Advanced Process Calculi

Lecture 4: higher-order psi-calculi

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Joachim Parrow

Psi-calculi

T	(Data) Terms	M, N
A	Assertions	Ψ, Ψ'
C	Conditions	φ, φ'

Channel Object Pattern Test (aka guard)

$\overline{M}\, N.P$	Output
$\underline{M}(\lambda\widetilde{x})N.P$	Input
$\mathbf{case}\ \varphi_1 : P_1 \,[\!]\, \cdots \,[\!]\, \varphi_n : P_n$	Case
$(\nu a)P$	Restriction
$P \mid Q$	Parallel
$!P$	Replication
$(\!\vert \Psi \vert\!)$	Assertion

To mean tell Ψ

How to cook a Psi-calculus

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   can be absolutely anything

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Compositional semantics
Algebraic laws
Bisimulation theory

Machine certified

- **Can capture**
  - **Applied pi-calculus** (Abadi, Fournet 2001)
  - **Explicit fusion calculus** (Wischik, Gardner 2005)
  - **Concurrent constraint pi** (Buscemi, Montanari 2007)
  - **Polyadic synchronization** (Carbone, Maffeis 2003)
  - **Pattern matching** and **higher order values** (Various)
- **And moreover**
  - **Higher-order concurrent constraints**
  - **Algebraic operators on communication channels**

For every application there is a suitable psi-calculus?

Of course not

Current extensions

- **Higher-order** psi: Agents can be sent around as data objects.
- **Broadcast** psi: an output action can be received by many inputs
- **Sorted** psi: A sort system regulates what can be substituted, sent on channels etc
- **Priority** psi: actions carry priorties, lower are preempted by higher

Psi and sorts

Problem: Substitution must be total, and all terms can act as both subjects and objects

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Effect: over-expressiveness. It is difficult to restrict a calculus to avoid useless agents, aka **junk**

Psi and sorts

Example: polyadic pi. Objects of prefixes are name tuples, as in $a(x_1,\ldots,x_n).P$

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Tuples as channels

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Tuples as channels

$$(x, y, z)[y := (u, w)] \quad \stackrel{?}{=} \quad (x, (u, w), z)$$

Nested tuples (aka trees)

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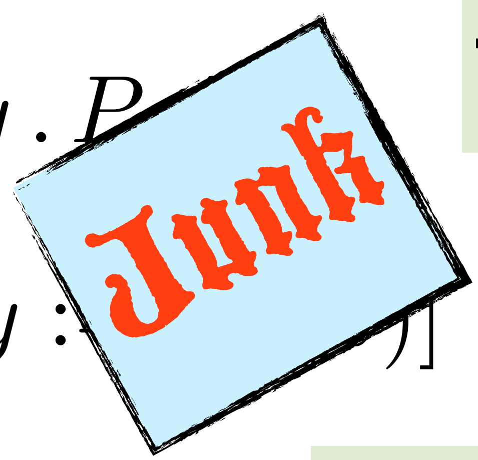

$$(x, y, z)[y : \ldots)] \quad \stackrel{?}{=} \quad (x, (u, w), z)$$

Nested tuples (aka trees)

Dealing with junk

Allow it, using (ad hoc) invariants
to ensure it never arises

or

Disallow it, using a formal sort system

Sorts

Assume a set of sorts S

Names and data terms have unique sort

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Names and data terms have unique sort

$$
\begin{array}{llll}
\underline{\propto} & \subseteq & \mathcal{S} \times \mathcal{S} & \text{Can be used to receive} \\
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Well-formedness criteria in writing agents, eg in

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Input rule: substitution conforms to $\prec$

