Modeling and Parameter Estimation of the Diffusion Equation

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Abstract

In many applications such as heat diffusion and flow problems, it is of interest to describe the process behavior inside a particular medium. An example can be the strive for estimating certain parameters related to the material. These processes are often modeled by a partial differential equation. Certain methods for identifying unknown material constants require the model to be of finite order. This thesis describes how the diffusion process can be approximated with finite order model, and how the accuracy of an estimated model depends on the model order. In particular, a detailed analysis is carried out for the case when the approximate model accounts for solving the diffusion by a difference method.
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Contents

1 Introduction 1

2 A short review of system identification 3

3 The heat diffusion model 5
  3.1 Statement of the problem ................................. 5
  3.2 The system dynamics ................................ 6
  3.3 Solution close to the boundary $x = 0$, $t = 0$ .......... 10

4 Approximate models 17
  4.1 Difference approximation .................................. 17
  4.2 Thermal networks ......................................... 23
  4.3 Alternative approaches .................................... 26

5 Approaches for parameter estimation 27
  5.1 Method 1: A direct approach in the time domain .......... 27
  5.2 Method 2: A direct approach in the frequency domain .... 28
  5.3 Methods 3-5: Indirect approaches ....................... 28

6 Numerical evaluation 31
  6.1 Affecting the bias in the parameter estimates .......... 31
  6.2 Choice of input signals .................................. 33
  6.3 Computational complexity ................................ 34
  6.4 Sensitivity to parameter variations ...................... 36
  6.5 Existence of local minima ................................ 36
  6.6 Model validation .......................................... 43
  6.7 Results of numerical evaluation .......................... 46
      6.7.1 Identification from noise-free data .............. 48
      6.7.2 Identification from noise-corrupted data .......... 55

7 Bias contribution for general approximate models 59

8 Approximation errors of auxiliary parameters 65

9 A theoretical analysis of the parameter bias 73
  9.1 Estimating the bias in the frequency domain .......... 73
  9.2 Estimating the bias in the time domain ............... 75

10 Numerical illustrations of the bias contribution 77

11 Conclusions 85

A Proofs of results in Chapter 3 89
  A.1 Proof of Lemma 3.1 ...................................... 89
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>Proofs of results in Chapter 4</td>
<td>93</td>
</tr>
<tr>
<td>B.1</td>
<td>Proof of Lemma 4.1</td>
<td>93</td>
</tr>
<tr>
<td>B.2</td>
<td>Proof of Lemma 4.2</td>
<td>100</td>
</tr>
<tr>
<td>C</td>
<td>Proofs of results in Chapter 8</td>
<td>111</td>
</tr>
<tr>
<td>C.1</td>
<td>Proof of Lemma 8.1</td>
<td>111</td>
</tr>
<tr>
<td>C.2</td>
<td>Proof of Lemma 8.2</td>
<td>112</td>
</tr>
<tr>
<td>C.3</td>
<td>Proof of Lemma 8.3</td>
<td>116</td>
</tr>
<tr>
<td>C.4</td>
<td>Proof of Lemma 8.4</td>
<td>117</td>
</tr>
<tr>
<td>C.5</td>
<td>Proof of Lemma 8.5</td>
<td>123</td>
</tr>
<tr>
<td>C.6</td>
<td>Proof of Lemma 8.6</td>
<td>128</td>
</tr>
<tr>
<td>C.7</td>
<td>Proof of Lemma 8.7</td>
<td>130</td>
</tr>
<tr>
<td>C.8</td>
<td>Proof of Lemma 8.8</td>
<td>131</td>
</tr>
<tr>
<td>C.9</td>
<td>Proof of Lemma 8.9</td>
<td>132</td>
</tr>
<tr>
<td>C.10</td>
<td>Proof of Lemma 8.10</td>
<td>133</td>
</tr>
<tr>
<td>C.11</td>
<td>Proof of Lemma 8.11</td>
<td>141</td>
</tr>
<tr>
<td>C.12</td>
<td>Proof of Lemma 8.12</td>
<td>145</td>
</tr>
<tr>
<td>C.13</td>
<td>Proof of Lemma 8.13</td>
<td>146</td>
</tr>
<tr>
<td>C.14</td>
<td>Proof of Lemma 8.14</td>
<td>147</td>
</tr>
<tr>
<td>C.15</td>
<td>Proof of Lemma 8.15</td>
<td>149</td>
</tr>
<tr>
<td>D</td>
<td>Proofs of results in Chapter 9</td>
<td>151</td>
</tr>
<tr>
<td>D.1</td>
<td>Proof of Theorem 9.1</td>
<td>151</td>
</tr>
<tr>
<td>D.2</td>
<td>Proof of Lemma 9.1</td>
<td>159</td>
</tr>
<tr>
<td>D.3</td>
<td>Proof of Theorem 9.2</td>
<td>186</td>
</tr>
</tbody>
</table>
Symbols

\( C \) thermal capacitance
\( C(kh) \) proportional constant
\( c \) specific heat capacity
\( d \) thickness of wall element
\( e(t) \) white noise
\( G(q, \theta), G(i\omega) \) transfer function of system
\( G_0(q) \) true system
\( G^n(q, \theta), G^n(i\omega) \) model of \( G(q, \theta) \)
\( G^n(i\omega), \Delta G(i\omega) \) approximation error \( G(i\omega) - G^n(i\omega) \)
\( g(t) \) weighting function
\( \tilde{g}^n(t) \) estimate of weighting function
\( \hat{g}^n(t) \) approximation error \( g(t) - \tilde{g}^n(t) \)
\( H(i\omega) \) derivative of \( G(i\omega) \) w. r. t. \( \theta \)
\( H(q, \theta) \) noise modeling filter
\( H_0(q) \) true noise model
\( h \) sampling interval
\( h(t) \) derivative of weighting function \( g(t) \) w. r. t. \( \theta \)
\( k \) a specific layer in the wall element
\( L(q) \) prefilter
\( N \) number of data samples
\( n \) model order
\( O(x) \) \( O(x)/x \) is bounded when \( x \to 0 \)
\( p \) differentiation operator
\( p_m \) pole number \( m \) of transfer function
\( Q_i \) weighting function for method \( i \)
\( q \) shift operator \( qu(kh) = u(kh + h) \)
\( q_i(t) \) (interior) heat flux
\( R \) thermal resistance
\( R(n) \) relative error
\( r_u(t) \) covariance function of input signal
\( T(x,t) \) temperature at position \( x \) and time \( t \)
\( T(x,s) \) Laplace transform of \( T(x,t) \)
\( T_e(t) \) exterior temperature
\( T_i(t) \) interior temperature
\( u(t) \) input signal, continuous time
\( u(kh) \) input signal, discrete time
\( V(\theta) \) loss function
\( \nu(t) \) noise, continuous time
\( x \) coordinate across the wall
\( y(t) \) system output, continuous time
\( y(kh) \) system output, discrete time
\( y^n(t) \) model output, continuous time
Symbols cont’d

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{y}(kh</td>
<td>kh-h;\theta)$</td>
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<tr>
<td>$z_m$</td>
<td>zero number $m$ of transfer function</td>
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<tr>
<td>$\alpha_k$</td>
<td>parameter (normed pole) of the systems $G_1(s)$, $G_2(s)$</td>
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<tr>
<td>$\tilde{\alpha}_k$</td>
<td>approximation of $\alpha_k$</td>
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<td>$\tilde{\alpha}_k$</td>
<td>approximation error $\tilde{\alpha}_k - \alpha_k$</td>
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<td>$\beta_{1,k}$</td>
<td>parameter in $G_1(s)$</td>
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<tr>
<td>$\tilde{\beta}_{1,k}$</td>
<td>approximation of $\beta_{1,k}$</td>
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<tr>
<td>$\beta_{2,k}$</td>
<td>parameter in $G_2(s)$</td>
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<tr>
<td>$\tilde{\beta}_{2,k}$</td>
<td>approximation of $\beta_{2,k}$</td>
</tr>
<tr>
<td>$\tilde{\beta}_{2,k}$</td>
<td>approximation error $\tilde{\beta}<em>{2,k} - \beta</em>{2,k}$</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>thickness of each wall layer</td>
</tr>
<tr>
<td>$\varepsilon(t)$</td>
<td>error caused by noise and approximation error</td>
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<td>$\varepsilon(kh)$</td>
<td>prediction error</td>
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<td>$\varepsilon(t,\tau_0,\tau)$</td>
<td>error function</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>diffusivity</td>
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<tr>
<td>$\theta$</td>
<td>parameter vector</td>
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<tr>
<td>$\tilde{\theta}$</td>
<td>estimate of the parameter vector $\theta$</td>
</tr>
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<td>$\hat{\theta}$</td>
<td>bias of parameter vector</td>
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<tr>
<td>$\theta_0$</td>
<td>true value of parameter vector</td>
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<tr>
<td>$\kappa$</td>
<td>thermal conductivity</td>
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<tr>
<td>$\lambda^2$</td>
<td>variance of white noise</td>
</tr>
<tr>
<td>$\rho$</td>
<td>density</td>
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<tr>
<td>$\sigma^2_d$</td>
<td>variance of input signal</td>
</tr>
<tr>
<td>$\tau$</td>
<td>time constant</td>
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<tr>
<td>$\Phi$</td>
<td>spectrum</td>
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<tr>
<td>$\omega$</td>
<td>angular frequency</td>
</tr>
<tr>
<td>$\omega_k$</td>
<td>element number $k$ in a vector of frequencies</td>
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<tr>
<td>$\omega_0$</td>
<td>single frequency</td>
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</tbody>
</table>
Acronyms

AIC  Akaike’s information criterion
AR   autoregressive
ARX  autoregressive with exogenous variables
BIBO bounded input bounded output
cond condition number
cov covariance matrix
dim  dimension
DSP  digital signal processor
FFT  fast fourier transform
OEM  output error model
PDE  partial differential equation
PEM  prediction error model
SISO single input single output
w. r. t. with respect to

Notational conventions

\(\Delta\) defined as
\(\sim\) distributed as
\(\rightarrow\) converges to
\(\forall\) for all
\((\cdot)^T\) transpose of a vector or matrix
\(\arg\min_x f(x)\) the value of \(x\) that minimizes \(f(x)\)
\(E[\cdot]\) expectation of
\(V^{\text{\text{HH}}}\) Hessian (matrix of second order derivatives) of the loss function \(V\)
1 Introduction

In environmental systems, diffusion processes often play a central role. Applications such as diffusion of heat and of pollution, \cite{1-3} are of practical importance. An example is heat transfer through a wall element, which then can be viewed as a solid bounded by two parallel planes. Another example is the transfer of pollution in still waters, or the mixing of two gases in a closed container. All these examples can be referred to as \textit{diffusion or conduction}, which means that the heat or the specific blending material passes through the substance of the body, which itself is still.

Two other methods by which the transfer can be performed are \textit{convection} and \textit{radiation}. Convection takes place when e.g. heat or pollution is transported by motion of the body itself, which would happen in for example a moving water stream. The second method refers to when for example heat transfers between two parts of a body by electromagnetic radiation. All of these three modes of transfer are discussed more in the classical book \cite{4} or other related literature. Only diffusion will be treated in this thesis, and implicitly, the material considered are solids.

The diffusion is generally modeled by a partial differential equation (PDE), which may contain unknown parameters. In order to model such a system, these parameters need to be determined. System identification is a statistical technique for modelling dynamic systems \cite{5,6}. It is then assumed that input-output data are available from an experiment. A model can be fitted to these data using a variety of methods. The dominating effort in system identification has dealt with \textit{black-box} models, in which no a priori information about the system dynamics is used. It is also possible to use \textit{grey-box} models, where physical insight is used to find an appropriately parameterized model \cite{7}. Such models are typically strongly application dependent. So far, it is rather exceptional to fit distributed models by system identification techniques. Some examples in the field include \cite{8-11}. By ‘distributed models’ is meant models with a spatial distribution, mostly described by means of partial differential equations. In contrast, by ‘lumped models’ is meant models where the spatial dimension does not appear.

In physics, the system identification problem is often referred to as an \textit{inverse problem}, that is, given input and output data, determine the underlying system. The \textit{direct problem} is correspondingly in physics defined as the problem of determining the output when the model and the input data are given.

In the field of system identification, finite order models are often used \cite{5}. Since PDEs are of infinite order, it can be necessary to approximate the original model by a finite order model in order to enable usage of some standard system identification technique. Even if the approximate model shows a good fit to the original model, it is inevitable to introduce some error due to the finite order. To be able to verify if the parameter identification was a success or not, it is crucial to know the impact of the reduced model order. The main purpose of
this thesis is to describe what influence the selection of model order has on the estimated parameters. The effect of so-called variance error due to noisy and finite data sequences, is separated from the bias error, and not treated here. The contribution is thus to evaluate how the bias error of the estimated parameters decrease with the model order.

As a ‘case study’, the identification of a simple diffusion process is considered, where the model output of the PDE is approximated using a finite difference scheme.

A numerical evaluation of different methods for parameter estimation has been published in [12, 13].

**Thesis outline**
The work is organized as follows. Chapter 2 summarizes some results in the field of system identification. In Chapter 3, the heat diffusion model is described. Transfer functions of infinite orders are developed, as well as corresponding weighting functions.

The heat diffusion can be modeled in various ways. In this work, one particular approach is considered, namely the difference approximation model. This is discussed in Section 4.1. The difference approximation model offers a conceptually simple way to approximate an infinite order system by an arbitrary complex model. Expressions of the approximated transfer functions and weighting functions, corresponding to the ones in Chapter 3, are given. An alternative approximation method is presented briefly in Section 4.2.

Chapter 5 introduces some approaches for parameter estimation including methods both in the time domain as well as in the frequency domain. In Chapter 6, a numerical evaluation of the methods presented in Chapter 5 is performed. Also, a discussion is carried out how to improve the results by various techniques, for example prefiltering. Chapter 7 discusses the modeling of parameter bias caused by undermodeling of the system. This chapter is general, and applies to other problems than the heat diffusion model. Chapter 8 is of auxiliary character, containing some results that will be useful in the following chapter. In Chapter 9, the parameter bias of the material constants is studied theoretically in the frequency domain and time domain, respectively. In Chapter 10, numerical illustrations of the findings in Chapter 9 are presented. Finally, conclusions are given in Chapter 11, as well as some ideas for future research.
2 A short review of system identification

In what follows, a brief description of model fitting by system identification techniques is given. Consider a dynamic system with input $u(t)$ and output $y(t)$. Assume that discrete time data are available at times $t = h, 2h, \ldots, Nh$, where $h$ is the sampling interval. Further, it is assumed that a linear model of the form

$$y(kh) = G(q; \theta)u(kh) + H(q; \theta)e(kh)$$  \hspace{1cm} (2.1)

is to be fitted to the data. Here $q$ denotes the shift operator $(qu(kh) = u(kh + h)$, etc), and $\theta$ is a vector of the unknown parameters to be determined. In (2.1), $G(q; \theta)$ and $H(q; \theta)$ are linear filters, and $e(kh)$ denotes a white noise sequence, that is, a sequence of uncorrelated random variables. The purpose of the term $H(q; \theta)e(kh)$ is to account for disturbances such as sensor noise and modeling errors. In black-box models the filters $G(q; \theta)$ and $H(q; \theta)$ are rational, i.e. ratios of polynomials in $q$. The polynomial coefficients are taken as elements of $\theta$ in such a case. The filters $G(q; \theta)$ and $H(q; \theta)$ may have some common parameters.

For the output error method (OEM) the noise part of the model is disregarded and $\theta$ is determined by minimizing the squared sum of output errors. In case a prefiltering of the data is included and a multi-variable systems is allowed, i.e. systems where $u$ and $y$ are vectors, this generalizes to

$$\hat{\theta} = \arg \min_{\theta} \sum_{k=1}^{N} || L(q) \{ y(kh) - G(q; \theta)u(kh) \} ||^2$$  \hspace{1cm} (2.2)

where $\theta$ is an estimate of the parameter vector $\theta$, and where $L(q)$ is a prefilter. Still a further extension can be to minimize another norm of the output error.

For the prediction error method (PEM) the noise properties of the model are taken into account, and the averaged one step prediction error is minimized. The estimate reads

$$\hat{\theta} = \arg \min_{\theta} \sum_{k=1}^{N} \| y(kh) - \hat{y}(kh|kh-h;\theta) \|^2$$  \hspace{1cm} (2.3)

where $\hat{y}(kh|kh-h;\theta)$ is the one-step ahead optimal prediction of the output signal. It turns out that the prediction error for the model (2.1) can be written as

$$y(kh) - \hat{y}(kh|kh-h;\theta) = H^{-1}(q; \theta) \{ y(kh) - G(q; \theta)u(kh) \}$$  \hspace{1cm} (2.4)

There is a rich experience in the literature about various techniques for model validation, which concerns assessment whether or not a fitted model reasonably well describes the underlying dynamics. Traditional techniques include various statistical tests, formed from the residuals

$$\varepsilon(kh) = H^{-1}(q; \theta)[y(kh) - G(q; \theta)u(kh)]$$  \hspace{1cm} (2.5)
which for a perfect model should behave as a white noise sequence. As a complement some graphically based approaches can be used for assessing the validity of the model.

