Review questions on the π-calculus

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Questions come in two categories: basic (unmarked) and advanced (marked by $\bigstar$). A student may decide to go for only the basic questions. Solving them requires no great creative effort, yet they are a good test that you have mastered the fundamentals. The advanced questions require more thought and correlation between different sections and are suitable for students who suspect that they might actually use the calculus in their line of research.

Chapter 2

1. Write an agent that
   (a) reads something from port a and sends it twice along port b.
   (b) reads two ports and sends the first along the second.
   (c) reads three ports. If all three are the same it does nothing. If two are the same it sends one of the two along the third. If all are distinct it recurs and again reads three ports.
   (d) contains three agents P, Q, R such that P can communicate with both Q and R but there is no communication possible between Q and R.
2. Are the following two agents structurally congruent? Motivate your answer.
   (a) $x(y) . y(z) \equiv x(z) . y(y)$
   (b) $\texttt{if } x = x \texttt{ then } x(y) \equiv x(y)$
   (c) $x|y \equiv x . y + y . x$
   (d) $x(y) . x(z) \,|\, y(z) . z(y) \equiv y(y) . y(z) \,|\, x(z) . x(y)$
   (e) $x(y) . A(y, x) \equiv x(y) . \overline{y}x . x(z) . A(z, x)$ where $A(y, z) \stackrel{\mathrm{def}}{=} \overline{y}z . z(y) . A(y, z)$
3. Simplify the following agents as far as possible using structural congruence.
   (a) $((\boldsymbol{\nu} x)\overline{a}x + \mathbf{0}) \,|\, \mathbf{0} \,|\, (\boldsymbol{\nu} x)\overline{b}x$

(b) $(\boldsymbol{\nu} x\, y\, z)(a(x) . \overline{x}y \mid \overline{a}z \mid (\boldsymbol{\nu} z)\overline{a}z)$

(c) $(\boldsymbol{\nu} x)\overline{a}x . (\boldsymbol{\nu} x)A(a,x)$ where $A(a,x) \overset{\mathrm{def}}{=} (\boldsymbol{\nu} x)\overline{a}x . A(a,x)$ (hint: it simplifies a lot!)

★4 Which of the following are plausible SC laws, i.e., you think they could be added to the SC laws while keeping the same intuition. Discuss the merits of each proposed law. Suggest a law not mentioned here or in the text and discuss its suitability for SC.

(a) $P|P \equiv P$

(b) $(\boldsymbol{\nu} x)\overline{x}y . P \equiv \mathbf{0}$

(c) $(\boldsymbol{\nu} x)\overline{u}w . P \equiv \overline{u}w . (\boldsymbol{\nu} x)P$ if $x \neq u$ and $x \neq w$

(d) $\texttt{if}\ x = y\ \texttt{then}\ \overline{x}a . P \equiv \texttt{if}\ x = y\ \texttt{then}\ \overline{y}a . P$

(e) $(\boldsymbol{\nu} x)P \equiv P$ if x does not occur syntactically in P

(f) $(\boldsymbol{\nu} x)P \equiv P\{z/x\}$ for some name z not occurring in any agent under consideration.

5 Write down the following system in the pi-calculus. A printer is an agent that can receive something and then print it, signified by outputing it on port named "print". A controller contains exclusive access to the printer and can distribute this acceess to clients. There are two clients. One wishes to print A followed by B and one wishes to print C followed by D. Set up the system so that it is possible that the printer will print A B, and also possible it will print C D, and that any other combination is impossible. Explain how the system evolves as it executes.

★6 Change the system above so that there are three clients and two printers. Also make sure that once a client has finished using the printer its control is returned to the controller.

Chapter 3

7 Explain how booleans can be encoded in the pi-calculus without match and sum.

8 Give two reasons for including the Sum operator in the calculus.

9 For each of the following agents, give a sort to which it conforms or prove that it cannot conform to any sort.

(a) $x(y) . \overline{y}x$

(b) $x(y) . y(z) . z(x)$

(c) $\overline{x}\langle y\, z\rangle \,.\, \overline{x}\langle u\, v\rangle + \overline{y}u + \overline{u}\langle v\, v\rangle$

(d) $\overline{x}\langle y\, z\rangle \,.\, x(u\, v) + \overline{y}u + \overline{u}\langle v\, v\rangle$

★10 Give a few nontrivial examples of agents with restriction where it is possible to infer the sort of the restricted name. Do you think, in this sort system, that it is always possible to infer a sort if one exists?