Example: polyadic pi

$$\text{Names } \mathcal{N} = a, b, \ldots$$

$$\mathbf{T} = \mathcal{N} \uplus \mathcal{N}^*$$

$$\mathcal{S} = \{\mathbf{chan}, \mathbf{tup}\}$$

$$\textsc{sort}(a) = \mathbf{chan}$$

$$\textsc{sort}(\tilde{a}) = \mathbf{tup}$$

$$\mathbf{chan} \mathrel{\overline{\propto}} \mathbf{tup}$$

$$\mathbf{chan} \mathrel{\underline{\propto}} \mathbf{tup}$$

$$\mathbf{chan} \prec \mathbf{chan}$$

Higher-order psi

Higher-order

logic: quantify over predicates

functions: can have functions as parameters

process calculi: agents can be transmitted in communications

Higher-order pi

$\overline{a}P \,.\, Q$	Send the agent P along a and continue as Q
$a(X) \,.\, R$	Receive for the agent variable X along a and continue as R

New syntactic category!

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$$\overline{a}P \,.\, Q \mid a(X) \,.\, R \xrightarrow{\tau} Q \mid R[X := P]$$

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Eg

$$\overline{a}P\,.\,Q \mid a(X)\,.\,(R' \mid X)$$

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Agents constitute a nominal set!

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So we choose $\mathbf{T}$ = the set of agents!

$$\overline{M}\,P\,.\,Q \mid a(x)\,.\,R \xrightarrow{\tau} Q \mid R[x := P]$$

R receives the agent P, substituting x

$$\overline{M}P \,.\, Q \mid a(x) \,.\, R \xrightarrow{\tau} Q \mid R[x := P]$$

Problem: How can R get to 'execute' the newly received P?

Where can x occur in R?

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x can only occur in data terms, assertions and conditions :(

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- Psi **already** can accommodate agents as data values
- Psi **lacks** a notion of higher order variable that can stand for agents and be substituted.
- **Introducing** that is more complicated than you would think.
- There is a way that is both **easier and more general**!

Clauses

A **clause** is of the form $\boxed{M \Leftarrow P}$

Means that the data term M can be used as a **handle** for the agent P

The handle can be **invoked** in the new agent form $\boxed{\textbf{run}\ M}$

Intuition

Assume a clause $M \Leftarrow P$

Sending P along a is then $\overline{a}M \,.\, Q$

Receiving a process along a is $a(x) \,.\, (R \mid \mathbf{run}\ x)$

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$$\overline{a}M\,.\,Q \mid a(x)\,.\,(R \mid \mathbf{run}\ x) \xrightarrow{\tau} Q \mid R \mid \mathbf{run}\ M$$

Ho-pi vs HO-psi

pi

$$a(X)\,.\,(R \mid X)$$

New kind of variable
New kind of substitution

psi

$$a(x)\,.\,(R \mid \mathbf{run}\ x)$$

New syntactic construct
(Notion of clause)

Where do clauses live?

One possibility: introduce a new instance parameter as a set of clauses

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Again, there is an easier and more general way!

Hint: transitions always depend on environmental assertions.

$$\Psi \rhd P \xrightarrow{\alpha} P'$$

Entailed by assertions

A clause can be **entailed** by assertions, as in

$$\Psi \vdash M \Leftarrow P$$

Formally, clauses can be a **subset of the conditions**

Clearer terminology: extend $\vdash$ to also relate assertions with conditions **and clauses**

Semantics of **run**

$$\frac{\Psi \vdash M \Leftarrow P \qquad \Psi \;\rhd\; P \;\xrightarrow{\alpha}\; P'}{\Psi \;\rhd\; \mathbf{run}\, M \;\xrightarrow{\alpha}\; P'}$$

Universal clauses

In some applications it might be sufficient with **universal** clauses, entailed by **all** assertions

Example: universal clauses can express recursive definitions!