In the case when the model parameterization is rich enough to cover the true system dynamics, the parameter estimate \( \hat{\theta} \) is, under weak assumptions, consistent, i.e. it converges to the true value, say \( \theta_0 \), as the number of data samples, \( N \), tends to infinity. For large \( N \) it can in such cases be said that there is no significant systematic error (i.e. no bias) in \( \hat{\theta} \). However, there is a stochastic error due to the disturbances. Assuming the true dynamics fulfill

\[
y(kh) = G_0(q)u(kh) + H_0(q)e(kh)
\]

and that the estimated parameters \( \hat{\theta} \) are close to the true ones (hence \( G(q; \hat{\theta}) \approx G_0(q), H(q; \hat{\theta}) \approx H_0(q) \)). Then the covariance matrix of \( \hat{\theta} \) can for large \( N \) be approximated as

\[
\text{cov}(\hat{\theta}) = \frac{\lambda^2}{N} \left[ E \left( \frac{dz(kh)}{d\hat{\theta}} \right)^T \left( \frac{dz(kh)}{d\hat{\theta}} \right) \right]^{-1}
\]

(2.6)

where \( \lambda^2 = E e^2(kh) \) [5]. The standard deviation of \( \hat{\theta} \) is thus of magnitude \( \lambda/\sqrt{N} \). Note that the covariance expression (2.7) in general can be estimated from the data.

In the last decade research has been more focused to system identification as a form of model approximation or model reduction. If the true system dynamics is too complex to be covered by the chosen model parameterization, there will be a systematic error, a bias, in \( \hat{\theta} \) even for large data sets. For such cases the prefiltering of the data can have a significant impact on for what frequency range the model fit is good, i.e. for what frequencies \( \omega \), \( G(e^{j\omega k}; \hat{\theta}) \approx G_0(e^{j\omega k}) \). In general terms, it is often desirable to choose the model order so that the bias term is not (much) larger than the standard deviation term.
3 The heat diffusion model

There are several applications where diffusion models can be used, as described in Chapter 1. Heat propagation in different media, e.g. solid materials such as building elements, liquid media such as water, and gas, is of both practical and theoretical interest. It is clear that the heat propagation differs for different media and applications. In this thesis, the study is limited to the case of heat diffusion in an homogeneous wall. The main results will be valid for other applications. However, the procedure of taking the Laplace transform of other models of heat propagation and further modeling, might not be obvious.

3.1 Statement of the problem

As an illustrative example, the heat flow through an homogeneous wall of thickness $d$ is examined in some detail. Let $x$ be the coordinate across the wall, $T_e$ the temperature on the exterior side, $T_i$ the temperature on the interior side and $q_i$ the supplied heat on the interior side, cf Figure 3.1.

Let $T = T(x, t)$ denote the temperature at position $x$ and at time $t$. Then the heat flow and the temperature are described by

$$\frac{\partial T(x, t)}{\partial t} = \kappa \frac{\partial^2 T(x, t)}{\partial x^2}$$  \hspace{0.5cm} (3.1)

$$q(x, t) = -\kappa \frac{\partial T(x, t)}{\partial x}$$  \hspace{0.5cm} (3.2)

where

$$\zeta = \frac{\kappa}{\rho c}$$  \hspace{0.5cm} (3.3)

Figure 3.1: Heat diffusion through an homogeneous wall.
The quantity $\kappa$ is the thermal conductivity, $\rho$ is the density, and $c$ is the specific heat capacity. Finally, $\zeta$ denotes the diffusivity\footnote{In some literature, $\zeta$ is called the thermometric conductivity of the substance.}. The heat equation (3.1) is derived from the rule of preservation of energy. A derivation can be found in e. g. [14].

Remark 1. The thermal conductivity $\kappa$ is in reality not a constant for an homogeneous substance, but depends on the temperature. However, when the range of temperature is limited, it can be assumed that $\kappa$ does not vary with the temperature. The neglect of the changes in $\kappa$ is introduced in order to formulate a simplified model.

Remark 2. The analysis in this thesis is restricted to isotropic media, which means that the structure and properties of the substance in the neighbourhood of any point are the same relative to all directions through the point. Hence, the heat flux vector is in the direction of falling temperature, and it is a valid method to simplify the problem to one dimension.

Consider the wall in Figure 3.1 as a dynamic system with $T_e$ and $q_i$ as input variables and $T_i$ as output. These parameters are related to (3.1), (3.2) as

$$T_e(t) = T(d, t) \quad (3.4)$$
$$T_i(t) = T(0, t) \quad (3.5)$$
$$q_i(t) = q(0, t) \quad (3.6)$$

The problem to be considered is thus the following.

**Given:** Discrete-time data $T_e(h), q_i(h), T_i(h), T_e(2h), q_i(2h), \ldots, T_i(Nh)$.

**Find:** Parameter estimates of $\zeta$ and $\kappa$ and evaluate the estimates in terms of bias.

The above problem is apparently an example of grey-box modeling, where physical interpretation has been included in the choice of model.

Needless to say, the geometry in the chosen example is very simple. Nevertheless, the problem is certainly nontrivial, in particular if the data is somewhat corrupted by noise. Note that some of the analysis and identification approaches that will be described are strongly tied to this simple geometry.

Before presenting identification schemes for the stated problem, the model (3.1), (3.2) will be examined from a system theoretic perspective.

### 3.2 The system dynamics

The defined system is linear and has two inputs, $q_i$ and $T_e$, and one output, $T_i$. There are several questions that need to be answered before identification of the unknown parameters.

- What is the system order?
3.2 The system dynamics

- How can the state vector be defined?

The order is in fact infinite. It is not possible to exactly characterize the dynamics by a finite number of state variables. Instead the whole temperature distribution through the wall, \( T(x, t) \), acts as a state vector. This means also that in order to determine the output \( T_i(t) \) the initial state \( T(x, 0) \) needs to be known for all \( x \).

Interestingly enough, one can rather easily derive the transfer function of the system, which will give an input-output description. To do this, first take the Laplace transform of (3.1) which gives

\[
sT(x, s) = \zeta \frac{d^2}{dx^2} T(x, s)
\]

where \( T(x, s) \) is the Laplace transform of the temperature \( T(x, t) \). It was assumed that \( T(x, t)|_{t=0} = 0 \). Regarding \( s \) as a fixed parameter, (3.7) is a standard ordinary differential equation. Then it is found that the general solution to (3.7) can be written as

\[
T(x, s) = F_1(s) \sinh \left( x \sqrt{\frac{s}{\zeta}} \right) + F_2(s) \cosh \left( x \sqrt{\frac{s}{\zeta}} \right)
\]

for some ‘constants’ \( F_1(s) \) and \( F_2(s) \). Further, using the Laplace transform on (3.2) and utilizing (3.8) gives

\[
q(x, s) = -\kappa F_1(s) \sqrt{\frac{s}{\zeta}} \cosh \left( x \sqrt{\frac{s}{\zeta}} \right) - \kappa F_2(s) \sqrt{\frac{s}{\zeta}} \sinh \left( x \sqrt{\frac{s}{\zeta}} \right)
\]

The functions \( F_1(s) \) and \( F_2(s) \) can be found from the boundary conditions, where (3.5), (3.6), (3.8) and (3.9) have been used

\[
\begin{align*}
T_1(s) &= T(0, s) = F_2(s) \\
q_1(s) &= q(0, s) = -\kappa F_1(s) \sqrt{\frac{s}{\zeta}}
\end{align*}
\]

The temperature at the exterior side is thus given by, cf (3.4),

\[
T_e(s) = T(d, s)
= -\frac{q_1(s)}{\kappa \sqrt{s/\zeta}} \sinh \left( d \sqrt{\frac{s}{\zeta}} \right) + T_1(s) \cosh \left( d \sqrt{\frac{s}{\zeta}} \right)
\]

To proceed, introduce some transformed parameters. Set

\[
\begin{align*}
R & \triangleq \frac{d}{\kappa [\text{°Cm}^2/\text{W}]} \quad \text{thermal resistance} \\
C & \triangleq \frac{d \rho c}{[\text{Wh/°Cm}^2]} \quad \text{thermal capacitance} \\
\tau & \triangleq RC \quad \text{time constant}
\end{align*}
\]
Next note that, using (3.3), (3.12)

\[
\begin{align*}
\sqrt{\frac{s}{\zeta}} &= \sqrt{\frac{d^2 s}{\zeta}} = \sqrt{\frac{d^2 \rho c}{\kappa}} \frac{s}{\sqrt{\sigma}} \\
\kappa \sqrt{\frac{s}{\zeta}} &= \frac{\kappa}{d} \sqrt{\sigma r} = \sqrt{\frac{\sigma r}{R}}
\end{align*}
\]

Inserting this into (3.11) gives

\[
T_e(s) = -\frac{R q_i(s)}{\sqrt{\sigma r}} \sinh(\sqrt{\sigma r}) + T_i(s) \cosh(\sqrt{\sigma r})
\]
or, rewritten,

\[
T_i(s) = R \frac{\tanh(\sqrt{\sigma r})}{\sqrt{\sigma r}} q_i(s) + \frac{1}{\cosh(\sqrt{\sigma r})} T_e(s) \tag{3.13}
\]

Equation (3.13) is a transfer function model that relates $q_i$ and $T_e$ as inputs to $T_i$ as an output.

Modeling the heat diffusion equation provides a multiple input single output system, cf (3.13). In the frequency domain, the system structure is thus $G(s) = [G_1(s) \ G_2(s)]$ and the input signal $u(s) = [q_i(s) \ T_e(s)]^T$.

Note that the transfer functions appearing in (3.13) are transcendental and cannot be written as finite order models. Also note in (3.13) that $G_1(s)$ and $G_2(s)$ are defined as some functions of $\sqrt{s}$, not of $s$. However, this is not a contradiction. By series expansion, odd powers of $\sqrt{s}$ will disappear. For example,

\[
\cosh(\sqrt{\sigma r}) = \sum_{j=0}^{\infty} \frac{1}{j!} (\sqrt{\sigma r})^j = \sum_{k=0}^{\infty} \frac{1}{(2k)!} (s\tau)^k
\]

which illustrates that $G_2(s)$ is indeed a function of $s$ rather than of $\sqrt{s}$. It is also crucial to note that $G_1(s)$ and $G_2(s)$ are both of low pass character, see Figure 3.2. This is an attractive property, as this facilitates approximation with finite order models. (Such an approximation appears inherently during the parameter estimation phase.) Choosing $q_i$ or $T_e$ as the output, i.e. dependent, variable, would though lead to a high pass transfer function.

The transfer functions of (3.13) can also be characterized in terms of poles, zeros and static gain. They have the same set of poles, given by

\[
\cosh(\sqrt{\sigma r}) = 0 \Rightarrow e^{\sqrt{\sigma r}} + e^{-\sqrt{\sigma r}} = 0 \Rightarrow e^{2\sqrt{\sigma r}} = -1 \tag{3.15}
\]

leading to

\[
s\tau = -\frac{\pi^2}{4} (2m - 1)^2, \quad m = 1, 2, \ldots \tag{3.16}
\]
and

$$p_m = -\frac{\pi^2}{\tau} \left( m - \frac{1}{2} \right)^2, \quad m = 1, 2, \ldots$$  \hspace{1cm} (3.17)

Only $G_1(s)$ has any zeros. These are given by

$$\sinh(\sqrt{\tau s}) = 0 \quad (s \neq 0) \quad \Rightarrow \quad e^{\sqrt{\tau s}} - e^{-\sqrt{\tau s}} = 0 \quad \Rightarrow \quad e^{\sqrt{\tau s}} = 1$$  \hspace{1cm} (3.18)

leading to

$$z_m = -\frac{\pi^2}{\tau} m^2, \quad m = 1, 2, \ldots$$  \hspace{1cm} (3.19)

Finally, the static gains are

$$G_1(0) = R, \quad G_2(0) = 1$$  \hspace{1cm} (3.20)

The transfer functions in (3.13) can be rewritten as shown in the following lemma.
**Lemma 3.1.** Let $G_1(s)$ and $G_2(s)$ be given by (3.13). Then the transfer functions may be rewritten, using partial decomposition, as

\[
G_1(s) = R \sum_{k=1}^{\infty} \frac{\beta_{1,k}}{s\tau + \alpha_k} + RD_1 \tag{3.21}
\]

\[
G_2(s) = \sum_{k=1}^{\infty} \frac{\beta_{2,k}}{s\tau + \alpha_k} + D_2 \tag{3.22}
\]

where

\[
\alpha_k = \frac{\pi^2(2k-1)^2}{4}, \quad k = 1, 2, \ldots \tag{3.23}
\]

\[
\beta_{1,k} = 2 \tag{3.24}
\]

\[
\beta_{2,k} = \pi(2k-1)(-1)^{k+1}, \quad k = 1, 2, \ldots \tag{3.25}
\]

\[
D_1 = 0 \tag{3.26}
\]

\[
D_2 = 0 \tag{3.27}
\]

**Proof** See Appendix A.

Using Laplace transform, (3.21) and (3.22) can be transformed to time domain, resulting in the following weighting functions,

\[
g_1(t) = \frac{R}{\tau} \sum_{k=1}^{\infty} \beta_{1,k} e^{-\frac{\alpha_k t}{\tau}} \tag{3.28}
\]

\[
g_2(t) = \frac{1}{\tau} \sum_{k=1}^{\infty} \beta_{2,k} e^{-\frac{\alpha_k t}{\tau}} \tag{3.29}
\]

### 3.3 Solution close to the boundary $x = 0, \ t = 0$

The system (3.13) rewritten in the time domain equals

\[
T_1(t) = T(0, t) = \int_{0}^{t} g_1(t - t')q_i(t')dt' + \int_{0}^{t} g_2(t - t')T_x(t')dt' \tag{3.30}
\]

It was assumed that the initial value $T(x, 0) = 0$ for all $x$. The behaviour of the weighting functions at $t = 0$ is illustrated in the following example.

**Example 3.1.** Consider the expressions for the weighting functions as given by (3.28) and (3.29). For simplicity, let $R = 1 \ [\text{°Cm}^2/\text{W}], \ \tau = 1 \ [\text{h}],$ which gives a
dimensionless system. Then

\begin{align}
  g_1(t) &= \sum_{k=1}^{\infty} \beta_{1,k} e^{-\alpha x t} \\
  g_2(t) &= \sum_{k=1}^{\infty} \beta_{2,k} e^{-\alpha x t}
\end{align}

(3.31) (3.32)

As long as \( t > 0 \), the exponential terms in (3.31) and (3.32) guarantee that the weighting functions are bounded. However, when inserting \( t = 0 \) the equations become

\begin{align}
  g_1(0) &= \sum_{k=1}^{\infty} \beta_{1,k} = \sum_{k=1}^{\infty} 2 \to \infty \\
  g_2(0) &= \sum_{k=1}^{\infty} \beta_{2,k} = \sum_{k=1}^{\infty} \pi(2k - 1)(-1)^{k+1} = -\sum_{k \text{ odd}}^{\infty} 2\pi \to -\infty
\end{align}

(3.33) (3.34)

The weighting functions at \( t = 0 \) can also be determined using the limit theory, [15].

\[
\lim_{t \to 0^+} g(t) = \lim_{s \to \infty} sG(s)
\]

For \( g_1(t) \), the following result is obtained,

\[
\lim_{t \to 0^+} g_1(t) = \lim_{s \to \infty} sG_1(s) = \lim_{s \to \infty} \frac{\tanh(\sqrt{s\tau})}{\sqrt{s\tau}} \to \infty
\]

(3.35)

which gives the same result as (3.33). But \( g_2(t) \) gives

\[
\lim_{t \to 0^+} g_2(t) = \lim_{s \to \infty} sG_2(s) = \lim_{s \to \infty} s \frac{1}{\cosh(\sqrt{s\tau})} \to 0
\]

(3.36)

which is different from (3.34)!