11 Let $A(x) \stackrel{\text{def}}{=} x(y)\,.\,(\overline{y}x\,.\,A(y) + \overline{x}y\,.\,A(x))$. Encode $A(x)$ using replication.

★12 Encode $!a\,.\,b\,.\,c$ using no replication and only guarded recursion. (With unguarded recursion the encoding is trivial as $A \stackrel{\text{def}}{=} a\,.\,b\,.\,c \mid A$. Guarded recursion means that a recurrence of A in its definition must lie under a prefix operator.)

13 Which of the following agents are asynchronous?

(a) $x(y)\,.\,(\overline{a}x \mid y(z))$

(b) $x(y)\,.\,(\overline{a}x + y(z))$

(c) $x(y)\,.\,(\overline{a}x\,.\,y(z))$

(d) $\overline{a}x\,.\,(\mathbf{0} + \mathbf{0})$

(e) $\overline{a}x \mid \overline{b}x$

(f) $\overline{a}x\,.\,\overline{b}x + \overline{b}x\,.\,\overline{a}x$

★14 Consider the scheme in section 2.3 to encode polyadic interactions as a sequence of monadic interactions. It is not asynchronous, since outputs are followed by non-nil agents. Devise a similar scheme in the asynchrous calculus.

15 Encode $!x\,.\,y\,.\,\mathbf{0}$ into the higher-order calculus without using recursion or replication. Develop a few transitions from the encoding.

16 Encode the following higher-order agents into the first order calculus. Develop a few transitions from the encodings.

(a) $\overline{a}\langle \overline{b}c\rangle\,.\,b(x)\,.\,\overline{x} \mid a(X)\,.\,X$

(b) $a(X)\,.\,(X \mid \overline{a}\langle X\rangle)$

★17 Consider the system in question 6, containing printers, clients and users. Describe a similar system in the higher order calculus where the printers move between controller and clients. Discuss the relative merits of using higher order and first order for this example. Use the translation from higher order to first order on your higher order model. Do you get precisely your first order model or what is the difference?

★18 A commonly occurring variant of the π-calculus is asynchronous, polyadic and higher order. Think of one situation where these choices are appropriate. Then determine if it would be good to have Sum in that situation.

Chapter 4

19 What are all the possible transitions (not only the τ transitions) from the following agents?

(a) $x(y).y(z) \mid \overline{x}a$

(b) $(x(y)\,.\,\overline{y}u) \mid (\boldsymbol{\nu} u)\overline{x}u$

(c) $(a(x)\,.\,x(a) + a(x)\,.\,a(x)) \mid \overline{a}u \mid (\boldsymbol{\nu} u)\overline{a}u$

(d) $!(\overline{a} \mid a\,.\,\overline{b})$ (recall that ! has a structural congruence unfolding)

(e) $A(a, a)$ where $A(x, y) \stackrel{\mathrm{def}}{=} \overline{x} \mid y$

(f) $(\texttt{if } x = y \texttt{ then } \overline{a}u) \mid a(x)$

(g) $a(x)\,.\,(\texttt{if } x = y \texttt{ then } \overline{a}u \mid a(x)) \mid \overline{a}y$

(h) $a(x)\,.\,(\texttt{if } x = y \texttt{ then } \overline{a}u \mid a(x)) \mid (\boldsymbol{\nu} y)\overline{a}y$

20 An agent P has the only transition $P \xrightarrow{\overline{a}\boldsymbol{\nu} u} P$ (note that it leads back to P). Thus P must be some kind of recursive definition. A student once suggested that it be $P = (\boldsymbol{\nu} u)A(a, u)$ where $A(a, u) \stackrel{\mathrm{def}}{=} \overline{a}u\,.\,A(a, u)$. Demonstrate that this is wrong and give a correct definition of P.

