topology.

EXAMPLE 31. The smallest (that is, the coarsest) topology is represented by $\mathcal{O}(X) = \{\emptyset, X\}$. On the other hand, the finest topology is the *discrete topology*, where $\mathcal{O}(X) = \wp(X)$. It is easy to prove that a topology is discrete if and only if $\{x\} \in \mathcal{O}(X)$ for all $x \in X$, i.e., if every point is separated from the others (hence the name). Clearly, every function is continuous with respect to the discrete topology.

*Remark 6. (product of spaces)* The category **top** is complete and cocomplete (Mac Lane, 1971, Section V.9). In particular, given a family of topological spaces $(X_i, \mathcal{O}(X_i)) \in \mathbf{top}$, indexed by $i \in I$, the product $\prod_{i \in I}(X_i, \mathcal{O}(X_i))$ is the topological space whose space is $X = \prod_{i \in I} X_i$, and the topology is the smallest topology such that the projections $\pi_i : X \to X_i$ are continuous. If $I$ is finite, then $\mathcal{O}(X) = \prod_{i \in I} \mathcal{O}(X_i)$. This does not hold for $I$ infinite, in general.

Finally, we recall the last standard definition we need for our development, which generalizes Definition 29.

DEFINITION 32. [topological groups and continuous $G$-sets] A group $G$ is a *topological group* if its carrier is equipped with a topology, and its multiplication and inverse are continuous with respect to this topology.

A $G$-set $(X, \cdot_X)$ is *continuous* if $G$ is topological and the action $\cdot_X : X \times G \to G$ is continuous with respect to $X$ equipped with the discrete topology.

A morphism $f : (X, \cdot_X) \to (Y, \cdot_Y)$ between continuous $G$-sets is a function $f : X \to Y$ which respects the actions.

For a given topological group $G$, continuous $G$-sets and their morphisms form a category, denoted by $\mathbf{B}G$.

Notice that for any group $G$, the category of *all* $G$-sets is the category of continuous $G$-sets where $G$ is taken with the discrete topology – hence the notation $\mathbf{B}G^\delta$ from (Mac Lane and Moerdijk, 1994) that we have used in Definition 29.

A useful characterization of continuous $G$-sets is given by the following lemma (Mac Lane and Moerdijk, 1994, I, Exercise 6).

LEMMA 33. *Let $G$ be a topological group, let $(X, \cdot_X)$ be a $G$-set, and for each $x \in X$ let $\mathrm{fix}_X(x) \triangleq \{g \in G \mid x \cdot_X g = x\}$ be denoted the* isotropy group *of $x$. Then, $(X, \cdot_X)$ is continuous iff all its isotropy groups are open sets in $G$.*

## 5.2. Permutation algebras as $G$-sets

For any countably infinite set of names $\mathcal{N}$, equipped with a total ordering, a permutation $\pi \in \mathrm{Aut}(\mathcal{N})$ is equivalent to a permutation over the set of natural numbers $\mathbf{N}$. Therefore, in the rest of this section we assume, without loss of generality, that $\mathcal{N} = \mathbf{N}$

Let us consider the $G$-sets when $G$ is either Aut or $\mathrm{Aut}^f$. Clearly, every Aut-set is also a $\mathrm{Aut}^f$-set (just by restricting the action to the finite permutations), mimicking the correspondence between $Alg_\pi$ and $Alg^f_\pi$. In fact, a stronger equivalence holds between the formalisms, as it is put in evidence by the next result.

PROPOSITION 34. $Alg_\pi \cong \mathbf{B}\mathrm{Aut}^\delta$ *and* $Alg^f_\pi \cong \mathbf{B}\mathrm{Aut}^{f^\delta}$.

*Proof.* Let $\mathcal{A}$ be a permutation algebra. The corresponding Aut-set is $G(\mathcal{A}) = (A, \cdot_{G(A)})$, where $a \cdot_{G(A)} \pi \triangleq \widehat{\pi}_A(a)$ for all $a \in A$. On the other hand, if $(X, \cdot_X)$ is a Aut-set, the corresponding algebra $\mathcal{X} = (X, \{\pi_X\})$ is defined by taking $\widehat{\pi}_X(x) \triangleq x \cdot_X \pi$ for $\pi \in \mathrm{Aut}$.