The result of Example 3.1 is not expected when considering the underlying physical system, at least not for \( g_2(t) \). An increase in the heat flux \( q_i(t) \) at \( t = 0 \) causes a quick increase in the indoor temperature \( T_i(t) \), giving that \( g_1(t) \) can be unbounded for a short moment. But, the weighting function \( g_2(t) \), connecting the outdoor temperature \( T_e(t) \) with \( T_i(t) \), cf (3.30), is expected to be bounded for all \( t \geq 0 \). In order to study the heat diffusion equation for \( t \) close to zero, it is not possible to simply insert \( t = 0 \) into (3.28) or (3.29). The reason for this is that there are singularities in (3.1), (3.2) which must be evaluated with limits of the functions, \( \lim_{t \to 0} \), \( \lim_{x \to 0} \).
In spite of the discouraging result of (3.33), (3.34), this does not cause problems in the calculations to come. The system is rather bounded input bounded output (BIBO) stable, which means that

$$\int_0^\infty |g(t)| dt < \infty$$  \hfill (3.37)

This is shown to hold for both weighting functions below. The weighting function $g_1(t)$ is treated first.

$$\int_0^\infty |g_1(t)| dt = \int_0^\infty \left| \frac{R}{\tau} \sum_{k=1}^\infty \beta_{1,k} e^{-\frac{\alpha_k}{R} t} \right| dt$$  \hfill (3.38)

It is assumed that $g_1(t)$ is uniformly convergent. Then it is allowed to switch the positions of the sum and the integral. Further, from (3.24) it is known that $\beta_{1,k} > 0$. Then

$$\int_0^\infty |g_1(t)| dt = \frac{R}{\tau} \sum_{k=1}^\infty \int_0^\infty \beta_{1,k} e^{-\frac{\alpha_k}{R} t} dt$$

$$= \frac{R}{\tau} \sum_{k=1}^\infty \frac{\beta_{1,k}}{\alpha_k}$$

$$= \frac{8R}{\pi^2} \sum_{k=1}^\infty \frac{1}{(2k-1)^2}$$

$$= \frac{8R}{\pi^2} \sum_{k=0}^\infty \frac{1}{(2k+1)^2}$$

$$= R$$  \hfill (3.39)

where theory of Bernoulli numbers was used \cite{15}. The expressions for $\alpha_k$ and $\beta_{1,k}$ were found in (3.23), (3.24). The assumption that $g_1(t)$ is uniformly convergent can be shown to hold for $t > 0$ using Weierstrass’ majorant test \cite{15}, since $g_1(t)$ converges and since all terms in $g_1(t)$ are positive.

The integral of $g_2(t)$ is not computed exactly, but an upper bound is determined. It is assumed that $g_2(t)$ is uniformly convergent.

$$\int_0^\infty |g_2(t)| dt = \frac{1}{\tau} \int_0^\infty \sum_{k=1}^\infty \beta_{2,k} e^{-\frac{\alpha_k}{R} t} dt$$  \hfill (3.40)

Introduce

$$p_k \triangleq \beta_{2,k} e^{-\frac{\alpha_k}{R} t}$$  \hfill (3.41)
Then

\[ \int_0^\infty |q_2(t)| \, dt \leq \frac{1}{\tau} \int_0^\infty \sum_{k = 1 \atop k \text{ odd}}^\infty |p_k + p_k+1| \, dt \tag{3.42} \]

The sum \( p_k + p_k+1 \) is computed next.

\[
p_k + p_k+1 = \pi(2k - 1)(-1)^{k+1}e^{-\frac{\pi^2 |2k+1|^2 t}{4\tau}} - \pi(2k + 1)(-1)^{k+1}e^{-\frac{\pi^2 |2k+1|^2 t}{4\tau}}
\]

\[
= (-1)^{k+1} \left[ \pi(2k - 1)e^{-\frac{\pi^2 |2k+1|^2 t}{4\tau}} - \pi(2k + 1)e^{-\frac{\pi^2 |2k+1|^2 t}{4\tau}} \right]
\]

\[
= (-1)^{k+1} \left[ \pi(2k - 1)e^{-\frac{\pi^2 |2k+1|^2 t}{4\tau}} - \pi(2k - 1)e^{-\frac{2\pi^2 |2k+1|^2 t}{4\tau}} - 2\pi e^{-\frac{\pi^2 |2k+1|^2 t}{4\tau}}e^{-\frac{2\pi^2 |2k+1|^2 t}{4\tau}} \right] \tag{3.43}
\]

It can be noted that the first term within the square parenthesis is larger or equal to the second term caused by the exponential \( e^{-2\pi^2 kt / \tau} \), which is less than or equal to one. The expression (3.43) is inserted into (3.42) giving

\[
\int_0^\infty |q_2(t)| \, dt \leq \frac{1}{\tau} \sum_{k = 1 \atop k \text{ odd}}^\infty \left[ \pi(2k - 1)e^{-\frac{\pi^2 |2k+1|^2 t}{4\tau}} - \pi(2k + 1)e^{-\frac{\pi^2 |2k+1|^2 t}{4\tau}} - 2\pi e^{-\frac{\pi^2 |2k+1|^2 t}{4\tau}}e^{-\frac{2\pi^2 |2k+1|^2 t}{4\tau}} \right] \, dt
\]

\[
\leq \frac{1}{\tau} \sum_{k = 1 \atop k \text{ odd}}^\infty \left( \pi(2k - 1)e^{-\frac{\pi^2 |2k+1|^2 t}{4\tau}} - \pi(2k - 1)e^{-\frac{2\pi^2 |2k+1|^2 t}{4\tau}} + 2\pi e^{-\frac{\pi^2 |2k+1|^2 t}{4\tau}}e^{-\frac{2\pi^2 |2k+1|^2 t}{4\tau}} \right) \, dt
\]

\[
= \frac{4}{\pi} \sum_{k = 1 \atop k \text{ odd}}^\infty \left( \frac{1}{2k - 1} - \frac{2k - 1}{(2k + 1)^2} \right) + \frac{8}{\pi} \sum_{k = 1 \atop k \text{ odd}}^\infty \frac{1}{(2k + 1)^2} \tag{3.44}
\]
The second sum in (3.44) is approximated with an upper bound as follows.

\[
I_2 = \sum_{k=1, \text{odd}}^{\infty} \frac{1}{(2k+1)^2} \\
\leq \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} \\
= \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} - 1 \\
= \frac{\pi^2}{8} - 1 \tag{3.45}
\]

where theory of Bernoulli numbers was used in order to determine the sum. The first sum in (3.44) is also approximated with an upper bound.

\[
I_1 = \sum_{k=1, \text{odd}}^{\infty} \left( \frac{1}{2k-1} - \frac{2k-1}{(2k+1)^2} \right) \\
= \sum_{k=1, \text{odd}}^{\infty} \frac{8k}{(2k-1)(2k+1)^2} \\
\leq \sum_{k=1, \text{odd}}^{\infty} \frac{8k}{2k(2k-1)(2k+1)} \\
= 4 \sum_{k=1, \text{odd}}^{\infty} \frac{1}{4k^2 - 1} \\
\leq 4 \sum_{k=1, \text{odd}}^{\infty} \frac{1}{(2k-1)^2} \\
\leq 4 \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \\
= \frac{\pi^2}{2} \tag{3.46}
\]

Inserting the expressions (3.45) and (3.46) into (3.44), an upper bound of the integral is obtained.

\[
\int_{0}^{\infty} |g_2(t)| dt \leq \frac{4\pi^2}{\pi} + \frac{8}{\pi} \left( \frac{\pi^2}{8} - 1 \right) \tag{3.47}
\]
The assumption that \( g_2(t) \) is uniformly convergent for \( t > 0 \) can be shown by Weierstrass’ majorant test. An upper bound of \( g_2(t) \) can be obtained from (3.44),

\[
|g_2(t)| \leq \frac{1}{T} \sum_{k \text{ odd}}^{\infty} \left( \pi (2k - 1) e^{-\frac{\pi^2 |g_{k+1}|^2}{4t}} \left( 1 - e^{-\frac{\pi^2 |g_k|^2}{4t}} \right) + 2\pi e^{-\frac{\pi^2 |g_{k+1}|^2}{4t}} \right)
\]

(3.48)

which converges for \( t > 0 \). Hence, both (3.39) and (3.47) are bounded and satisfy (3.37).

The important aspect (3.37) of the weighting functions implies that they do not cause an unbounded output \( T_i(t) \). Consider (3.30),

\[
|T_i(t)| = \left| \int_0^t g_1(t - t') q_i(t') dt' + \int_0^t g_2(t - t') T_e(t') dt' \right|
\]

\[
\leq \left| \int_0^t g_1(t - t') q_i(t') dt' \right| + \left| \int_0^t g_2(t - t') T_e(t') dt' \right|
\]

\[
\leq \int_0^t |g_1(t - t')||q_i(t')| dt' + \int_0^t |g_2(t - t')||T_e(t')| dt'
\]

\[
\leq \max_t |q_i(t)| \int_0^\infty |g_1(t - t')| dt' + \max_t |T_e(t)| \int_0^\infty |g_2(t - t')| dt'
\]

(3.49)

which is bounded when the inputs \( q_i(t) \) and \( T_e(t) \) are bounded. The system is hence BIBO stable.
3.3 Solution close to the boundary $x = 0, t = 0$
4 Approximate models

There are several ways to approximate the dynamics (3.1), (3.2) by a finite order model. These models are used in order to estimate the unknown material constants, cf for example (2.2), (2.3). The estimates will be non-consistent when the model order is smaller than that of the system. The size of the bias depends highly on the structure and order of the model.

Two alternative models will be considered here. The emphasize is put on the difference approximation method. Thermal networks are mainly presented as an alternative way of approximating the true system dynamics. These methods are application dependent. If a model for a general system is preferred, a black-box model might be a better choice.

4.1 Difference approximation

One way to go, that is often used when solving the PDE numerically, is to use a difference approximation scheme of order \( n \), which for large \( n \) gives small approximation errors for all frequencies of interest, [16],

\[
G_1^n(s) \approx G_1(s) \\
G_2^n(s) \approx G_2(s)
\]

Consider approximation of the original model (3.1), (3.2), by discretizing the spatial dimension. Divide the wall into \( (n+1) \) layers, each of thickness \( \Delta = d/(n+1) \). The order of the approximated model thus equals \( n \). The temperatures between the different layers can then be used as state variables in a finite order approximate model, compare Figure 4.1.

\[
\begin{array}{ccccccc}
0 & \Delta & 2\Delta & \ldots & d-\Delta & d \\
T_i & q_i & T_1 & T_2 & \ldots & T_n & T_e \\
\end{array}
\]

Figure 4.1: Discretization model of the heat propagation in an homogeneous wall.

The approximate model reads

\[
\begin{align*}
\dot{T}_k &= \frac{s}{\Delta^2}(T_{k+1} - 2T_k + T_{k-1}), \quad k = 1, \ldots, n \\
T_o &= T_i, \quad T_{n+1} = T_e \\
q_i &= -\sigma A(T_1 - T_i)
\end{align*}
\]

(4.1)
Introduce further
\[ \mu \triangleq \frac{\zeta}{\Delta^2} = \frac{(n+1)^2}{\tau}, \quad \gamma \triangleq \frac{\Delta}{\kappa} = \frac{R}{n+1} \] (4.2)

Then the approximated model, of order \( n \), becomes in standard state space form
\[
\begin{align*}
\dot{x} &= \mu A x + \mu \gamma e_1 u_1 + \mu e_n T_e \\
T_i &= e_1^T x + \gamma q_i \\
x &= (T_1 \ldots T_n)^T
\end{align*}
\] (4.3) (4.4) (4.5)

\[
A = \begin{pmatrix}
-1 & 1 & 0 \\
1 & -2 & 1 \\
0 & -2 & 1 \\
1 & -2 & \\
& & & \ddots \\
\end{pmatrix}
\] (4.6)

\[
e_1 = (1 \ 0 \ldots \ 0)^T, \quad e_n = (0 \ldots 0 \ 1)^T
\] (4.7)

Rewriting (4.3), (4.4) in input-output form gives the approximate transfer functions \( G_1^n(s) \) and \( G_2^n(s) \).

\[
\begin{align*}
\tilde{G}_1^n(s) &= \gamma + \mu \gamma e_1^T [sI - \mu A]^{-1} e_1, \\
\tilde{G}_2^n(s) &= \mu e_1^T [sI - \mu A]^{-1} e_n.
\end{align*}
\] (4.8) (4.9)

As an illustration, (4.8) and (4.9) are given for the case of model order 2.

**Example 4.1.** For model order \( n = 2 \) the following expressions of the approximate transfer functions are obtained,

\[
\begin{align*}
G_1^n(s) &= \gamma \frac{s^2 + 4s + 3\mu^2}{s^2 + 3\mu s + \mu^2} \\
G_2^n(s) &= \frac{\mu^2}{s^2 + 3\mu s + \mu^2}
\end{align*}
\] (4.10)

\[
\Box
\]

The approximate transfer functions (4.8), (4.9) can now be rewritten as shown by the following lemma. The transfer functions then obtain a structure similar to (3.21), (3.22).

**Lemma 4.1.** Let \( \tilde{G}_1^n(s) \) and \( \tilde{G}_2^n(s) \) be the approximate transfer functions of order \( n \), given by (4.8) and (4.9). They can be rewritten as

\[
\begin{align*}
\tilde{G}_1^n(s) &= R \sum_{k=1}^n \frac{\beta_{1,k}}{s \tau + \alpha_k} + R D_1 \\
\tilde{G}_2^n(s) &= \sum_{k=1}^n \frac{\beta_{2,k}}{s \tau + \alpha_k} + D_2
\end{align*}
\] (4.11) (4.12)
where

\[
\alpha_k = 2(n + 1)^2 \left(1 - \cos \left(\frac{\pi(2k - 1)}{2n + 1}\right)\right), \quad k = 1, 2, \ldots, n \tag{4.13}
\]

\[
\beta_{1,k} = \frac{(n + 1) \cos^2 \left(\frac{\pi(2k-1)}{2(2n+1)}\right)}{\sum_{k=1}^{n} \cos^2 \left(\frac{\pi(2k-1)}{2(2n+1)}\right)} \tag{4.14}
\]

\[
\beta_{2,k} = \frac{(n + 1)^2 \cos \left(\frac{\pi(2k-1)(n+1)}{2n+1}\right) \cos \left(\frac{\pi(2k-1)}{2(2n+1)}\right)}{\sqrt{\sum_{k=1}^{n} \cos^2 \left(\frac{\pi(2k-1)(n+1)}{2n+1}\right)}} \sqrt{\sum_{k=1}^{n} \cos^2 \left(\frac{\pi(2k-1)}{2(2n+1)}\right)} \tag{4.15}
\]

\[
\tilde{D}_1 = \frac{1}{n + 1} \tag{4.16}
\]

\[
\tilde{D}_2 = 0 \tag{4.17}
\]

**Proof** See Appendix B.

The indices \(k\) correspond to specific layers in the solid. Hence, \(\{k\}\) are integer values in the interval 1 to \(n\), as illustrated in Figure 4.2.

![Figure 4.2: Illustration of the possible values of the indices \(k\) as a function of \(n\).](image)

The expressions (4.14) and (4.15) can be simplified when the model order \(n\) is large. The result is presented in Lemma 4.2.
Lemma 4.2. Let $\hat{G}_1^n(s)$ and $\hat{G}_2^n(s)$ be the approximate transfer functions of order $n$, given by (4.11) and (4.12). For large $n$, the associated parameters $\beta_{1,k}$ and $\beta_{2,k}$ can be simplified, resulting in the following model

$$\hat{G}_1^n(s) = R \sum_{k=1}^{n} \frac{\beta_{1,k}}{s^r + \alpha_k} + R \tilde{D}_1$$  \hspace{1cm} (4.18)

$$\hat{G}_2^n(s) = \sum_{k=1}^{n} \frac{\beta_{2,k}}{s^r + \alpha_k} + \tilde{D}_2$$  \hspace{1cm} (4.19)

where

$$\tilde{\alpha}_k = 2(n+1)^2 \left(1 - \cos \left(\frac{\pi(2k-1)}{2n+1}\right)\right), \hspace{1cm} k = 1, 2, \ldots, n$$ \hspace{1cm} (4.20)

$$\tilde{\beta}_{1,k} = \frac{4(n+1)}{(2n+1)} \cos^2 \left(\frac{\pi(2k-1)}{2(2n+1)}\right) \hspace{1cm} (4.21)$$

$$\tilde{\beta}_{2,k} = \frac{4(n+1)^2(-1)^{k+1}}{(2n+1)} \sin \left(\frac{\pi(2k-1)}{2n+1}\right) \cos \left(\frac{\pi(2k-1)}{2(2n+1)}\right) \hspace{1cm} (4.22)$$

$$\tilde{D}_1 = \frac{1}{n+1} \hspace{1cm} (4.23)$$

$$\tilde{D}_2 = 0 \hspace{1cm} (4.24)$$

Proof \hspace{1cm} See Appendix B.

Some notes about the results of Lemma 4.1 and Lemma 4.2 might be necessary. Lemma 4.1 gives exact expressions of the approximate transfer functions, as obtained from the difference approximation model. This lemma should be used when an arbitrary model order is used. In numerical examples, where analytical solutions are not given, Lemma 4.1 is therefore to prefer. However, the normalization factors in $\tilde{\beta}_{1,k}$ and $\tilde{\beta}_{2,k}$ in Lemma 4.1 may give unnecessary complicated expressions in analytical solutions. Therefore, these may be approximated under the constraint that the model order $n$ is large. Then Lemma 4.2 results, which is a good approximation of the results of Lemma 4.1 when $n$ is large.