$$\forall\Psi. \quad \Psi \vdash M \Leftarrow a(x) \,.\, \overline{b}x \,.\, \mathbf{run}\ M$$

cf pi-calculus

$$A \Leftarrow a(x) \,.\, \overline{b}x \,.\, A$$

Local clauses

Clauses are entailed by assertions

Handles may **contain names** and be **scoped**

$$z \in \mathrm{n}(M), \quad \Psi \vdash M \Leftarrow a(x) \,.\, \overline{b}x \,.\, \mathbf{run}\; M$$

$$P \mid (\nu z)((\!|\Psi|\!) \mid Q)$$

Here Q but not P can use $\mathbf{run}\; M$

Mobile clauses

$$z \in \mathrm{n}(M), \quad \Psi \vdash M \Leftarrow a(x) \,.\, \overline{b}x \,.\, \mathbf{run}\ M$$

$$P \mid (\nu z)((\!|\Psi|\!) \mid Q)$$

The ability to use $\mathbf{run}\ M$ can be transmitted by Q by sending z

Or by sending M itself

In both cases extruding z

Multiple clauses

Nothing prevents the **same** handle to occur in **many** clauses

$$M \Leftarrow P_1$$
$$M \Leftarrow P_2$$
$$\vdots$$

The rule for **run** M is applicable to all

Nondeterminism (can represent +)

Requirement on clauses

In any clause $M \Leftarrow P$

we require $\mathrm{n}(P) \subseteq \mathrm{n}(M)$

ie, the support of the handle contains at least the support of the agent it represents.

Motivation: otherwise scope extension fails

$$\Psi \vdash M \Leftarrow \overline{b}N \, . \, \mathbf{0}$$

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$$\Psi \rhd \mathbf{run}\ M \xrightarrow{\overline{b}N} \cdots$$

$\Psi \rhd (\nu b)\mathbf{run}\ M \cdots$ has no transition

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Contradiction

$\Psi \rhd (\nu b)\mathbf{run}\ M \cdots$ has no transition

Example: stacks

Let assertions be sets of parametrised clauses

$$M(\lambda\tilde{x})N \Leftarrow P$$

$$M(\lambda\tilde{x})N \Leftarrow P \in \Psi \quad \Longrightarrow \quad \Psi \vdash M\langle N[\tilde{x} := \tilde{L}]\rangle \Leftarrow P[\tilde{x} := \tilde{L}]$$

$$\begin{array}{l} \textsc{Stack}(\lambda x)x \Leftarrow \underline{Push}(\lambda y)y\,.\,\mathbf{run}\ \textsc{Stack}\langle \mathrm{cons}(y,x)\rangle \\ \textsc{Stack}(\lambda x,y)\mathrm{cons}(x,y) \Leftarrow \overline{Pop}\ x\,.\,\mathbf{run}\ \textsc{Stack}\langle y\rangle \end{array}$$

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A stack factory

$$!\overline{a}\ \textsc{Stack}\,.\,\mathbf{0}$$

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But this uses the **same** push and pop channels for **all** stacks

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Alternative:

$$\textsc{Stack}(\lambda i,o,x)i,o,x \Leftarrow \underline{i}(\lambda y)y\,.\,\mathbf{run}\;\textsc{Stack}\langle i,o,\mathrm{cons}(y,x)\rangle$$
$$\textsc{Stack}(\lambda i,o,x,y)i,o,\mathrm{cons}(x,y) \Leftarrow \overline{o}\;x\,.\,\mathbf{run}\;\textsc{Stack}\langle i,o,y\rangle$$

$$\textsc{StackStart} \Leftarrow c(Push, Pop)\,.\,\mathbf{run}\;\textsc{Stack}\langle (Push, Pop, \mathrm{nil})\rangle$$

Canonical HO-calculi

The stack example can be generalised considerably

Thm *(paraphrased, see paper for details)* Any ordinary psi-calculus of nontrivial expressiveness can be systematically raised to a higher-order calculus by letting assertions be sets of parametrised clauses.

Representing replication

In HO-pi we can **encode replication**.
Can we do that in HO-psi?