Let $\mathcal{A}, \mathcal{B}$ be two permutation algebras. A function $f : A \to B$ is a morphism $f : \mathcal{A} \to \mathcal{B}$ in $Alg_\pi$ iff $f(\widehat{\pi}_A(a)) = \widehat{\pi}_B(f(a))$ for all permutations $\pi$ and $a \in A$, which in turn holds iff $f(a \cdot_{G(A)} \pi) = f(a) \cdot_{G(B)} \pi$ for all $\pi$ and $a$, which equivalently states that $f : (A, \cdot_{G(A)}) \to (B, \cdot_{G(B)})$ is a morphism in $\mathbf{B}\mathrm{Aut}^\delta$.

Using the same argument, we have also that $Alg^f_\pi \cong \mathbf{B}\mathrm{Aut}^{f^\delta}$. □

Also the categories of algebras with finite support, possibly over only finite permutations, can be recasted in the general setting of $G$-sets, but to this end we need to equip the groups Aut and $\mathrm{Aut}^f$ with a topology.

Let us consider the space $\mathbf{N}$, given as the set of natural numbers equipped with the discrete topology. The *Baire space* is the topological space $\prod_{i=0}^{\infty} \mathbf{N} = \mathbf{N}^\omega$, equipped with the infinite product topology. A base of this topology is given by the sets of the form $\prod_{i=0}^{\infty} X_i$ where $X_i \neq \mathbf{N}$ only for *finitely many* indexes $i$.

Let us now consider the groups Aut and $\mathrm{Aut}^f$. The carriers of these groups are subspaces of the Baire space, where each $\pi$ corresponds to the infinite list $(\pi(0), \pi(1), \pi(2), \dots)$, as described in (Mac Lane and Moerdijk, 1994, Section III.9) for Aut. Therefore, both Aut and $\mathrm{Aut}^f$ inherit a topology from $\mathbf{N}^\omega$: their open sets are of the form $U \cap \mathrm{Aut}$ and $U \cap \mathrm{Aut}^f$, for $U$ open set of $\mathbf{N}^\omega$.

We can thus consider the categories $\mathbf{B}\mathrm{Aut}$ and $\mathbf{B}\mathrm{Aut}^f$ of continuous Aut-sets and continuous $\mathrm{Aut}^f$-sets, respectively. For the former category there is a famous characterization result (Mac Lane and Moerdijk, 1994, Section III.9, Corollary 3).

PROPOSITION 35. *$\mathbf{B}\mathrm{Aut} \cong \mathrm{Sh}(\mathbb{I}^{op})$.*

By Theorem 12, we have that $FSAlg_\pi \cong \mathbf{B}\mathrm{Aut} \cong FSAlg^f_\pi$. But actually this equivalence can be extended to $\mathbf{B}\mathrm{Aut}^f$ as well, as a consequence of the following result.

THEOREM 36. *$FSAlg^f_\pi \cong \mathbf{B}\mathrm{Aut}^f$.*

*Proof.* We show that the functor $G$ of Proposition 34 maps finite permutation algebras with finite support to continuous $\mathrm{Aut}^f$-sets, and *vice versa*.