Figure 4.3 displays the true transfer function $G_1$, as given by (3.13), together with approximations $\hat{G}_1^n$ for different orders of the difference approximation method. Plot (a) shows the case of a second order model and plot (b) shows the result of a third order model. Plots (c) and (d) display the cases of higher order models, $n = 10$ and $n = 50$ respectively. Figure 4.4 illustrates the corresponding for $G_2$. It can be noted that it might be sufficient to use a rather low model order if the frequencies of interest are low. The deviation of the approximate transfer function from the true transfer function is larger for high
Figure 4.3: The difference approximation of $G_1(i\omega)$ for some different model orders ($n = 2, 3, 10, 50$). The true transfer function (of infinite order) is plotted with a dotted line and the approximate transfer function (of finite order) is plotted with a solid line. The parameter set is $R = 1 \text{ [°Cm}^2/\text{W}], \tau = 1 \text{ [hi]}$. 

Figure 4.4: The difference approximation of $G_2(i\omega)$ for some different model orders. The parameter set is $R = 1 \text{ [°Cm}^2/\text{W}], \tau = 1 \text{ [hi]}$. 

frequencies. Also note from Figures 4.3 and 4.4 that for a given model order, the approximation of \( G_2(s) \) is more accurate than the approximation of \( G_1(s) \).

Using Laplace transform, (4.18) and (4.19) can be transformed into the time domain, resulting in the following weighting functions

\[
\tilde{g}_k^1(t) = \frac{R}{\tau} \sum_{k=1}^{n} \beta_{1,k} e^{-\frac{\alpha_{1}}{\tau} t} + RD_1 \delta(t) \quad (4.25)
\]

\[
\tilde{g}_k^2(t) = \frac{1}{\tau} \sum_{k=1}^{n} \beta_{2,k} e^{-\frac{\alpha_{2}}{\tau} t} \quad (4.26)
\]

where \( \delta(t) \) is the Dirac function.

Figure 4.5 shows the approximated parameters \( \alpha_k, \beta_{1,k} \) and \( \beta_{2,k} \) together with the true ones from (3.23)-(3.25). The true parameters are plotted with dotted lines and the approximates are plotted with solid lines. The fit between the true and approximate parameters is best for small indices \( k \). Terms with medium-sized or large \( k \) are not equally well approximated. On the other hand, due to the exponential terms in the weighting functions, which will be small for large \( k \), the impact of the errors reduces for increasing indices \( k \).

\vspace{1cm}

**Figure 4.5:** The parameters \( \alpha_k, \beta_{1,k} \) and \( \beta_{2,k} \) from the original dynamics and the parameters \( \hat{\alpha}_k, \hat{\beta}_{1,k} \) and \( \hat{\beta}_{2,k} \) from the approximate model are shown. The parameters \( \alpha_k, \beta_{1,k} \) and \( \beta_{2,k} \) are plotted with dotted lines and the approximates are plotted with solid lines. The model order \( n \) is 50.
The approximation errors of the parameters,

\[ \alpha_{k} - \alpha_{k}, \quad \beta_{1,k} - \beta_{1,k}, \quad \beta_{2,k} - \beta_{2,k} \]

are displayed in Figure 4.6.

4.2 Thermal networks

Another way to approximate the system (3.1), (3.2) is to model it as a thermal network. This method is popular within building engineering [17]. This is the thermal equivalence of an electrical R\(C\) network. The heat flux corresponds to a current and the temperatures correspond to voltages. As an illustration consider a first order network, see Figure 4.7.

For the moment the question of how to choose the parameters \(R_{1}, R_{2}\) and \(C_{1}\) is left for future discussion. To derive the corresponding equations, Kirchoff’s law is applied,

\[
q_{i} = \frac{T_{i} - T}{R_{1}} = \frac{T - T_{e}}{R_{2}} + C_{1} \frac{dT}{dt} \tag{4.27}
\]

This can be reformulated into a first order model with \(T\) as the state, \(T_{e}\) and \(q_{i}\)
Figure 4.7: Structure of a first order thermal network.

as inputs and $T_i$ as output. The result is

$$
\begin{align*}
\dot{T} &= -\frac{1}{C_1 R_2} T + \frac{1}{C_1} q_i + \frac{1}{C_1 R_2} T_e \\
T_i &= T + R_1 q_i 
\end{align*}
$$

(4.28)

Writing this dynamics in input-output form as

$$
T_i(s) = G_1^o(s) q_i(s) + G_2^o(s) T_e(s)
$$

(4.29)

results in

$$
G_1^o(s) = \frac{R_1 C_1 s + (1 + R_1/R_2)}{C_1(s + 1/C_1 R_2)} \quad G_2^o(s) = \frac{1}{s C_1 R_2 + 1}
$$

(4.30)

One way to choose the network parameters $R_1$, $R_2$ and $C_1$ is to compare $G_1^o(s)$ and $G_2^o(s)$ to the true transfer functions $G_1(s)$ and $G_2(s)$. Require that the static gains as well as the first (dominating) pole and zero coincide, see (3.17), (3.19) and (3.20). Note that $G_2^o(0) = 1$ automatically. The above idea leads to

$$
\begin{align*}
\frac{1 + R_1/R_2}{1/R_2} &= R \\
-\frac{1}{C_1 R_2} &= -\frac{\pi^2}{4 \tau} \\
-\frac{1 + R_1/R_2}{R_1 C_1} &= -\frac{\pi^2}{\tau}
\end{align*}
$$

(4.31) \quad (4.32) \quad (4.33)

which has the solution

$$
R_1 = \frac{1}{4} R, \quad R_2 = \frac{3}{4} R, \quad C_1 = \frac{16}{3 \pi^2} C
$$

(4.34)

A slightly different approach is to require that the approximate model should resemble the true dynamics for low frequencies. Mathematically, this means that
the Taylor series expansions of the differences \( G_1(s) - \bar{G}_1(s) \) and \( G_2(s) - \bar{G}_2(s) \) around \( s = 0 \) should have no constant terms and zero coefficients for \( s \). Working out the details leads to the equations

\[
\begin{align*}
R_1 + R_2 &= R \\
-C_1R_2^2 &= -\frac{1}{3}R\tau \\
-C_1R_2 &= -\frac{1}{2}R\tau
\end{align*}
\]  
(4.35)

resulting in the solution

\[
R_1 = \frac{1}{3}R, \quad R_2 = \frac{2}{3}R, \quad C_1 = \frac{3}{4}C
\]  
(4.36)

Note that the parameter values of (4.34) and (4.36) differ somewhat.

Figure 4.8 shows the result of approximation of the transfer functions \( G_1 \) and \( G_2 \) using a thermal network of first order. It can be noted that the approximate transfer functions resemble the true ones well for low frequencies. Requiring the same dominating pole, zero and static gain seems to give a slightly better approximation for \( G_1 \) (the curve denoted by ‘b’ in the figure).

**Figure 4.8:** The approximation of \( G_1(i\omega) \) and \( G_2(i\omega) \) as a thermal network of first order. The upper plot shows approximations of \( G_1(s) \). The dashed line (a) shows the approximation obtained from (4.36) and the solid line (b) the approximation from (4.34). The true transfer function is drawn with a dotted line (c). The lower plot shows the corresponding for \( G_2(s) \). Parameter values used are \( R = 1 \, \text{[\Omega \cdot \text{cm}^2/\text{W]}}, \tau = 1 \, \text{[h]} \).
Thermal network models of higher order can also be derived in a similar way, but this thesis does not include the details. To indicate the principle, let it suffice to show the structure of a second order thermal network, see Figure 4.9.

![Diagram](image)

**Figure 4.9**: Structure of a second order thermal network.

### 4.3 Alternative approaches

Two methods for model reduction have been proposed in this chapter. There are of course several other possible ways of approximating the system. One approach is the collocation method [18], which forces the approximation to exactly satisfy the system equation at some arbitrarily chosen points. See for example [19] for a numerical evaluation of parameter estimation using collocation for model approximation. The Galerkin and the least squares method [20] are variants of collocation that could be used as well. All these three methods change the partial differential equation into a set of ordinary differential equations.

If the system dynamics is unknown, or if the user for some reason does not want to utilize knowledge about the dynamics, a black-box model can be used instead. Then the system can be approximated by an ARX model of some finite order. This approach has been studied extensively during the past years.
5 Approaches for parameter estimation

In this chapter a brief presentation of some approaches for the parameter estimation problem introduced in Section 3.1 is given. The next chapter is devoted to a comparison study using simulated data. It will be shown that the bias is larger than the standard deviation for all approaches. This indicates that it is more important to minimize the bias of the parameter estimates. For this reason, Chapter 9 is dedicated to an analytic analysis of the parameter bias.

To simplify the notations, and also use some standard symbols, the underlying model will mostly be written as

\[ Y(s) = G(s)U(s) \quad (5.1) \]

where

\[
\begin{align*}
Y(s) &= T_i(s) \\
G(s) &= \begin{bmatrix} G_1(s) & G_2(s) \end{bmatrix} \\
U(s) &= \begin{bmatrix} q_i(s) & T_e(s) \end{bmatrix}^T
\end{align*}
\quad (5.2)
\]

The quantities \( q_i(s) \) and \( T_e(s) \) thus act as input signals and \( T_i(s) \) as the output, compare with (3.13). Further let

\[ \theta = (R^T C)^T \quad (5.3) \]

denote the parameter vector to be estimated. Of course, the physical parameters \( \zeta \) and \( \kappa \) could be introduced as unknowns instead, but these parameters can easily be derived from (5.3) using (3.3) and (3.12).

5.1 Method 1: A direct approach in the time domain

Let \( y(\theta) \) denote the measured value of the output \( y(t) \). Then the following least squares criterion function may be defined

\[ V_1(\theta) = \frac{1}{N} \sum_{k=1}^{N} [y(kh) - y^m(kh, \theta)]^2 \quad (5.4) \]

where \( y^m(t, \theta) \) is the solution to the PDE at time \( t \) using the parameter values given by \( \theta \), and \( N \) is the number of data points available. The model output \( y^m(t, \theta) \) is obtained either from the infinite order model (3.13), or from the approximate finite order model (4.3)-(4.7). These are the approaches normally used to deal with the problem of an infinite order system [10]. The parameter estimate is taken as

\[ \hat{\theta}_1 = \arg \min_{\theta} V_1(\theta) \quad (5.5) \]
This estimate is conceptually simple, but computationally quite complex. As $V_1(\theta)$ depends on $\theta$ in a highly nonlinear way, a numerical search procedure is needed to compute $\theta_1$. For each function evaluation, the PDE is to be solved for the current value of $\theta$.

The approach can be extended in several ways, for example by

- penalizing another error, e.g. the $l^p$ norm of the error $y(kh) - y^m(kh, \theta)$.
- introducing some prefiltering of the data. As in ‘standard system identification’, this corresponds to modeling the measurement noise, and to give emphasis to certain frequency bands.

5.2 Method 2: A direct approach in the frequency domain

The idea of this approach is to convert the criterion $V_1(\theta)$ into the frequency domain using Parseval’s relation. Let $Y(\omega)$ denote the discrete Fourier transform of the time series $\{y(kh)\}_{k=1}^N$. Using a fast fourier transform (FFT) algorithm it can conveniently be computed for frequencies $\omega = \frac{2\pi k}{N}$, $k = 0, \ldots, N - 1$. Next introduce the criterion

$$V_2(\theta) = \frac{1}{K} \sum_{k=0}^{K} Q_2(j\frac{2\pi k}{N}) \left| Y(j\frac{2\pi k}{N}) - Y^m(j\frac{2\pi k}{N}, \theta) \right|^2$$

$$= \frac{1}{K} \sum_{k=0}^{K} Q_2(j\frac{2\pi k}{N}) \left| Y(j\frac{2\pi k}{N}) - G^n(j\frac{2\pi k}{N}, \theta) U(j\frac{2\pi k}{N}) \right|^2$$

(5.6)

where $K$ is the number of frequency points for which the criterion is evaluated. $K$ can be chosen much smaller than the number of data points $N$, which results in a lower computational complexity. Here, $Q_2(\omega)$ is a user chosen weighting function, with which the deviation $Y(\omega) - Y^m(\omega, \theta)$ can be penalized in different frequency regions. In the particular case study it would be natural to set $Q_2(\omega) \equiv 1$ (no weighting), or to let $Q_2(\omega)$ emphasize the low frequency region, of the frequency characteristics of the transfer functions in (3.13).

The parameter estimate for this approach is taken as

$$\hat{\theta}_2 = \arg\min_{\theta} V_2(\theta)$$

(5.7)

There is one potential advantage of this approach over (5.4), (5.5). The optimization problem to solve is still nonlinear. However, as the $\theta$-dependence of the function $G^n(i\omega, \theta)$ is known, the criterion evaluations are much simpler for $V_2(\theta)$ than for $V_1(\theta)$. On the other hand, this approach is limited to linear models for which $G^n(i\omega, \theta)$ is a known function of the unknown parameter vector $\theta$.

5.3 Methods 3-5: Indirect approaches

The main idea of these approaches is as follows. First the transfer function is estimated by using a standard black-box model of high order, or a smoothed
empirical transfer function estimate is used. In any case, by some method an estimated frequency function $\hat{G}^n(i\omega)$ is available. In the second step a parametric model is fitted to $G^n(i\omega)$.

Ideally the parameter vector $\theta$ is wanted for which

$$\hat{G}^n(i\omega) \approx G(i\omega, \theta) \quad (5.8)$$

The dilemma is, of course, that as $\hat{G}^n(i\omega)$ deviates from the true frequency function, there is no $\theta$ that satisfies (5.8) for all frequencies. There is hence a need to introduce some ways to solve the relation (5.8) with respect to $\theta$. This can be done in several ways.

- One possibility, that has some similarities to the direct approach in the frequency domain is to choose $\theta$ to minimize

$$V_3(\theta) = \frac{1}{K} \sum_{k=0}^{K-1} \left| G^n(i\omega(k)) - G(i\omega(k), \theta) \right|^2 Q_3(i\omega(k)) \quad (5.9)$$

where again $Q_3(i\omega(k))$ is a user-defined weighting function, and $\{\omega(k)\}$ are some selected frequency points. In fact, this approach may be interpreted as the frequency domain direct approach using a white noise as input. This approach will be regarded as Method 3.

- A second way of treating (5.8) is to evaluate the two transfer functions in terms of static gain, dominating poles and zeros, and so on. Equating such quantities for $\hat{G}^n(i\omega)$ and $G(i\omega, \theta)$ leads to a set of equations for determining $\theta$. A further possible quantity to compare is the squared $L_2$-norm of the transfer functions, $\int_0^\infty |G(i\omega)|^2 d\omega$. In the examples of Chapter 6, the following criterion is used for this approach,

$$V_4(\theta) = (\hat{G}^n_1(0) - G_1(0, \theta))^2 + (\hat{p}_1 - p_1)^2 + (\hat{z}_1 - z_1)^2 \quad (5.10)$$

where $\hat{G}^n_1(0)$ is the static gain of the black-box model and $G_1(0, \theta)$ is obtained from (3.20) using the estimated value of $R$. The quantities $p_1$ and $z_1$ are the dominating pole and zero from (3.17) and (3.19), respectively. Possibly some weighting of the static gain, the dominating pole and the zero can be introduced in (5.10) in order to emphasize the quantity which is supposed to give the best estimate. In the following, this approach is denoted Method 4.

- A third alternative can be to apply the so called indirect prediction error method, cf [21]. Assume that $\hat{G}^n(i\omega)$ is obtained from a black-box model, say of order $n$, with an estimated parameter vector $\theta$. $\theta$ is here chosen as the magnitude vector of $G^n(i\omega)$. In practice a rather good fit to the true transfer function can be expected also for a moderate value of $n$, as shown
in Figures 4.3 and 4.4. Next consider a difference approximation (4.3)–(4.7) of the same order $n$. When the difference approximation depends only on the parameter vector $\theta$ of dimension 2, the parameter vector of the $n$th order model (4.3)–(4.7) may in this case be regarded as a mapping $\vartheta(\theta)$. The parameter vector $\vartheta(\theta)$ is taken as the magnitude vector of the transfer function obtained from (4.3)–(4.7). The unknown parameters $\theta$ are next determined as the minimization elements of

$$V_5(\theta) = \left(\vartheta - \vartheta(\theta)\right)^T Q_5 \left(\vartheta - \vartheta(\theta)\right)$$

(5.11)

with $Q_5$ being a positive definite weighting matrix. The $\theta$ dependence of $V_5(\theta)$ can be highly nonlinear and a numerical search is needed. However, if some reasonable initial values are given, a few Gauss Newton iterations for improving the guessed minimum of $V_5(\theta)$ should be sufficient. Finally, this approach is denoted Method 5.
6 Numerical evaluation

The approaches for parameter estimation that were introduced in the previous chapter are evaluated using simulated data. The goal is to obtain knowledge of what results can be expected. Some questions can be

- Which method is the best, and under what circumstances?
- How does noise influence the estimates?
- What is largest: bias or standard deviation?
- What bias can be expected?
- Are the unknown parameters equally hard to estimate?
- When can numerical problems occur?

6.1 Affecting the bias in the parameter estimates

There are several problems involved in the estimation of material parameters in a distributed system. The first two factors listed below give bias on the parameter estimates. Different realizations of the data also give slightly different biases. Noise, which is mentioned in the third point, gives a variance contribution around the biased mean value of the estimates.