Yes - at least in enough expressive psi-calculi

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$\Psi^{M \Leftarrow P}$ is a **characteristic assertion** for M and P if

1. $\Psi \vdash M \Leftarrow Q$ implies $\mathrm{n}(M) \subseteq \mathrm{n}(\Psi)$
2. $\Psi \otimes \Psi^{M \Leftarrow P} \vdash \xi \quad \text{iff} \quad (\xi = M \Leftarrow P \;\vee\; \Psi \vdash \xi)$
3. $\mathrm{n}(\Psi^{M \Leftarrow P}) = \mathrm{n}(M)$

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This means that the only effect of the characteristic assertion is to entail the clause $M \Leftarrow P$

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Thm characteristic assertions always exist in canonical higher-order calculi

Representing replication

Let a be fresh and $a \in \mathrm{n}(M)$

Let $\Psi^{M \Leftarrow P \mid \mathbf{run}\ M}$ be characteristic for M and $P \mid \mathbf{run}\ M$

Then $\boxed{!P \stackrel{\bullet}{\sim} (\nu a)(\mathbf{run}\ M \mid (\!|\Psi^{M \Leftarrow P \mid \mathbf{run}\ M}|\!))}$

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Then $\boxed{!P \dot{\sim} (\nu a)(\mathbf{run}\ M \mid (\!|\Psi^{M \Leftarrow P \,|\, \mathbf{run}\ M}|\!))}$

Idea:

$$\mathbf{run}\ M \mid (\!|\Psi^{M \Leftarrow P \,|\, \mathbf{run}\ M}|\!)$$

$$(P \mid \mathbf{run}\ M) \mid (\!|\Psi^{M \Leftarrow P \,|\, \mathbf{run}\ M}|\!)$$

$$(P \mid (P \mid \mathbf{run}\ M)) \mid (\!|\Psi^{M \Leftarrow P \,|\, \mathbf{run}\ M}|\!)$$

$$\vdots$$

By the semantic rules these all have the **same** transitions!

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Then $\boxed{!P \stackrel{\centerdot}{\sim} (\nu a)(\mathbf{run}\ M \mid (\!| \Psi^{M \Leftarrow P \mid \mathbf{run}\ M} |\!))}$

Idea:

$$\mathbf{run}\ M \mid (\!| \Psi^{M \Leftarrow P \mid \mathbf{run}\ M} |\!)$$
$$(P \mid \mathbf{run}\ M) \mid (\!| \Psi^{M \Leftarrow P \mid \mathbf{run}\ M} |\!)$$
$$(P \mid (P \mid \mathbf{run}\ M)) \mid (\!| \Psi^{M \Leftarrow P \mid \mathbf{run}\ M} |\!)$$
$$\vdots$$

By the semantic rules these all have the **same** transitions!

Why the (νa)?

Representing replication

Let a be fresh and $a \in \mathrm{n}(M)$
Let $\Psi^{M \Leftarrow P \,|\, \mathbf{run}\ M}$ be characteristic for M and $P \mid \mathbf{run}\ M$

Then $\boxed{!P \dot{\sim} (\nu a)(\mathbf{run}\ M \mid (\!|\Psi^{M \Leftarrow P \,|\, \mathbf{run}\ M}|\!))}$

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Anyway, what is this in HO-Psi?

$$\mathbf{run}\ M \mid (\!|\Psi^{M \Leftarrow P \mid \mathbf{run}\ M}|\!)$$

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Bisimulation

Formally, the **only** new aspect of higher-order psi is the inclusion of the **run** construct with accompanying rule!

No new syntactic categories, substitution etc

Just one more case when doing induction proofs

So perhaps we can just re-use the old definition!

R is a *bisimulation* if $R(\Psi, P, Q)$ implies

1. $R(\Psi, Q, P)$
2. $\forall \alpha . \mathrm{bn}(\alpha) \# Q, \Psi.$
   $\Psi \rhd P \xrightarrow{\alpha} P'$ implies $\Psi \rhd Q \xrightarrow{\alpha} Q'$ and $R(\Psi, P', Q')$
3. $\Psi \otimes \mathcal{F}(P) \simeq \Psi \otimes \mathcal{F}(Q)$
4. $\forall \Psi'.\ R(\Psi \otimes \Psi', P, Q)$

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Thm: all laws and congruence properties that used to hold still hold!