★21 If only guarded recursion and no replication is used, argue that it is formally decidable if an agent has a transition. (If unguarded recursion is admitted this turns out to be formally undecidable, something the really ambitious student might prove!) Hint: argue by induction on the length of an agent.

★22 Prove that if $P \xrightarrow{\alpha} P'$ then $\texttt{fn}(P') \subseteq \texttt{fn}(P) \cup \texttt{bn}(\alpha)$. Hint: use induction on the length of the inference of $P \xrightarrow{\alpha} P'$. For the inductive step you need to consider one case for each rule. Why can you not use induction on the length of P instead?

Chapter 5

23 Compute a τ transition from the following agents using each of 1) the early sematics 2) the late sematics without structural congruence 3) the reduction semantics.

(a) $x(y).y(z)|\overline{x}a$

(b) $(x(y)\,.\,\overline{y}u)\,|\,(\boldsymbol{\nu}u)\overline{x}u$

24 Compute all the symbolic transitions from

(a) $x(y).y(z)|\overline{u}a$

(b) $(\texttt{if}\ x \neq y\ \texttt{then}\ \overline{x}y)\,|\,z(u)\,.\,\overline{u}w$

(c) $\texttt{if}\ x \neq y\ \texttt{then}\ (\overline{x}y\,|\,z(u)\,.\,\overline{u}w)$

25 Consider the following different kinds of semantics: 1) late with structural congruence 2) late without structural congruence 3) reductions 4) symbolic. For each of them, explain one situation where that semantics is suitable.

26 The symbolic semantics in 5.2 is a kind of late semantics. Define an early symbolic semantics. You only need to state the rules which are different from the ones in 5.2.

★27 Give a formal proof of the statement at the end of 5.2. As in question 22 it will use induction over length of inference, with one case for each rule. You need not write down the cases for all rules in detail, just pick a few that are representative.

★28 Give a symbolic semantics for the version of the calculus in section 5.6 (with abstractions and concretions).

Chapter 6

29 Give a strong bisimulation that relates

(a) $\overline{a}x\,.\,(x|\overline{u})$ and $\overline{a}x\,.\,(x\,.\,\overline{u} + \overline{u}\,.\,x)$

(b) $a(x)\,.\,(x|\overline{u})$ and $a(x)\,.\,(x\,.\,\overline{u} + \overline{u}\,.\,x + \texttt{if}\ x = u\ \texttt{then}\ \tau)$

(c) $A(u)$ and $B(u,u)$ where $A(x) \stackrel{\mathrm{def}}{=} x\,.\,A(x)$ and $B(x,y) \stackrel{\mathrm{def}}{=} x\,.\,B(x,y) + y\,.\,y\,.\,B(x,y)$

The bisimulations should be as small as possible, i.e., contain as few pairs as possible.

30 An anonymous researcher once suggested bisimulation to be defined as in Definition 1 on page 39 but with the phrase “where $\texttt{bn}(\alpha)$ is fresh” omitted. Demonstrate that this is not a good definition by giving two agents which then would be non bisimilar, even though the agents have the same operational behaviour.

31 Another researcher redefined bisimulation as in Definition 1 but without clause (i), in other words, clause (ii) applies to all actions. The resulting bisimilarity then relates more agents than the original definition. Give an example of two agents which are not operationally the same but related by the new definition.

32 Which of the following pairs are bisimilar?

(a) $a(x)\,.\,u$ and $a(x)\,.\,u + a(x)\,.\,\texttt{if}\ x = y\ \texttt{then}\ u$

(b) $a(x)\,.\,u$ and $a(x)\,.\,u + \texttt{if}\ x = y\ \texttt{then}\ a(x)\,.\,u$

33 Which of the following pairs are congruent?

(a) $a(x)\,.\,u$ and $a(x)\,.\,u + a(x)\,.\,\texttt{if}\ x = y\ \texttt{then}\ u$

(b) $a(x)\,.\,u$ and $a(x)\,.\,u + \texttt{if}\ x = y\ \texttt{then}\ a(x)\,.\,u$

34 Find an agent that is bisimilar but not congruent to $a(x)\,.\,x \,|\, \bar{b}y\,.\,\overline{y}$ and which does not contain a Parallell operator.