- Firstly, the system is time-continuous, whereas the data are sampled and thus given in time-discrete form. Since sampling has to be performed, some error is introduced. The size of this error depends on the sampling interval.

- Secondly, the continuous system is of infinite order. In order to approximate it using a black-box model, a finite order model must be used. The model order is a user choice. Different identification methods require different model orders in order to give reliable results. There is a trade-off between having a large model order that can describe the dynamics of the system well, and a low model order that does not give numerical problems and keeps the computational load at a reasonably low level. This is a matter of the amount of under-parameterization that can be accepted.

- Thirdly, there is probably some noise, e.g., measurement noise, in the data. This study deals with noisy outputs, but in reality even the input data may contain disturbances.

There are some known ways of reducing the influence of the modeling errors that occur caused by the above events. Concerning the first problem, sampling cannot be avoided. However, it is important to try to minimize the effect of it, through careful choice of the sampling interval. It is of importance both to
sample fast enough in order not to loose information about dynamics of the system or signal, and also to use data of sufficient length.

Under-parameterization is not possible to entirely avoid when a finite order model is used. Therefore, there will inevitably be a systematic error, a bias, on the estimates. The influence of under-parameterization on the estimates can however be reduced by e.g. prefiltering of the data as described in [22]. Also the effect of possible existence of noise on the data can be reduced by prefiltering.

The bias distribution is affected by, [22],
- spectrum of input signal
- prefiltering
- model order
- sampling interval
- noise model
- prediction horizon

That the input spectrum affects the distribution of the modeling error is a known fact in system identification, see (6.1) – (6.3). The effect of prefiltering and model order will to some extent be investigated in this thesis. The sampling interval is assumed to be short enough not to make the model too coarse. According to [22], there is no disadvantage in using a short sampling interval, except for an increase of the computational load. The measurement noise is assumed to be white Gaussian noise, hence contributing to the measurements \( y(kh) \) with a constant spectral density over the whole frequency range. For the effect of prediction horizon, see [22].

The output \( T_i \) is obtained through the output, \( y(kh) \) in (5.4). Its estimate is obtained from filtering the measured input signals \( q_i \) and \( T_e \) through estimates of the transfer functions,

\[
T_i(s) = G_1^m(s)q_i(s) + G_2^m(s)T_e(s)
\]  

(6.1)

The spectral density of the model output signal is

\[
\Phi_{T_i}(i\omega) = \begin{pmatrix} G_1^m(i\omega) & G_2^m(i\omega) \end{pmatrix} \begin{pmatrix} \Phi_{q_i}(i\omega) & 0 \\ 0 & \Phi_{T_e}(i\omega) \end{pmatrix} \begin{pmatrix} G_1^m(-i\omega) \\ G_2^m(-i\omega) \end{pmatrix}
\]  

(6.2)

where \( \Phi \) denotes spectrum and where \( q_i \) and \( T_e \) are assumed to be uncorrelated.

The error spectrum becomes

\[
\Phi_{\hat{T}_i}(i\omega) = \begin{pmatrix} \hat{G}_1^m(i\omega) \\ \hat{G}_2^m(i\omega) \end{pmatrix} \begin{pmatrix} \hat{\Phi}_{q_i}(i\omega) & 0 \\ 0 & \hat{\Phi}_{T_e}(i\omega) \end{pmatrix} \begin{pmatrix} \hat{G}_1^m(-i\omega) \\ \hat{G}_2^m(-i\omega) \end{pmatrix} + \Phi_e(i\omega)
\]  

(6.3)
where $\tilde{G}_j(i\omega) = G_j(i\omega) - \hat{G}_j(i\omega)$, $j = \{1, 2\}$, and where $\Phi_q(\omega)$ equals the spectrum of possible measurement noise on the output. The error spectrum is thus dependent on the spectra of the input signal and the measurement noise as well as on the error in the transfer function approximation.

In some cases the user has the opportunity to choose the input signal. In this application, however, the signals are assumed to be given. Prefiltering of the signals was used above to specify for what frequencies the estimated $G^o(s)$ should be a good approximation of the true transfer function. In this case, for example, it is more important to correctly describe the transfer functions for low frequencies up to and including the first pole and zero. It is of importance to have a correct approximation of the static gain and the dominating pole and zero if these specifically are to be used in the identification as is the case for Method 4. The difficulty is then to know where to put the cut-off frequency in the low pass filter, since the dominating pole which specifies the break point in the Bode diagram is not known to the user. (The dominating pole and zero are highly dependent on the unknown product $RC$, see (3.17)). Here it might be possible to use a fairly moderate filter and obtain a rough estimate of the dominating pole and zero, and then perform a new identification with a prefilter adjusted to the obtained approximate model.

When a system is approximated from data, as is done in Section 5.3, the system must be observable. If several modes of the system are badly excited and only weakly observable, it can be hard to make a good approximation of the system. To obtain a good model requires the input data to have energy in the frequency band corresponding to the system, which here is of low-pass character. The input signals thus have to contain energy components in the lower frequency region.

### 6.2 Choice of input signals

Data are generated and used in order to illustrate the behavior of the different methods for parameter estimation. The signals acting as inputs to the system, $T_0$ and $q_0$, are chosen to resemble possible outdoor temperature and heat flux, respectively. The data series are of length $N = 16384$ samples and the sampling interval equals $h = 1$ [h]. Then the system output, $T_i$, is generated using the continuous-time system (3.13) in frequency domain with the parameters

\[
R = 10 \, [\text{°C m}^2/\text{W}] \quad (6.4)
\]

\[
C = 15 \, [\text{Wh/°C m}^2] \quad (6.5)
\]

Finally $T_i$ is transformed back to time domain. The system (3.13) with the chosen parameter values will thus be considered as the ‘true system’ in the following.

An example of the signals $T_0$, $T_i$ and $q_0$ is given in Figure 6.1. Only parts of the signals are shown. The outdoor temperature is modeled as an AR process.
with a spectrum corresponding to daily variations in temperature. The heat flux $q_i$ is modeled in such a way that it somewhat stabilizes the indoor temperature $T_i$. Figure 6.2 shows the spectra of the outdoor temperature, $T_o$, the heat flux, $q_i$, the indoor temperature $T_i$ and the measurement noise $e$, which in some examples is added to the $T_i$ measurements.

### 6.3 Computational complexity

The computational load is next considered. The algorithms are implemented in Matlab, and the function *flops* has been used for complexity estimation. The result must be considered with some care, since the implementation may be somewhat ‘programmer dependent’. Further, implementation in a digital signal processor (DSP) would give other values of the computational load. Also, different implementations give a variation of the resulting complexity. Rough estimates are determined for each method, and are given in Table 6.1. The model order $n$ in Table 6.1 is used in the following ways in the algorithm implementations. For Methods 1, 2 and 5, $n$ equals the model order when the finite order model (4.3)−(4.7) is used to produce the model output. If the model (3.13) is used in order to obtain the model output, $n$ is infinite. For Methods 3−5, $n$ determines the order of the ARX-model estimating the transfer function. Note that Method 5 utilizes both (4.3)−(4.7) and an ARX model in order to approximate the transfer functions in two different ways. The same order $n$ is, however, used.
### 6.3 Computational complexity

![Spectra of $T_e$, $T_i$, $q_i$ and $e$](image)

**Figure 6.2:** Spectra of temperatures, heat flux and the measurement noise of the indoor temperature. The temperatures are plotted with solid lines.

<table>
<thead>
<tr>
<th>Model order $n$</th>
<th>Computational complexity for Methods 1–5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\infty$</td>
<td>$4.4 \times 10^2$            $6.5 \times 10^2$</td>
</tr>
<tr>
<td>10</td>
<td>$1.8 \times 10^2$  $3.1 \times 10^2$  $3.6 \times 10^3$  $3.5 \times 10^3$  $3.9 \times 10^3$</td>
</tr>
<tr>
<td>20</td>
<td>$4.2 \times 10^2$  $6.7 \times 10^2$  $1.6 \times 10^3$  $1.3 \times 10^3$  $1.7 \times 10^3$</td>
</tr>
<tr>
<td>30</td>
<td>$8.7 \times 10^2$  $8.6 \times 10^2$  $3.5 \times 10^3$  $2.8 \times 10^3$  $3.9 \times 10^3$</td>
</tr>
<tr>
<td>40</td>
<td>$2.4 \times 10^3$  $1.4 \times 10^3$  $6.2 \times 10^3$  $4.9 \times 10^3$  $7.2 \times 10^3$</td>
</tr>
</tbody>
</table>

**Table 6.1:** Computational complexity for the different methods in Mflops (million floating point operations). For the model output for the infinity order model, (3.13) is used. For finite order models, $y^n(kh, \theta)$ is taken from (4.3)–(4.7).

A comparison of the complexity of the approaches shows that the indirect approaches require less computations than the direct approaches for equal model orders. The difference increases further when taking into account that the direct approaches require larger model orders for similar results, see the simulation results in Section 6.7.
6.4 Sensitivity to parameter variations

It was shown in Section 4.1 that the transfer function $G_2(s)$ gives a better approximation for a low model order than $G_1(s)$ does. This can be analytically verified through study of the transfer functions, (3.13). The transfer function $G_2(i\omega) = 1 / \cosh(\sqrt{\omega/\tau})$ is decreasing slowly for $\omega$ varying from zero to $1/\tau$ which corresponds to the frequencies of main interest. For $G_1(s)$ on the other hand, the magnitude depends also on $R$, which makes the approximation more sensitive to modeling errors. It is also noted that estimation of $R$ and $C$, using only $G_2(s)$, cannot be successful, since $G_2(s)$ depends on the product of the unknowns $R$ and $C$, and not on these parameters individually. A summary of how estimation errors in $R$ and $C$ affect the modeling error of the approximations of $G_1(s)$ and $G_2(s)$ is given in Table 6.2.

<table>
<thead>
<tr>
<th>Error in parameter</th>
<th>Affect on $G_1(s)$</th>
<th>Affect on $G_2(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x%$ in $R$</td>
<td>$x%$ in $G_1(0)$,</td>
<td>$0%$ in $G_2(0)$,</td>
</tr>
<tr>
<td></td>
<td>less for higher freq.</td>
<td>more for higher freq.</td>
</tr>
<tr>
<td>$x%$ in $C$</td>
<td>$0%$ in $G_1(0)$,</td>
<td>$0%$ in $G_2(0)$,</td>
</tr>
<tr>
<td></td>
<td>more for higher freq.</td>
<td>more for higher freq.</td>
</tr>
<tr>
<td>$x%$ in $R$ or $C$</td>
<td>$\approx x%$ in dominating pole, zero</td>
<td>$\approx x%$ in dominating pole, zero</td>
</tr>
<tr>
<td>$x%$ in $R$ and $C$</td>
<td>$\leq 2x%$ in dominating pole, zero</td>
<td>$\leq 2x%$ in dominating pole, zero</td>
</tr>
<tr>
<td></td>
<td>$x%$ in $G_1(0)$</td>
<td>$x%$ in $G_1(0)$</td>
</tr>
</tbody>
</table>

Table 6.2: Effect of modeling errors.

The conclusion of these observations is that it is more difficult to estimate the unknown parameters using $G_2(s)$ when the indirect approaches described in Section 5.3 are considered. In principle, both $G_1(s)$ and $G_2(s)$ could be used at the same time for determination of $\theta$, but here only the approximations of $G_1(s)$ have been used.

6.5 Existence of local minima

When identifying the parameters using a non-linear least squares criterion, the algorithm might get stuck at a local minimum. If this is a risk, an alternative search algorithm, for example an exhaustive search, might be considered. It is thus important to know if the loss function never provides any false minima, i.e. the resulting estimated parameters are always the best possible. However, an example contradicting this utopia is easily produced.

Consider the second transfer function only, $G_2(s) = 1 / \cosh(\sqrt{\omega/\tau})$. In the degenerated case, the input signal $T_i(s)$ can be chosen as a sinusoidal. This assumption is not entirely unrealistic, since the temperature $T_e$ mainly varies.
with the day rhythm and thus has a time period of 24 h. The minimization criterion, cf (5.6), can be written as

\[
V(\tau) = E \left[ Y(s) - Y^m(s, \tau) \right]^2
= E \left[ \{G_2(s, \tau_0) - G_2^m(s, \tau)\} U(s) \right]^2
= E \left[ \varepsilon(\tau_0, \tau) \right]^2
\]

(6.6)

where \(\tau_0\) equals the true parameter and \(\tau\) is the model parameter. Note that \(G_2(s)\) depends only on \(\tau\) and not on the whole parameter vector \(\theta\). The following input signal is used,

\[
u(t) = A_u \sin(\omega t) \quad (6.7)
\]

Filtering the input (6.7) results in a sinusoid with new amplitude and phase. The error function \(\varepsilon(\tau_0, \tau)\) can thus be rewritten in the time domain as

\[
\varepsilon(t, \tau_0, \tau) = \int_0^t (g_2(t - v, \tau_0) - g_2^m(t - v, \tau)) u(v) dv = A_1 \sin(\omega t + \varphi_1) - A_2 \sin(\omega t + \varphi_2)
\]

(6.8)

where \(g_2^m(t, \tau)\) is the weighting function, that is, the inverse Laplace transform of transfer function \(G_2^m(s, \tau)\). In order to determine \(A_1, A_2, \varphi_1\) and \(\varphi_2\), the transfer function \(G_2(s)\) is rewritten as

\[
G_2(s, \tau_0) = \frac{1}{\cosh \sqrt{i \omega \tau_0}}
= \frac{1}{\cosh(x_0) \cos(x_0) + i \sinh(x_0) \sin(x_0)}
= \frac{\cosh(x_0) \cos(x_0) - i \sinh(x_0) \sin(x_0)}{(\cosh(x_0) \cos(x_0))^2 + (\sinh(x_0) \sin(x_0))^2}
\]

(6.9)

where

\[
x_0 \triangleq \sqrt{\frac{\omega \tau_0}{2}}
\]

(6.10)

is used. Similarly, \(G_2^m(s, \tau)\) is expressed using \(x \triangleq \sqrt{\omega \tau/2}\). Using (6.9), the amplitude \(A_1\) now can be expressed as

\[
A_1 = A_u |G_2(s, \tau_0)| = A_u \frac{1}{\sqrt{\cosh^2(x_0) \cos^2(x_0) + \sinh^2(x_0) \sin^2(x_0)}}
\]

(6.11)
and the phase equals

$$\varphi_1 = -\arctan\left(\frac{\sinh(x_0) \sin(x_0)}{\cosh(x_0) \cos(x_0)}\right) \quad (6.12)$$

The corresponding holds for $A_2$ and $\varphi_2$ (substitute $\tau_0$ by $\tau$, and $x_0$ by $x$). The error function $\varepsilon(t, \tau_0, \tau)$ can now be further developed.

$$\varepsilon(t, \tau_0, \tau) = A_1 \sin(\omega t + \varphi_1) - A_2 \sin(\omega t + \varphi_2)$$
$$= (A_1 \cos(\varphi_1) - A_2 \cos(\varphi_2)) \sin(\omega t)$$
$$+ (A_1 \sin(\varphi_1) - A_2 \sin(\varphi_2)) \cos(\omega t)$$
$$= A_4 \sin(\omega t) + A_5 \cos(\omega t)$$
$$= \sqrt{A_4^2 + A_5^2} \sin(\omega t + \arctan(A_5/A_4)) \quad (6.13)$$

The criterion (6.6) can now be expressed as

$$V(\tau) = E[\varepsilon(t, \tau_0, \tau)]^2$$
$$= \frac{1}{2} (A_4^2 + A_5^2)$$
$$= \frac{1}{2} (A_1 \cos(\varphi_1) - A_2 \cos(\varphi_2))^2 + \frac{1}{2} (A_1 \sin(\varphi_1) - A_2 \sin(\varphi_2))^2$$
$$= \frac{1}{2} [A_1^2 + A_2^2 - 2A_1 A_2 \cos(\varphi_1 - \varphi_2)] \quad (6.14)$$

In the following example, $V(\tau)$ is computed for different values of $\omega \tau$ and $\omega \tau_0$. Figure 6.3 displays the contour plot of $V(\tau)$ for varying $\omega \tau$ and $\omega \tau_0$. The frequency $\omega$ should in a real situation equal $\omega = 2\pi f \approx 2\pi/24 \approx 0.26$ [rad/h]. If the true parameter $\tau_0$ is assumed to be in the interval $\tau_0 \in [6 \ 150]$ [h], then $\omega \tau_0 \in [1.57 \ 40]$ [rad]. The exact value of $\tau_0$ depends on the wall element. In order to cover the range of interest, the contour plot is studied in the interval $[1 \ 100]$. The minima of $V(\tau)$ (corresponding to true and false estimates $\tau$) for different $\tau_0$ where found. With the resolution used in Figure 6.3, it is not possible to detect any local minima, only the global minimum is clearly visible. For each fixed $\tau_0$, the global minimum appears at $\tau = \tau_0$, as $V(\tau_0) = 0$. A better resolution of Figure 6.3, however, shows the existence of local minima. Such a result is illustrated in Figure 6.4. The criterion function $V(\tau)$ is computed for values of $\omega \tau$ up to 500. It is clear that several minima exist. The global minimum is always the minimum corresponding to the smallest value of $\omega \tau$. If it is assumed that the input signal, i.e. the outdoor temperature, varies mainly with a 24 h period, values of $\omega \tau$ larger than 40 can be considered as unrealistic. Therefore, only the global, true minimum remains when evaluating the criterion $V(\tau)$, cf Figure 6.4 in the layer $\omega \tau < 40$. 
Figure 6.3: Contour plot of $V(\tau)$ (6.14).