Isabelle proof in approx one day!

So we are done?

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In normal HO-calculi we would expect, as part of compositionality, that

$$P \stackrel{\bullet}{\sim} Q \quad \Rightarrow \quad \overline{a}P\,.\,R \stackrel{\bullet}{\sim} \overline{a}Q\,.\,R$$

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$$P \mathrel{\dot\sim} Q \quad \Rightarrow \quad \overline{a}P\,.\,R \mathrel{\dot\sim} \overline{a}Q\,.\,R$$

In HO-psi the counterpart could be

$$P \mathrel{\dot\sim} Q \quad \Rightarrow \quad \overline{a}M\,.\,R \mid \Psi^{M \Leftarrow P} \;\mathrel{\dot\sim}\; \overline{a}M\,.\,R \mid \Psi^{M \Leftarrow Q}$$

You believe this?

$$P \stackrel{\bullet}{\sim} Q \quad \Rightarrow \quad \overline{a}M\,.\,R \mid \Psi^{M \Leftarrow P} \stackrel{\bullet}{\sim} \overline{a}M\,.\,R \mid \Psi^{M \Leftarrow Q}$$

R is a *bisimulation* if $R(\Psi, P, Q)$ implies

1. $R(\Psi, Q, P)$
2. $\forall \alpha.\mathrm{bn}(\alpha) \# Q, \Psi.$
   $\Psi \triangleright P \xrightarrow{\alpha} P'$ implies $\Psi \triangleright Q \xrightarrow{\alpha} Q'$ and $R(\Psi, P', Q')$
3. $\Psi \otimes \mathcal{F}(P) \simeq \Psi \otimes \mathcal{F}(Q)$
4. $\forall \Psi'.\ R(\Psi \otimes \Psi', P, Q)$

$$P \stackrel{\bullet}{\sim} Q \quad \Rightarrow \quad \overline{a}M\,.\,R \mid \Psi^{M \Leftarrow P} \;\stackrel{\bullet}{\sim}\; \overline{a}M\,.\,R \mid \Psi^{M \Leftarrow Q}$$

$\mathcal{R}$ is a *bisimulation* if $\mathcal{R}(\Psi, P, Q)$ implies

1. $\mathcal{R}(\Psi, Q, P)$
2. $\forall \alpha.\mathrm{bn}(\alpha) \# Q, \Psi.$
   $\Psi \rhd P \xrightarrow{\alpha} P'$ implies $\Psi \rhd Q \xrightarrow{\alpha} Q'$ and $\mathcal{R}(\Psi, P', Q')$
3. $\Psi \otimes \mathcal{F}(P) \simeq \Psi \otimes \mathcal{F}(Q)$
4. $\forall \Psi'.\ \mathcal{R}(\Psi \otimes \Psi', P, Q)$

$$\overline{a}M\,.\,R \mid \Psi^{M \Leftarrow P} \;\stackrel{\bullet}{\not\sim}\; \overline{a}M\,.\,R \mid \Psi^{M \Leftarrow Q}$$

since the frames are different

The culprit

$$\Psi \otimes \mathcal{F}(P) \simeq \Psi \otimes \mathcal{F}(Q)$$

$$\Psi \otimes \mathcal{F}(P) \vdash \varphi \quad \text{implies} \quad \Psi \otimes \mathcal{F}(Q) \vdash \varphi$$

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Relax this condition so that for clauses it suffices with bisimilar ones!