35 Find an agent that is congruent to $a(x)\,.\,x \,|\, \bar{b}y\,.\,\overline{y}$ and which does not contain a Parallel operator.

36 The last sentence on page 42 says that "following the τ-transition from both sides ..." Write out these τ-transitions and argue that the conclusion $P\sigma \dot{\sim} Q\sigma$ follows.

37 Prove formally that $P|Q \dot{\sim} Q|P$. (This is easy since the semantics contains the rule STRUCT.)

★38 A communication protocol is given as a collection of communicating processes. All names in the protocol are restricted except two: an input port and an output port. The intention of the designer is that the service of the protocol is that of a buffer between these ports. A graduate student managed to prove that the protocol is bisimilar but not congruent to such a buffer. Do you think that the protocol then is "correct", in an intuitive sense of the word? Motivate your answer carefully.

★39 True or False? Motivate carefully! "If bisimilarity is decidable for some subset of the calculus, then also congruence is decidable for the same subset."

Chapter 7

40 Give an example of two agents that do not contain the Match operator such that they are early bisimilar but not late bisimilar. Hint: they contain a Parallel operator.

★41 Prove the claim that a relation is an early bisimilation if and only if it is an early bisimulation with the late semantics (def 4 on page 44).

42 Prove that $(\boldsymbol{\nu}x)\overline{a}x\,.\,\overline{b}x$ and $(\boldsymbol{\nu}x\,y)\overline{a}x\,.\,\overline{b}y$ are not barbed congruent by demonstrating a context which makes them not barbed equivalent.

43 Prove that an open bisimulation is also a late bisimulation in the subcalculus without Restriction. Does the converse hold?

44 Prove that in the absence of Restriction, open bisimilarity as defined in Def 8 is the same as open bisimilarity as defined in Def 7.

★45 Prove that an open bisimulation is also a late bisimulation in the full calculus (that is, also including Restriction).

★46 Prove that a dynamic bisimulation is also an open bisimulation (you need only consider the subcalculus without Restriction).

47 Prove from the definitions that a strong late bisimulation is also a weak late bisimulation.

★48 Write out the definition of early weak bisimilarity and prove that early weak bisimilarity includes late weak bisimilarity.

Chapter 8

49 Prove that the laws R1–R3 are sound.

50 Use the axioms in Table 6 to simplify the agent $(\boldsymbol{\nu}x)(\overline{a}x + (\boldsymbol{\nu}y)\overline{a}y)$ as far as possible.

51 Write out a full proof of Prop 7 (about head normal forms).

52 The proof of Prop 8 given in the text gives one inductive step for α being an input action. Give another case where α is a bound output action. Hint: it is easier!

★53 The expansion law in Table 7 apparently does not take bound output actions into account. Suppose $\alpha_i = (\boldsymbol{\nu}u)\overline{a}u$ and $\beta_j = a(x)$, what would you expect R_{ij} to be? How can it be that the expansion law in Table 7 is complete even though it does not mention bound outputs?

54 Prove that the law GM1 is sound.

★55 Which of the laws GM2-GM6 also hold for strong late bisimilarity? For those laws that do hold for late bisimilarity, give a direct deriviation using the axioms in Table 6.

56 Derive the law GM6* from the axioms in Table 8.

★57 Write out the proof of Prop 9.

★58 Read the proof sketch of Prop 10 and then explain in your own words why the head normal form of Prop 7 is not sufficient in this proof.

★59 Some of the laws of structural congruence are redundant in the sense that if they were removed then all instances of them could still be inferred from the remaining axioms for equivalence. Determine which of the laws of structural congruence can be removed in this way.

Chapter 9

60 Prove that the law EARLY is sound.

★61 A student claims to have found a complete axiomatisation of the finite fragment of the higher-order calculus. Explain why this is unlikely.

★62 Consider the subcalculus without Mismatch. One might easily believe that complete axiomatisations for bisimilarity and congruence can be obtained from Table 6 and Table 8 by simply striking all axioms where Mismatch occurs. Is this true? Motivate carefully!

★63 Similarly, consider the subcalculus without Sum. Do you obtain complete axiomatisations by simply striking all axioms where Sum occurs?