Let $\mathcal{A} = (A, \{\hat{\pi}_A\})$ be an algebra in $FSAlg^f_\pi$; the corresponding $\mathrm{Aut}^f$-set is $(A, \cdot_{G(A)})$, where $a \cdot_{G(A)} \pi \triangleq \hat{\pi}_A(a)$ for all $a \in A$. For Lemma 33, $G(\mathcal{A})$ is continuous if and only if $\mathrm{fix}_A(a)$ is open for all $a \in A$: this is proved by a suitable characterization of $\mathrm{fix}(a)$, given by

$$\begin{aligned}
\mathrm{fix}_A(a) &= \bigcup_{\pi \in \mathrm{fix}_A(a)} \prod_{i=0}^{\infty} \{\pi(i)\} \\
&= \bigcup_{\pi \in \mathrm{fix}_A(a)} \left( \prod_{i=0}^{\infty} A_i^\pi \right) \cap \mathrm{Aut}^f \text{ for } A_i^\pi \triangleq \begin{cases} \{\pi(i)\} & \text{if } i \in supp_A(a) \\ \mathbf{N} & \text{otherwise} \end{cases} \\
&= \left( \bigcup_{\pi \in \mathrm{fix}_A(a)} \prod_{i=0}^{\infty} A_i^\pi \right) \cap \mathrm{Aut}^f \qquad (1)
\end{aligned}$$

where the second equality holds since $fix_A(a) \subseteq sp_A(a)$, while the latter expression clearly denotes an open set in $\mathrm{Aut}^f$ because each $\prod_{i=0}^{\infty} A_i^\pi$ is open in $\mathbf{N}^\omega$ since $supp_A(a)$ is finite and thus only finitely many $A_i^\pi$'s are not equal to $\mathbf{N}$.

On the other hand, let $(X, \cdot_X)$ be a continuous $\mathrm{Aut}^f$-set; we prove that $\mathcal{X} = (X, \{\widehat{\pi}_X\})$ is in $FSAlg^f_\pi$. Clearly $\mathcal{X}$ is a finite permutation algebra. By Lemma 33, for any $x \in X$, $\mathrm{fix}_X(x)$ is an open set of $\mathrm{Aut}^f$, hence $\mathrm{fix}_X(x) = U \cap \mathrm{Aut}^f$ for some $U$ open set of $\mathbf{N}^\omega$. More explicitly, $\mathrm{fix}_X(x)$ can be written as

$$\mathrm{fix}_X(x) = \left( \bigcup_{i \in I} \prod_{j=0}^{\infty} X_{ij} \right) \cap \mathrm{Aut}^f$$

for some family of indexes $I$, and where for each $i \in I$ there exists a finite $J_i \subset \omega$ such that $X_{ij} \neq \mathbf{N}$ only for $j \in J_i$. Since $id \in \mathrm{fix}_X(x)$

(it is a group), there exists $i_0 \in I$ such that $id \in \prod_{j=0}^{\infty} X_{i_0 j}$. We prove that the finite set $J \triangleq J_{i_0}$ supports $x$. Let $\pi \in \text{fix}_X(J) \cap \text{Aut}^f$. For all $j \in \omega$, if $j \in J$ then $\pi(j) = j \in X_{i_0 j}$, otherwise $X_{i_0 j} = \mathbf{N}$. In both cases, $\pi(j) \in X_{i_0 j}$. So $\pi \in \prod_{j=0}^{\infty} X_{i_0 j}$, and therefore $\pi \in \text{fix}_X(x)$, i.e. $\widehat{\pi}_X(x) = x \cdot_X \pi = x$, hence the thesis. □

As a corollary we have

COROLLARY 37. $\mathbf{B}\text{Aut}^f \cong \text{Sh}(\mathbb{I}^{op})$.

Actually, the proof of Theorem 12 suggests a direct proof of the result above. Corollary 37 can be proved along the same pattern of the argument following (Mac Lane and Moerdijk, 1994, III.9, Theorem 2), just restricting to finite permutations. The argument works in the restricted case because any monomorphism $\beta : L \rightarrowtail K$ in $\mathbb{I}$ can be extended to a *finite kernel* isomorphism on $\mathbf{N}$, that is, to an object $\bar{\beta} \in \text{Aut}^f$, e.g. as

$$\bar{\beta}(i) \triangleq \begin{cases} \beta(i) & \text{if } i \in L \\ (i+1-j)\text{-th element of } \mathbf{N} \setminus \beta(L) & \text{otherwise,} \end{cases}$$

where $j = |\{l \in L \mid l < i\}|$. Clearly $\bar{\beta}$ is a permutation, and it is easy to see that $|\ker(\bar{\beta})| \leq \max(L \cup K)+1$, and hence it is finite. See (Gadducci et al., 2003) for a detailed description of this proof.