Figure 6.4: Global and local minima of $V(\tau)$ (6.14) corresponding to values of $\omega \tau_0 \in [1 \, 50]$. 
If the usage of a sinusoid as the input signal is considered to be a too rough simplification, the input signal can instead be modeled as a sum of sinusoids,

\[ u(t) = \sum_{j=1}^{k} A_{uj} \sin(\omega_j t + \varphi_j) \]  

(6.15)

The criterion function would then correspondingly be a sum of squared amplitudes

\[ V(\tau) = \sum_{j=1}^{k} \frac{A_{uj}^2(\omega_j)}{2} \]  

(6.16)

where the dominating term in (6.16) corresponds to the main periodicity of the input signal.

In the following, the assumptions made above, that only the second part of the transfer function, \( G_2(s) \), is used, and that the input signal is a clean sinusoid, are withdrawn. The input signals are generated according to Section 6.2, and the complete minimization criteria in Chapter 5 are used in the remaining of this chapter.

In order to determine the possibility of local minima in this case, the shape of the criteria is examined by evaluating them for various values of \( R \) and \( C \). If negative parameter values are allowed, it can be shown that several minima exist for some of the criteria, one of which corresponds to the true solution. Since it is known that the material constants are positive, the minima resulting from one or two negative parameters can be eliminated. The restrictions \( R > 0, C > 0 \) are therefore put on the estimates. Since values of \( \tau_0 \in [6 \text{ 150}] \) [11] are assumed to be of main interest, \( R \) varying from 1 to 30 [C/m²/W] and \( C \) varying from 1 to 30 [Wh/m²^2] have been used in this study. This restriction was also made in order to limit the computational load.

As examples, the criteria (5.4) and (5.9) are plotted for different integer values of \( R \) and \( C \) in the estimated model. The model order was here chosen as \( n = 50 \) and no noise was applied to any of the signals. Results are shown in Figures 6.5 – 6.7. Note that the restriction of using integer values of \( R \) and \( C \) is of course not used when estimating the parameters \( R \) and \( C \) in Section 6.7.

**Criterion for direct approach in the time domain**

In Figure 6.5 the criterion \( V_1 \) (5.4) for the direct time domain approach in Section 5.1 is shown. Criterion values above 10 are ignored in the plot, in order to show the shape of the function. The criterion function has a smooth surface leaning towards the minimum point, which corresponds to the true parameter values.

**Criterion for indirect approach in the frequency domain**

Figure 6.6 shows the criterion \( V_3 \) associated with the transfer function \( G_1(s) \)
for positive values of $\theta$ only. The weighting function $Q_3 \equiv 1$. The minimum is at $R = 10$ $[\text{Cm}^2/\text{W}]$, $C = 15$ $[\text{Wh}/\text{Cm}^2]$, which are also the true parameter values. Criterion values above 9 are ignored in the plot.

Figure 6.7 displays the criterion $V_3$ associated with the transfer function $G_2(s)$, where $Q_3 \equiv 1$. Several global minima exist for different positive values of $R$ and $C$. In fact, a minimum is found whenever $RC \approx R_C$. Some minima, corresponding to integer values of $R$ and $C$, are marked by circles in the figure. The fact that there are more than one minimum, is also obvious from the second part of (3.13). The same estimated transfer function results for different values of the parameters, since only the product $RC = \tau$ affects $G_2(s)$.

As a conclusion, analyzing the criteria (5.4), (5.6), (5.9), (5.10) and (5.11) for some values of the parameters, shows that the criteria are well-behaved if only positive parameters are considered. The minimum is then given, as expected, close to the true parameters in the examples. The deviation, due to the approximation introduced by using $n < \infty$, is small. For the indirect approach uniqueness problems occur if only $G_2(s)$ is used. This is natural as $G_2(s)$ depends only on one parameter, namely $\tau$, and not on $R$ and $C$ individually, see (3.13).
Figure 6.6: Mesh plot of criterion function $V_3$, (5.9), using $G_1(i\omega)$ corresponding to Section 5.3. The plot shows the criterion function around the minimum only. The parameters corresponding to the minimum point are $R = 10$ [°Cm$^2$/W] and $C = 15$ [Wh/°Cm$^2$], which also are the true parameter values.

Figure 6.7: Mesh plot of criterion function $V_3$ using $G_2(i\omega)$ corresponding to Section 5.3. The minimum points are indicated by circles in the figure. The true parameter values are $R = 10$ [°Cm$^2$/W] and $C = 15$ [Wh/°Cm$^2$].
6.6 Model validation

If \( y^m(t, \theta) \) is obtained from the approximate finite order model (4.3)–(4.7), some numerical errors are introduced due to the discretization in space. The difference between the approximated temperature \( T_k \) and the exact temperature in each point \( k \) converges as \( O(\Delta^2) \), i.e., the introduced error decreases quadratically as the number of layers increases. One way to estimate the error more accurately, is to use Richardson extrapolation [16], i.e., derive the model for two different model orders, e.g., \( n \) and \( 2n \), using (4.3)–(4.7). After conversion to discrete time, which is assumed to introduce negligible errors compared to the errors due to finite order models, \( y^m(n, k) \) and \( y^m(2n, k) \) result. The model outputs can then, according to the statement above, be approximated as

\[
y^m(n, k) \approx y^m(k) + \frac{C(kh)}{n^2} \quad (6.17)
\]

\[
y^m(2n, k) \approx y^m(k) + \frac{C(kh)}{4n^2} \quad (6.18)
\]

where \( y^m(k) \) is an improved approximation of the PDE and \( \{C(kh)\} \), \( k = 1, 2, \ldots, N \), is a vector of proportional constants. Comparing the results of (6.17) and (6.18), \( C(kh) \) and \( y^m(k) \) are computed as

\[
C(kh) \approx \frac{4n^2}{3} (y^m(n, k) - y^m(2n, k)) \quad (6.19)
\]

\[
y^m(k) \approx \frac{4}{3} y^m(2n, k) - \frac{1}{3} y^m(n, k) \quad (6.20)
\]

The relative error may be defined as

\[
R(n) = \sqrt{\frac{\sum_{k=1}^{N} C^2(kh)}{\sum_{k=1}^{N} y^{m2}(k)}} \approx \frac{2}{\sqrt{\sum_{k=1}^{N} \left( y^m(n, k) - \frac{C(kh)}{n^2} \right)^2}} \quad (6.21)
\]

Inserting the expression of \( C(kh) \) from (6.19) gives the relative error as function of model outputs,

\[
R(n) = 4\sqrt{\frac{\sum_{k=1}^{N} (y^m(n, k) - y^m(2n, k))^2}{\sum_{k=1}^{N} (4y^m(2n, k) - y^m(n, k))^2}} \quad (6.22)
\]

The approximate model order required to obtain a certain error can be determined from \( R(n) \), see Figure 6.8, which shows the relative error \( R(n) \) as a function of the model order \( n \). Here the simulation is made using one realization of
the data with $\theta^T = (R, C)$ where $R = 10 \text{[\,Cm}^2/\text{W}\,]$ and $C = 15 \text{[Wh/}^{\circ}\text{Cm}^2\,]$, generated according to Section 6.2, is the parameter vector defined in (5.3). The relative error is determined from (6.21), where $C(kh)$ is obtained from (6.19) using $n = 10$. In order to obtain a relative error of, for example, 1%, a model order larger than 17 must be chosen for the particular data realization used here, of Figure 6.8. Other realizations give, of course, slightly different values of $R(n)$.

It would be useful to have methods to determine an appropriate model order, using available data. The resulting ‘best’ model order differs depending on which estimation method that is used. It is extra important to determine a good model order when there is a risk for numerical problems when a too high model order is used. One way to compare two different model structures, is using the $F$-test [5]. Another way to determine the appropriate model order is to utilize the Akaike’s information criterion (AIC). This test also penalizes high-order models. The AIC criterion is defined as

$$AIC = N \log V(n) + 2p \quad (6.23)$$

where $V(n)$ is the minimum value of the loss function for model order $n$. The quantity $p$ equals the number of unknown parameters in the scheme, which is $p = 2n$ when an ARX model of order $n$ is fitted. The following validation scheme is used.
Scheme for model validation

1. Choose a maximum model order $N_{\text{MAX}}$
2. Choose model order $n = 1$.
3. Determine the AIC (6.23) for the specific model order $n$.
4. If warnings are given that the condition number when solving the normal equation is too large, go to Step 6.
5. Set $n = n + 1$. If $n \leq N_{\text{MAX}}$, go to Step 3, else go to Step 6.
6. Choose the model order $n$ which gives the smallest AIC value, but does not give numerical problems.

This test has been performed for Method 3, see Section 5.3, (5.9), where it is important not to choose a too large model order which would result in numerical problems. Numerical problems may occur when the condition number of the normal equation is large which typically happens for black-box models of high order. The loss function $V(n)$ in the AIC criterion is in this study chosen to be implemented as (cf (5.9))

$$V(n) = \sum_{k=0}^{K-1} [G_k^n(i\omega(k)) - G_1(i\omega(k), \hat{\theta})]^2$$ (6.24)

where $G_k^n(i\omega)$ originates from the black-box model approximation of order $n$. The estimate $G_1(i\omega, \hat{\theta})$ comes from the first part of the infinite order model (3.13) using approximated parameters $\hat{R}$ and $\hat{C}$. The parameter vector $\hat{\theta} = (\hat{R}, \hat{C})^T$ was determined by minimization of (3.9) for model order $n$, and $\{\omega(k)\}$ is a sequence of selected frequency points. Figure 6.9 shows the resulting AIC values for different model orders up to order 17, where the first warning about numerical problems occurred. Model order $n = 12$ gives the smallest AIC value, which then is the best model order according to the scheme above. For larger model orders it can be observed that the criterion (6.23) increases, probably due to large condition number of the normal equation, leaving too few bits for a good numerical precision in the calculations. Note also that a local minimum occurs for $n = 4!$.

The model validation scheme was also applied on Methods 1–2. Here, there is no risk for numerical problems due to large model orders. The AIC criterion is implemented using the loss function

$$V(n) = \frac{1}{N} \sum_{k=1}^{N} [y(kh) - y^m(kh, \theta)]^2$$ (6.25)
where \( y(kh) \) is the measured system output and where \( y^m(kh, \theta) \) is the model output, cf (5.4). Figure 6.10 shows the resulting AIC values for Method 1 for model orders varying from 2 to 54. Only values for every fourth order are computed. It is not surprising that the AIC decreases for increasing model orders. Since \( \dim \theta = 2 \), it is not likely that the Hessian \( V'' \) becomes badly conditioned, even for large \( n \). There is thus no reason to expect any numerical problems for the direct methods. The result indicates that as \( n \) grows, the model output \( y^m(kh, \theta) \) will resemble the system output \( y(kh) \) better and better. The computational load increases of course accordingly, cf Section 6.3.

### 6.7 Results of numerical evaluation

The estimation methods are tested using data generated according to Section 6.2 under various conditions, namely with and without noise on the output and with and without prefiltering of the signals. A summary of the different cases are given in Table 6.3 together with the section numbers in which they are treated.

<table>
<thead>
<tr>
<th>No prefiltering</th>
<th>Noise free data</th>
<th>Noisy data</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prefiltering</td>
<td>Section 6.7.1</td>
<td>Section 6.7.2</td>
</tr>
<tr>
<td></td>
<td>Section 6.7.1</td>
<td>Section 6.7.2</td>
</tr>
</tbody>
</table>

*Table 6.3: The different cases treated.*
6.7 Results of numerical evaluation

![Graph with AIC values for Methods 1-2](image)

**Figure 6.10:** AIC for determination of model order for Methods 1-2.

Simulations have been performed using ten different realizations of the data. For each realization, both a noise free and a noise corrupted output have been considered. Thus, 20 different signal sets have been used, where the same noise free data have been used in every two sets, one of which was corrupted by noise on the output. This means that comparison between the cases can be made without concern that the input signals have been different. An average of the resulting parameter estimates is given in Tables 6.4-6.7 and 6.9-6.12. When an approximate transfer function has been computed (indirect approaches), logarithmic distributed frequencies have been considered in the interval $\omega \in [10^{-4}, 1] \text{ [rad/h]}$, where the frequency band is supposed to cover frequencies from a value close to zero to frequencies well above $1/\tau$. In the examples $\tau = RC$ equals 150 [h], giving $1/\tau = 0.0067 \text{[h}^{-1}]$.

For a few data realizations and model orders, the algorithms give quite bad estimates. This can typically happen for Methods 3-5, where a large model order may give rise to bad numerical accuracy due to a large condition number of the normal equation. A scheme for evaluation of parameter estimates is implemented in order to detect such situations. If the scheme indicates that for a particular data set, the estimate of the frequency function $\hat{G}(i\omega)$ is not reliable, that particular realization is not used in the final result presented in the tables in the following sections. It is indicated in the tables if there are any rejected realizations.
Scheme for evaluation of parameter estimates

1. Estimate the parameters $R$ and $C$ using the whole data series of length $N$, resulting in $\tilde{R}$, $\tilde{C}$.

2. Estimate the parameters $R$ and $C$ using the first half of the data series, i.e. $1 : N/2$, resulting in $R_1$, $C_1$.

3. Estimate the parameters $R$ and $C$ using the second half of the data series, i.e. $N/2 + 1 : N$, resulting in $R_2$, $C_2$.

4. Since a long data series is used in the simulations, steps 2 and 3 should give similar results. Therefore a simple comparison of the resulting estimates gives an indication of the reliability and accuracy of the estimates. Compute

$$\Delta_R = \frac{\tilde{R} - \tilde{R}_1}{\tilde{R}} + \frac{\tilde{R} - \tilde{R}_2}{\tilde{R}}$$  \hspace{1cm} (6.26)$$

$$\Delta_C = \frac{C - C_1}{C} + \frac{C - C_2}{C}$$  \hspace{1cm} (6.27)$$

The result of the data realization is rejected if either $|\Delta_R| > 0.1$ or $|\Delta_C| > 0.1$, which corresponds to an allowed discrepancy of 10%.

6.7.1 Identification from noise free data

Results of the different estimation approaches for various model orders are summarized in Tables 6.4–6.7. In Tables 6.5 and 6.7, model orders up to 12 only are used. This is due to the numerical problems that can occur for higher model orders for Methods 3–5. The model (4.3)–(4.7) used in Methods 1–2 does not give rise to such numerical problems.

As can be seen, a bias appears in many of the estimates. Since no noise is present here, the estimation error is due to an imperfect approximation of the transfer functions, cf Chapter 5. There are two important aspects to consider when approximating the transfer functions.

i. A small model order $n$ cannot describe the system dynamics correctly and results in bad parameter estimates.

ii. For the black-box model used in Methods 3–5, a large model order often leads to numerical problems which in turn may give bad estimates.

The true transfer functions of infinite order are approximated using the difference approximation method in Methods 1, 2 and 5, and using an ARX model of finite order in Methods 3–5. Since finite order models are used, bias in the transfer
## 6.7 Results of numerical evaluation

<table>
<thead>
<tr>
<th>Model order</th>
<th>Results from Methods 1–2</th>
<th>Rejections</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>10</td>
<td>$R = 7.80$</td>
<td>$R = 9.72$</td>
</tr>
<tr>
<td></td>
<td>$\hat{C} = 17.81$</td>
<td>$\hat{C} = 23.70$</td>
</tr>
<tr>
<td>15</td>
<td>$\hat{R} = 8.62$</td>
<td>$\hat{R} = 10.10$</td>
</tr>
<tr>
<td></td>
<td>$\hat{C} = 17.01$</td>
<td>$\hat{C} = 20.68$</td>
</tr>
<tr>
<td>20</td>
<td>$\hat{R} = 9.04$</td>
<td>$\hat{R} = 10.07$</td>
</tr>
<tr>
<td></td>
<td>$\hat{C} = 16.57$</td>
<td>$\hat{C} = 18.85$</td>
</tr>
<tr>
<td>30</td>
<td>$\hat{R} = 9.44$</td>
<td>$\hat{R} = 10.00$</td>
</tr>
<tr>
<td></td>
<td>$\hat{C} = 16.08$</td>
<td>$\hat{C} = 17.21$</td>
</tr>
<tr>
<td>40</td>
<td>$\hat{R} = 9.62$</td>
<td>$\hat{R} = 9.95$</td>
</tr>
<tr>
<td></td>
<td>$\hat{C} = 15.80$</td>
<td>$\hat{C} = 16.48$</td>
</tr>
<tr>
<td>50</td>
<td>$\hat{R} = 9.71$</td>
<td>$\hat{R} = 10.00$</td>
</tr>
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<td></td>
<td>$\hat{C} = 15.62$</td>
<td>$\hat{C} = 17.21$</td>
</tr>
<tr>
<td>∞</td>
<td>$\hat{R} = 10.00$</td>
<td>$\hat{R} = 10.00$</td>
</tr>
<tr>
<td></td>
<td>$\hat{C} = 15.00$</td>
<td>$\hat{C} = 15.00$</td>
</tr>
</tbody>
</table>

### Table 6.4: Result of parameter estimation, Methods 1–2, no noise, no prefiltering of the signals. In the true system, $R$ equals 10 [°Cm²/W] and $C$ equals 15 [Wh/°Cm²]. The column Rejections indicates how many realizations that were rejected for each method.