$$\text{(a)} \;\; \forall \varphi \in \mathbf{C}. \quad \Psi \otimes \mathcal{F}(P) \vdash \varphi \;\Rightarrow\; \Psi \otimes \mathcal{F}(Q) \vdash \varphi$$

$$\text{(b)} \;\; \boxed{\begin{array}{l} \forall (M \Leftarrow P') \in \mathbf{Cl}. \quad \Psi \otimes \mathcal{F}(P) \vdash M \Leftarrow P' \;\Rightarrow \\ \quad \exists Q'.\, \Psi \otimes \mathcal{F}(Q) \vdash M \Leftarrow Q' \wedge (\mathbf{1}, P', Q') \in \mathcal{R} \end{array}}$$

A *strong HO-bisimulation* $\mathcal{R}$ is a ternary relation between assertions and pairs of agents such that $(\Psi, P, Q) \in \mathcal{R}$ implies all of

1. Static equivalence:

   (a) $\forall \varphi \in \mathbf{C}. \quad \Psi \otimes \mathcal{F}(P) \vdash \varphi \ \Rightarrow \ \Psi \otimes \mathcal{F}(Q) \vdash \varphi$

   (b) $\forall (M \Leftarrow P') \in \mathbf{Cl}. \quad \Psi \otimes \mathcal{F}(P) \vdash M \Leftarrow P' \ \Rightarrow$
   $\exists Q'.\ \Psi \otimes \mathcal{F}(Q) \vdash M \Leftarrow Q' \wedge (\mathbf{1}, P', Q') \in \mathcal{R}$

   The only new thing

2. Symmetry: $(\Psi, Q, P) \in \mathcal{R}$

3. Extension of arbitrary assertion: $\forall \Psi'.\ (\Psi \otimes \Psi', P, Q) \in \mathcal{R}$

4. Simulation: for all α, P' such that $\mathrm{bn}(\alpha) \# \Psi, Q$ there exists a Q' such that

$$\text{if } \Psi \vartriangleright P \xrightarrow{\alpha} P' \text{ then } \Psi \vartriangleright Q \xrightarrow{\alpha} Q' \wedge (\Psi, P', Q') \in \mathcal{R}$$

We define $\Psi \vartriangleright P \stackrel{\bullet}{\sim}^{\text{HO}} Q$ to mean that there exists a strong HO-bisimulation $\mathcal{R}$ such that $\Psi \vartriangleright P \mathrel{\mathcal{R}} Q$, and write $P \stackrel{\bullet}{\sim}^{\text{HO}} Q$ for $\mathbf{1} \vartriangleright P \stackrel{\bullet}{\sim}^{\text{HO}} Q$.

Thm

$$P \stackrel{\bullet}{\sim}^{\mathrm{HO}} Q \quad \Rightarrow \quad \Psi^{M \Leftarrow P} \stackrel{\bullet}{\sim}^{\mathrm{HO}} \Psi^{M \Leftarrow Q}$$

Thm

$$P \overset{\bullet}{\sim}{}^{\mathrm{HO}}\, Q \quad \Rightarrow \quad \Psi^{M \Leftarrow P} \ \overset{\bullet}{\sim}{}^{\mathrm{HO}}\ \Psi^{M \Leftarrow Q}$$

Thm: all laws and congruence properties that used to hold still holds!

The proof took forever to complete (several months)

$$\text{(a)}\ \forall \varphi \in \mathbf{C}.\quad \Psi \otimes \mathcal{F}(P) \vdash \varphi \ \Rightarrow\ \Psi \otimes \mathcal{F}(Q) \vdash \varphi$$

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Why not instead $\boxed{(\Psi, P', Q') \in \mathcal{R}}$

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Why not instead $\boxed{(\Psi, P', Q') \in \mathcal{R}}$

With this we fail to prove compositionality
(still unknown if it holds)

Conclusion

Psi-calculi is a family of process calculi

Accommodates a wide variety of data terms, functions and properties etc, based on nominal sets

Meta-theory proved once and for all in Isabelle

Outlook

- Extensions
- Combinations
- Applications
- Tool support

Thank you for your attention