It is interesting to notice that both the inclusion functor $\mathbf{B}\text{Aut} \hookrightarrow \mathbf{B}\text{Aut}^\delta$ and its counterpart for finite permutations have a right adjoint; the latter is e.g. defined on the objects as follows

$$\begin{aligned} r : \mathbf{B}\text{Aut}^\delta &\to \mathbf{B}\text{Aut} \\ (X, \cdot_X) &\mapsto (\{x \in X \mid \text{fix}(x) \text{ open for Aut}\}, \cdot_X) \end{aligned}$$

and it is the restriction on morphisms. Therefore, $r$ maps every $\mathbf{B}\text{Aut}^\delta$-set to the largest continuous $\mathbf{B}\text{Aut}$-set contained in it. Translating $r$ to permutation algebras along the equivalences, this is equivalent to state that there exists a functor

$$r' : \mathit{Alg} \to \mathit{FSAlg} \qquad (A, \{\hat{\pi}_A\}) \mapsto (B, \{\hat{\pi}_A|_B\})$$

where $B \triangleq \{a \in A \mid \text{fix}_A(a) \text{ open for Aut}\}$. Now, $\text{fix}_A(a)$ is open iff there exists a finite $J \subset \omega$ such that for any $\pi$, if $\pi(i) = i$ for all $i \in J$ then $\pi \in \text{fix}_A(a)$ (see the proof of Theorem 36). This corresponds exactly to say that $a$ has finite support, hence we can define directly $r'(A) = \{a \in A \mid \mathit{supp}_A(a) \text{ finite}\}$.

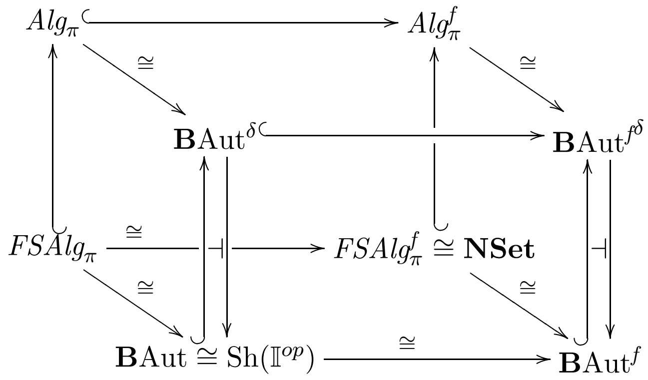


*Figure 1.* The Permutation Algebra Cube

## 6. Conclusions

In this paper we surveyed three main approaches to the treatment of nominal calculi. We compared metamodels based on (pre)sheaf categories, on permutation algebras (which subsume nominal sets, FM-sets and alike), and on named sets, that is sets enriched with names and permutation structures. We proved that the category of named sets are equivalent to the categories of permutation algebras with finite support (either on the signature with all permutations or with only finite ones) which in turn are equivalent to the category of sheaves over $\mathbb{I}$, that is the Schanuel topos. Figure 1 summarizes these relationships. These results confirm that permutation algebras and named sets can be used as algebraic specifications and “implementation versions” of sheaves of Schanuel topos. Moreover, using these equivalence, we can “import” the known final coalgebra machinery and constructions from the Schanuel topos into the category of finitely supported permutation algebras.

As a future work, it would be interesting to investigate a suitable internal language for the models analyzed here. The connection with Fraenkel-Mostowski set theory, would lead us to consider some variant (e.g., higher-order) of Pitts’ Nominal Logic (Pitts, 2003), or the Theory of Contexts (Honsell et al., 2001). Another interesting future work is to investigate how, and under which conditions, we can extend the basic (finite) permutation signature with other operators and axioms; for instance, these operators may represent object language constructors, or other operations over names such as (non-injective) substitutions.

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