<table>
<thead>
<tr>
<th>Model order</th>
<th>Results from Methods 3–5</th>
<th>Rejections</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>$R = 10.39$</td>
<td>$R = 10.38$</td>
</tr>
<tr>
<td></td>
<td>$\hat{C} = 10.47$</td>
<td>$\hat{C} = 5.08$</td>
</tr>
<tr>
<td>6</td>
<td>$\hat{R} = 15.86$</td>
<td>$\hat{R} = 15.86$</td>
</tr>
<tr>
<td></td>
<td>$\hat{C} = 23.33$</td>
<td>$\hat{C} = 18.47$</td>
</tr>
<tr>
<td>8</td>
<td>$\hat{R} = 9.59$</td>
<td>$\hat{R} = 9.57$</td>
</tr>
<tr>
<td></td>
<td>$\hat{C} = 10.36$</td>
<td>$\hat{C} = 1.90$</td>
</tr>
<tr>
<td>10</td>
<td>$\hat{R} = 10.30$</td>
<td>$\hat{R} = 10.30$</td>
</tr>
<tr>
<td></td>
<td>$\hat{C} = 15.13$</td>
<td>$\hat{C} = 14.84$</td>
</tr>
<tr>
<td>11</td>
<td>$\hat{R} = 10.16$</td>
<td>$\hat{R} = 10.30$</td>
</tr>
<tr>
<td></td>
<td>$\hat{C} = 14.88$</td>
<td>$\hat{C} = 14.84$</td>
</tr>
<tr>
<td>12</td>
<td>$\hat{R} = 10.01$</td>
<td>$\hat{R} = 10.01$</td>
</tr>
<tr>
<td></td>
<td>$\hat{C} = 14.71$</td>
<td>$\hat{C} = 14.86$</td>
</tr>
</tbody>
</table>

### Table 6.5: Result of parameter estimation, Methods 3–5, no noise, no prefiltering of the signals. In the true system, $R$ equals 10 [°Cm²/W] and $C$ equals 15 [Wh/°Cm²].
function estimates cannot be avoided. Exceptions are Method 1 and 2 for the case when infinite order models (3.13) are used, cf Table 6.4.

For Method 3, the whole transfer function estimate is used in the criterion. This gives, compared to Method 4 where only three points in the transfer function are used, some drawbacks:

(1) The estimated transfer function must be fitted for all frequencies, also higher frequencies for which it is more difficult to estimate it correctly.

(2) The whole transfer function approximated using the ARX model must be converted to continuous time in order to build the criterion. For Method 4, only three points in the frequency function need to be transformed. These points correspond to the dominating pole, the dominating zero and the static gain.

In Tables 6.6–6.7 the result is displayed for the noise free case, but when the data was prefiltered through a low pass filter. The filter is implemented as a second order Butterworth filter with a cut-off frequency of $1/(12t)$.

Comparing the results presented in Tables 6.4, 6.5 and 6.6, 6.7, it is obvious that a better estimate can be gained by prefiltering of the data. Note also that

<table>
<thead>
<tr>
<th>Model order n</th>
<th>Results from Methods 1–2</th>
<th>Rejections 1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$\hat{R} = 8.64$ $\hat{R} = 9.42$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>$\hat{C} = 18.05$ $\hat{C} = 18.93$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>15</td>
<td>$\hat{R} = 9.18$ $\hat{R} = 9.70$</td>
<td>-</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>$\hat{C} = 17.10$ $\hat{C} = 17.63$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>20</td>
<td>$\hat{R} = 9.42$ $\hat{R} = 9.72$</td>
<td>-</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>$\hat{C} = 16.59$ $\hat{C} = 16.81$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>30</td>
<td>$\hat{R} = 9.65$ $\hat{R} = 9.83$</td>
<td>-</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>$\hat{C} = 16.06$ $\hat{C} = 16.16$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>40</td>
<td>$\hat{R} = 9.75$ $\hat{R} = 9.83$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>$\hat{C} = 15.78$ $\hat{C} = 15.75$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>50</td>
<td>$\hat{R} = 9.90$ $\hat{R} = 9.83$</td>
<td>-</td>
<td>-</td>
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<tr>
<td></td>
<td>$\hat{C} = 15.61$ $\hat{C} = 16.16$</td>
<td>-</td>
<td>-</td>
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<tr>
<td>$\infty$</td>
<td>$\hat{R} = 10.00$ $\hat{R} = 10.00$</td>
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<td>-</td>
</tr>
<tr>
<td></td>
<td>$\hat{C} = 15.00$ $\hat{C} = 15.00$</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 6.6: Result of parameter estimation in the noise free case, Methods 1–2. The data are filtered through a low pass filter. In the true system, $R$ equals 30 $[\text{Vcm}^2/\text{W}]$ and $C$ equals 15 $[\text{Wh}^{\text{cm}^{-2}}]$. The column Rejections indicates how many realizations that were rejected for each method.
### Table 6.7: Result of parameter estimation in the noise free case, Methods 3-5.
The data are filtered through a low pass filter. In the true system, $R$ equals 10 $\degree\text{C}\cdot\text{m}^2/\text{W}$ and $C$ equals 15 [Wh/$\text{C}^2\cdot\text{m}^2$].

<table>
<thead>
<tr>
<th>Model order</th>
<th>Results from Methods 3–5</th>
<th>Rejections</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>$\hat{R} = 11.00$</td>
<td>$\hat{R} = 11.02$</td>
</tr>
<tr>
<td>&amp; $\hat{C} = 17.61$</td>
<td>$\hat{C} = 28.38$</td>
<td>$\hat{C} = 24.95$</td>
</tr>
<tr>
<td>6</td>
<td>$\hat{R} = 10.00$</td>
<td>$\hat{R} = 10.00$</td>
</tr>
<tr>
<td>&amp; $\hat{C} = 14.18$</td>
<td>$\hat{C} = 12.37$</td>
<td>$\hat{C} = 17.95$</td>
</tr>
<tr>
<td>8</td>
<td>$\hat{R} = 10.34$</td>
<td>$\hat{R} = 10.34$</td>
</tr>
<tr>
<td>&amp; $\hat{C} = 15.09$</td>
<td>$\hat{C} = 14.79$</td>
<td>$\hat{C} = 18.13$</td>
</tr>
<tr>
<td>10</td>
<td>$\hat{R} = 10.13$</td>
<td>$\hat{R} = 10.10$</td>
</tr>
<tr>
<td>&amp; $\hat{C} = 14.81$</td>
<td>$\hat{C} = 15.61$</td>
<td>$\hat{C} = 17.18$</td>
</tr>
<tr>
<td>11</td>
<td>$\hat{R} = 10.44$</td>
<td>$\hat{R} = 10.11$</td>
</tr>
<tr>
<td>&amp; $\hat{C} = 15.40$</td>
<td>$\hat{C} = 15.67$</td>
<td>$\hat{C} = 17.17$</td>
</tr>
<tr>
<td>12</td>
<td>$\hat{R} = 10.10$</td>
<td>$\hat{R} = 10.09$</td>
</tr>
<tr>
<td>&amp; $\hat{C} = 14.82$</td>
<td>$\hat{C} = 13.87$</td>
<td>$\hat{C} = 16.78$</td>
</tr>
</tbody>
</table>

The estimated parameters remain unbiased if the infinite order model is used (Methods 1–2).

In Figures 6.11 and 6.13, the bias and standard deviation of estimates of Method 1 and 3 are presented, as a complement to the tables. Figure 6.12 displays the bias as in Figure 6.11, but in logarithmic scale. The curves have the slope $–1$, indicating that the bias decreases with the model order as $O(1/n)$. This will be theoretically verified in Chapter 9.

It can be noted that $\hat{R}$ is more accurate than $\hat{C}$ for equal model orders. The fact that it is easier to estimate $R$ than $C$ can be theoretically verified by considering the second derivative of the loss function. The bias, as well as the covariance, of the estimates are dependent on the inverse of the Hessian, $(V''n)^{-1}$, [5]. Therefore, $(V''n)^{-1}$ gives a good indication of the variance of $\hat{R}$ and $\hat{C}$. For determination of $\hat{\theta}$ it is important that $V''n$ is nonsingular. The requirement of non-singularity holds under quite general conditions when the system is in the model set. For underparameterized models, as in this case, the condition number gives important information about the solution uncertainty. The condition number of the Hessian is given by

$$\text{cond}(V'') = \|V''\| \|V''n^{-1}\|$$

If the condition number of the Hessian is small, the Hessian is said to be well conditioned. If the condition number is large, the Hessian is badly conditioned,
Figure 6.11: Bias (solid lines) and standard deviation (dashed lines) of $\hat{R}$ and $\hat{C}$ obtained from Method 1 in the noise free case. Lines with stars (*) indicate no prefiltering and circles (o) indicate prefiltering of data.

Figure 6.12: Bias of $\hat{R}$ (plot (a)) and $\hat{C}$ (plot (b)) from Method 1 in the noise free case. Stars (*) indicate no prefiltering and circles (o) indicate prefiltering of data.
and the relative uncertainty of the solution is large. This means that a small perturbation of the data can give a large error in the solution, which of course in undesirable.

The Hessian,

$$V'' = \begin{pmatrix}
\frac{\partial^2 V}{\partial R^2} & \frac{\partial^2 V}{\partial R \partial C} \\
\frac{\partial^2 V}{\partial C \partial R} & \frac{\partial^2 V}{\partial C^2}
\end{pmatrix}$$

(6.28)

can be approximated as

$$\begin{cases}
\frac{\partial^2 V}{\partial R^2} \approx \frac{1}{(\Delta R)^2} [V(R - \Delta R, C) - 2V(R, C) + V(R + \Delta R, C)] \\
\frac{\partial^2 V}{\partial C^2} \approx \frac{1}{(\Delta C)^2} [V(R, C - \Delta C) - 2V(R, C) + V(R, C + \Delta C)] \\
\frac{\partial^2 V}{\partial R \partial C} \approx \frac{1}{(\Delta R)(\Delta C)} [V(R + \Delta R, C + \Delta C) - V(R + \Delta R, C - \Delta C) - V(R - \Delta R, C + \Delta C) + V(R - \Delta R, C - \Delta C)]
\end{cases}$$

(6.29)

where $\frac{\partial^2 V}{\partial R^2}$ and $\frac{\partial^2 V}{\partial C^2}$ are small changes in the $R$ and $C$ directions, respectively. The derivatives in (6.29) are derived using a central difference
quotient in $R$ and $C$, respectively. After series expansion, the approximation (6.29) can be shown to give the error

$$\text{err}(V^\prime) = \begin{pmatrix}
O(\Delta R^2) & \frac{\Delta R^2}{\Delta C} + O\left(\frac{\Delta C^2}{\Delta R^2}\right) \\
O\left(\frac{\Delta C^2}{\Delta R^2}\right) & O(\Delta C^2)
\end{pmatrix}$$

Choosing e.g. $\Delta R = 0.1$ [$^\circ$Cm$^2$/W] and $\Delta C = 0.1$ [Wh/$^\circ$Cm$^2$] gives for one particular realization of the data for the different methods the second derivatives according to Table 6.8. Also, the condition numbers of $V^\prime$ for the methods are displayed. For Methods 1–2, model order 50 was used, and for Methods 3–5, model order 12 was used. Note that the condition number is equal for the Hessian and its inverse. It can be observed that Method 1 is numerically the most well conditioned method. Method 2 also indicates good numerical behavior, while Method 4 shows a large condition number. A large model order may cause numerical problems for Methods 3–5, cf Section 6.6 and the fact that Method 4 gives comparably many rejections for large model orders in the noise free case, see Table 6.7. However, a smaller condition number results for Method 4 when a low model order is used.

<table>
<thead>
<tr>
<th>Method</th>
<th>$\sigma^2_{R \mid V}$</th>
<th>$\sigma^2_{C \mid V}$</th>
<th>$\sigma^2_{R \mid V}$</th>
<th>Condition number</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.201</td>
<td>0.080</td>
<td>-0.051</td>
<td>3.58</td>
</tr>
<tr>
<td>2</td>
<td>275</td>
<td>104</td>
<td>-154</td>
<td>26.5</td>
</tr>
<tr>
<td>3</td>
<td>1.30</td>
<td>0.013</td>
<td>-0.028</td>
<td>100</td>
</tr>
<tr>
<td>4</td>
<td>2.00</td>
<td>3.99 x 10$^{-5}$</td>
<td>6.06 x 10$^{-5}$</td>
<td>5.01 x 10$^4$</td>
</tr>
<tr>
<td>5</td>
<td>65.9</td>
<td>0.681</td>
<td>-1.52</td>
<td>102</td>
</tr>
</tbody>
</table>

**Table 6.8:** The Hessians and their condition numbers for different methods.

Studying the estimated Hessians for the different methods, reflects the fact that the variance of the estimates of $C$ is expected to be larger than that of $R$. As an example, study Method 1 and Method 3 using $n = 50$ and $n = 12$, respectively. Here the inverse of the Hessians are given by, cf (6.28),

$$\begin{pmatrix}
(V_1^\prime(\theta))^{-1} = & \begin{pmatrix} 5.94 & 3.78 \\
3.78 & 14.91 \end{pmatrix} \\
(V_2^\prime(\theta))^{-1} = & \begin{pmatrix} 0.807 & 1.74 \\
1.74 & 80.67 \end{pmatrix}
\end{pmatrix}$$

It can be noted that the $(2, 2)$ elements of (6.31)–(6.32) are larger than the $(1, 1)$ elements. Therefore the bias as well as the variance of $C$ is expected to be greater than that of $R$, according to the discussion above.
6.7.2 Identification from noise-corrupted data

In this test, the system output, $T_i$, contains additional noise. The noise level is 15 dB below the average level of $T_i$, cf Figure 6.2. The noise is white, while the signal is of low pass character. Therefore, the noise affects the higher frequencies more. Tables 6.9–6.10 display the estimated parameters in the case of noisy output signal, $T_i$. (No prefiltering of signals was performed when obtaining the results in the tables).

It is clear that presence of stochastic measurement noise results in parameter estimates with a stochastic errors. Especially Methods 3–5 seem to be sensitive to additive noise. There is hence a need for decreasing the influence of the disturbance. Filtering of the signals is a commonly used way of reducing the relative amount of noise. Since the data are of low pass character (slow variations of the temperature) and since it is known that the transfer functions to be estimated are of low pass character too, it is appropriate to emphasize the lower frequencies.

A simple Butterworth filter of second order with the cut-off frequency 1/12 [h$^{-1}$], corresponding to half the signal period, was applied to the signals involved in the identification procedure. The results are shown in Tables 6.11–6.12. The

<table>
<thead>
<tr>
<th>Model order</th>
<th>Results from Methods 1–2</th>
<th>Rejections</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>10</td>
<td>$R = 7.80$</td>
<td>$R = 9.45$</td>
</tr>
<tr>
<td></td>
<td>$\hat{C} = 17.82$</td>
<td>$\hat{C} = 22.47$</td>
</tr>
<tr>
<td>15</td>
<td>$\hat{R} = 8.62$</td>
<td>$\hat{R} = 9.89$</td>
</tr>
<tr>
<td></td>
<td>$\hat{C} = 17.08$</td>
<td>$\hat{C} = 19.99$</td>
</tr>
<tr>
<td>20</td>
<td>$\hat{R} = 9.05$</td>
<td>$\hat{R} = 9.84$</td>
</tr>
<tr>
<td></td>
<td>$\hat{C} = 16.59$</td>
<td>$\hat{C} = 18.47$</td>
</tr>
<tr>
<td>30</td>
<td>$\hat{R} = 9.45$</td>
<td>$\hat{R} = 9.94$</td>
</tr>
<tr>
<td></td>
<td>$\hat{C} = 16.09$</td>
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</tr>
<tr>
<td>40</td>
<td>$\hat{R} = 9.63$</td>
<td>$\hat{R} = 9.91$</td>
</tr>
<tr>
<td></td>
<td>$\hat{C} = 15.81$</td>
<td>$\hat{C} = 16.35$</td>
</tr>
<tr>
<td>50</td>
<td>$\hat{R} = 9.73$</td>
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</tr>
<tr>
<td>$\infty$</td>
<td>$\hat{R} = 10.01$</td>
<td>$\hat{R} = 9.90$</td>
</tr>
<tr>
<td></td>
<td>$\hat{C} = 15.04$</td>
<td>$\hat{C} = 14.88$</td>
</tr>
</tbody>
</table>

Table 6.9: Result of parameter estimation for Methods 1–2, with noise on the output signal and no prefiltering. In the true system, $R$ equals 10 [°Cm$^2$/W] and $C$ equals 15 [Wh/K°Cm$^2$].
### 6.7 Results of numerical evaluation

<table>
<thead>
<tr>
<th>Model order $n$</th>
<th>Results from Methods 3–5</th>
<th>Rejections $3$</th>
<th>$4$</th>
<th>$5$</th>
<th>$3$</th>
<th>$4$</th>
<th>$5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4$</td>
<td>$R = 4.66 \quad \hat{R} = 6.67 \quad \hat{R} = 6.66$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td></td>
<td>$\hat{C} = 6.27 \quad \hat{C} = 3.78 \quad \hat{C} = 6.58$</td>
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<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>$6$</td>
<td>$\tilde{R} = 6.48 \quad \hat{R} = 6.48 \quad \hat{R} = 6.49$</td>
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<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td></td>
<td>$\tilde{C} = 7.69 \quad \hat{C} = 4.64 \quad \hat{C} = 9.62$</td>
<td>$-$</td>
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<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
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<td>$-$</td>
<td>$-$</td>
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<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td></td>
<td>$\tilde{C} = 8.86 \quad \hat{C} = 5.22 \quad \hat{C} = 10.54$</td>
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<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>$10$</td>
<td>$\tilde{R} = 8.48 \quad \hat{R} = 8.47 \quad \hat{R} = 8.48$</td>
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<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td></td>
<td>$\tilde{C} = 9.92 \quad \hat{C} = 6.09 \quad \hat{C} = 11.43$</td>
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<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>$11$</td>
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<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td></td>
<td>$\tilde{C} = 10.55 \quad \hat{C} = 7.03 \quad \hat{C} = 12.01$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>$12$</td>
<td>$\tilde{R} = 8.99 \quad \hat{R} = 8.98 \quad \hat{R} = 8.99$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td></td>
<td>$\tilde{C} = 10.97 \quad \hat{C} = 7.22 \quad \hat{C} = 12.37$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
</tbody>
</table>

**Table 6.10:** Result of parameter estimation for Methods 3–5, noisy output signal and no prefiltering. In the true system, $R$ equals $30 \left[\text{\textdegree Cm}^2/\text{W}\right]$ and $C$ equals $15 \left[\text{Wh}/\text{Cm}^2\right]$.

<table>
<thead>
<tr>
<th>Model order $n$</th>
<th>Results from Methods 1–2</th>
<th>Rejections $1$</th>
<th>$2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10$</td>
<td>$R = 8.63 \quad \hat{R} = 9.71$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td></td>
<td>$\tilde{C} = 17.95 \quad \hat{C} = 19.31$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>$15$</td>
<td>$\tilde{R} = 9.16 \quad \hat{R} = 9.90$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td></td>
<td>$\tilde{C} = 17.03 \quad \hat{C} = 17.85$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>$20$</td>
<td>$\tilde{R} = 9.41 \quad \hat{R} = 9.93$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td></td>
<td>$\tilde{C} = 16.55 \quad \hat{C} = 17.03$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>$30$</td>
<td>$\tilde{R} = 9.63 \quad \hat{R} = 9.93$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td></td>
<td>$\tilde{C} = 16.03 \quad \hat{C} = 16.22$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>$40$</td>
<td>$\tilde{R} = 9.73 \quad \hat{R} = 9.92$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td></td>
<td>$\tilde{C} = 15.76 \quad \hat{C} = 15.82$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>$50$</td>
<td>$\tilde{R} = 9.78 \quad \hat{R} = 9.93$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td></td>
<td>$\tilde{C} = 15.60 \quad \hat{C} = 16.22$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>$\infty$</td>
<td>$\tilde{R} = 10.03 \quad \hat{R} = 9.91$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td></td>
<td>$\tilde{C} = 15.03 \quad \hat{C} = 14.82$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
</tbody>
</table>

**Table 6.11:** Result of parameter estimation for Methods 1–2, with noise on the output signal. The data has been filtered through a low pass filter prior to the parameter estimation. In the true system, $R$ equals $30 \left[\text{\textdegree Cm}^2/\text{W}\right]$ and $C$ equals $15 \left[\text{Wh}/\text{Cm}^2\right]$. 
Table 6.12: Result of parameter estimation for Methods 3–5, with noise on the output signal. The data has been filtered through a low pass filter prior to the parameter estimation. In the true system, $R$ equals 10 [°Cm²/W] and $C$ equals 15 [Wh/m²Cm²].

The same data series was used here as was used in the case when no filtering was involved.

In Figures 6.14 and 6.15, the bias and standard deviation of estimates of Method 1 and Method 3 are presented, as a complement to the tables.

For all methods, a considerable improvement of identification results was obtained when the signals were low-pass filtered prior to identification. Especially, advantage of filtering could be shown for Methods 3–5.
Figure 6.14: Bias (solid lines) and standard deviation (dashed lines) of $\hat{R}$ and $\hat{C}$ obtained from Method 1 for noisy data. Lines with stars (*) indicate no prefiltering and circles (o) indicate prefiltering of data.

Figure 6.15: Bias (solid lines) and standard deviation of $\hat{R}$ and $\hat{C}$ from Method 3 for noisy data. Stars (*) indicate no prefiltering, circles (o) indicate prefiltering of data.
7 Bias contribution for general approximate models

In the previous chapter the approximation errors in terms of bias and standard deviation were investigated for one particular parameter set $\theta$. The results illustrated the dominance of the bias over the variance of the error, see for example Figure 6.11. It is hence of primary importance to investigate the bias, in order to obtain knowledge on how to keep it at an acceptably low level. This chapter is therefore dedicated to an investigation of how the bias on the estimated parameters can be determined from the available transfer functions or weighting functions. The theory presented in this chapter is general, and applies thus to other problems than the specific application exemplified in this thesis.

A system ('the true dynamics') of infinite order is considered. A model of large, but finite, order is to be fitted to the system. The model is large enough to make the estimated parameter vector approximately equal to the true one, $\hat{\theta} \approx \theta_0$. The purpose of this chapter is to investigate how the bias error $\hat{\theta} - \theta_0$ of a general system is affected by the model order.

A formula for estimation of bias in the parameters is derived first, using standard techniques. Consider the system description

$$y(t) = G(p, \theta_0)u(t) + \nu(t)$$

(7.1)

where $G(p, \theta_0)$ is a transfer function of infinite order and $\theta_0$ is the parameter vector. Here $p$ is the differentiation operator, $p = \frac{d}{dt}$. The quantity $y(t)$ is the system output and $u(t)$ is the input. Further, $\nu(t)$ is the noise. The system is modeled by a finite order model of order $n$,

$$y^m(t) = \hat{G}^n(p, \theta)u(t) + \varepsilon(t)$$

(7.2)

where $\theta$ is the vector of estimated parameters, $y^m(t)$ is the model output and $\varepsilon(t)$ is the error caused by noise and the approximation error. The goal is to obtain a model such that

$$\hat{G}^n(p, \theta) \approx G(p, \theta_0)$$

(7.3)

from which a good estimate of $\theta_0$ may be obtained. The relation (7.3) may be interpreted to hold with good approximation for $p = i\omega$, and for all frequencies of interest. The parameter vector $\theta$ is estimated with an output error method as

$$\hat{\theta}(N) = \arg\min_{\theta} \frac{1}{N} \sum_{i=1}^{N} \left[y(t) - \hat{G}^n(p, \theta)u(t)\right]^2$$

(7.4)

where $N$ is the number of data. If $N$ is large and the data are stationary, the following asymptotic criterion can be used for analyzing the parameter estima-
tion,

\[ V(\theta) = \mathbb{E} \left[ y(t) - G^n(p, \theta) u(t) \right]^2 \]

\[ = \mathbb{E} \left[ \left( G(p, \theta_0) - G^n(p, \theta) \right) u(t) \right]^2 + \mathbb{E} [v(t)]^2 \]

\[ \triangleq W(\theta) + \mathbb{E} [v(t)]^2 \quad (7.5) \]

where it is assumed that \( u(t) \) and \( v(t') \) are uncorrelated for all \( t \) and \( t' \). This assumption is essentially the same as considering the system in open loop operation. The estimated parameter vector \( \hat{\theta} \) is then the minimizing argument of the criterion \( W(\theta) \),

\[ \hat{\theta} = \arg \min_{\theta} W(\theta) \quad (7.6) \]

Assuming that the model order is large enough so that \( \hat{\theta} \approx \theta_0 \), then \( G^n(p, \theta_0) \approx G(p, \theta_0) \) holds. A Taylor expansion of \( W(\theta) \) around \( \theta = \theta_0 \), retaining the first two terms, gives how the model approximation influences the bias \( \hat{\theta} - \theta_0 \), see for example [5, 6]. Then

\[ 0 = \frac{\partial W(\theta)}{\partial \theta} \approx \frac{\partial W(\theta_0)}{\partial \theta} + \frac{\partial^2 W(\theta_0)}{\partial \theta^2} (\hat{\theta} - \theta_0) \quad (7.7) \]

is obtained, which gives the expression for the parameter bias

\[ \hat{\theta} = \hat{\theta} - \theta_0 \approx - \left( \frac{\partial^2 W(\theta_0)}{\partial \theta^2} \right)^{-1} \frac{\partial W(\theta_0)}{\partial \theta} \quad (7.8) \]

where

\[ \frac{\partial W(\theta_0)}{\partial \theta} = 2E \left[ \left( - \frac{\partial G^n(p, \theta_0)}{\partial \theta} u(t) \right) \left( G(p, \theta_0) - G^n(p, \theta_0) \right) u(t) \right] \]

\[ \approx 2E \left[ \left( - \frac{\partial G(p, \theta_0)}{\partial \theta} u(t) \right) \left( G(p, \theta_0) - G^n(p, \theta_0) \right) u(t) \right] \quad (7.9) \]

\[ \frac{\partial^2 W(\theta_0)}{\partial \theta^2} = 2E \left\{ \left( - \frac{\partial G^n(p, \theta_0)}{\partial \theta} u(t) \right)^T \left( - \frac{\partial G^n(p, \theta_0)}{\partial \theta} u(t) \right) \right\} \]

\[ - 2E \left\{ \left( G(p, \theta_0) - G^n(p, \theta_0) \right) u(t) \left( - \frac{\partial^2 G^n(p, \theta_0)}{\partial \theta^2} u(t) \right) \right\} \]

\[ \approx 2E \left\{ \left( - \frac{\partial G(p, \theta_0)}{\partial \theta} u(t) \right)^T \left( - \frac{\partial G(p, \theta_0)}{\partial \theta} u(t) \right) \right\} \]

\[ \approx 2E \left\{ \left( - \frac{\partial G(p, \theta_0)}{\partial \theta} u(t) \right)^T \left( - \frac{\partial G(p, \theta_0)}{\partial \theta} u(t) \right) \right\} \quad (7.10) \]
For a single input single output (SISO) system, the transfer function $G(p, \theta_0)$ is a scalar, but for a system with more than one input or output, $G(p, \theta_0)$ becomes a vector. If the parameter $\theta$ is a vector, which holds true if there are more than one parameter to be estimated, the gradient

$$\frac{\partial G^n(p, \theta_0)}{\partial \theta}$$

in (7.10) becomes a matrix. In this case

$$\frac{\partial^2 G^n(p, \theta_0)}{\partial \theta^2}$$

in (7.10) becomes a tensor. For a definition of tensors, see e. g. [23]. Vectors and matrices are, for example, special classes of tensors.

The first approximation in (7.10) is motivated by the assumption that $G(p, \theta_0) - G^n(p, \theta_0)$ is small as compared to (7.11) for $p = i\omega$. The second approximation follows since (7.11) changes slowly with $n$, which was assumed also in (7.9). This assumption will be left without proof for now. It will be modified for the diffusion model, see Lemma 9.1. The approximation error in (7.8) tends to zero faster than the error $||\theta - \theta_0||$ which makes (7.8) justice. Using (7.8) and the expressions for the gradient and Hessian, the bias contribution $||\theta - \theta_0||$ will be examined for the heat equation. The gradient (7.9) and the Hessian (7.10) can be expressed either in the time domain or in the frequency domain.

The expressions for the gradient and the Hessian can be rewritten into weighting function representations using the transformation

$$\frac{\partial G(p, \theta)}{\partial \theta} u(t) = \int_{-\infty}^{t} h(t - s)u(s)ds \quad (7.13)$$

$$\left(G(p, \theta) - G^n(p, \theta)\right) u(t) = \int_{-\infty}^{t} g^n(t - s)u(s)ds \quad (7.14)$$

In the following, the general relation $||r_u(t)|| \leq ||r_u(0)|| \frac{\delta}{\sigma^2_u}$, where $\sigma^2_u$ is the variance of the input signal $i$, is used. No assumption is here taken regarding the size of the input signal $u(t)$. It may be a scalar or a vector. In the example considered in later chapters, the diffusion model gives that the input signal is of order two, but the theory given here is general.

The gradient and the Hessian are then expressed as the following in the time domain.
\[
\left\| \frac{\partial W(\theta)}{\partial \theta} \right\| \approx 2 \left\| \int_{-t}^{t} h(t-s) r_u(s-s') \hat{g}^{n'}(t-s') \, ds \, ds' \right\|
\]
\[
\leq 2 \int_{-\infty}^{\infty} \int_{-\infty}^{t} \| h(t-s) \| \| r_u(0) \| \| \hat{g}^{n'}(t-s') \| \, ds \, ds'
\]
\[
= 2\sigma_u^2 \int_{-\infty}^{t} \| h(t-s) \| \, ds \int_{-\infty}^{t} \| \hat{g}^{n'}(t-s') \| \, ds'
\]
\[
= 2\sigma_u^2 \int_{0}^{\infty} \| h(t) \| dt \int_{0}^{\infty} \| \hat{g}^{n'}(t) \| dt \tag{7.15}
\]

\[
\frac{\partial^2 W(\theta)}{\partial \theta^2} \approx 2 \int_{-\infty}^{\infty} \int_{-\infty}^{t} h(t-s) r_u(s-s') h^T(t-s') \, ds \, ds' \tag{7.16}
\]

Again, the approximations in (7.15) and (7.16) are assumed to give small errors for large \(n\).

The gradient in (7.9) and the Hessian in (7.10) can alternatively be formulated in the frequency domain as

\[
\frac{\partial W(\theta_0)}{\partial \theta} \approx -2 \int_{-\infty}^{\infty} \left[ \frac{\partial G(-i\omega)}{\partial \theta} \right] \Phi_u(\omega) \left[ G(i\omega) - \hat{G}^n(i\omega) \right]^T d\omega \tag{7.17}
\]

\[
\frac{\partial^2 W(\theta_0)}{\partial \theta^2} \approx 2 \int_{-\infty}^{\infty} \left( \frac{\partial G(-i\omega)}{\partial \theta} \right) \Phi_u(\omega) \left( \frac{\partial G(i\omega)}{\partial \theta} \right)^T d\omega \tag{7.18}
\]

where \(\Phi_u(\omega)\) is the spectral density of the input \(u(t)\). In (7.17), \(G(i\omega) - \hat{G}^n(i\omega)\) is small for large \(n\).

Several special cases can be considered. Different types of input signals enables approximations of the gradient. For convenience, denote

\[
H(i\omega) \triangleq \frac{\partial G(i\omega)}{\partial \theta} \tag{7.19}
\]

\[
\hat{G}^n(i\omega) \triangleq G(i\omega) - \hat{G}^n(i\omega) \tag{7.20}
\]

**Case 1:** Let \(u(t)\) be a sinusoid with angular frequency. Then \(\Phi_u(\omega) = C_u \delta(\omega - \omega_0)\) and

\[
\left\| \frac{\partial W(\theta_0)}{\partial \theta} \right\| \approx 2 \|H(-i\omega_0)\| \|C_u\| \|\hat{G}^n(i\omega_0)\| \tag{7.21}
\]
Case 2: Let \( u(t) \) be white noise, \( \Phi_u(\omega) = I \). Then

\[
\left\| \frac{\partial W(\theta_0)}{\partial \theta} \right\| \approx 2 \int_{-\infty}^{\infty} \| H(-i \omega) \| |\hat{G}^n(i \omega)| d\omega \\
\leq 2 \int_{-\infty}^{\infty} \| H(-i \omega) \| \| \hat{G}^n(i \omega) \| d\omega \\
\leq 2 \left( \max_{\omega} \| H(-i \omega) \| \right) \int_{-\infty}^{\infty} \| \hat{G}^n(i \omega) \| d\omega \tag{7.22}
\]

Case 3: Let the input spectrum be bounded in the sense \( \| \Phi_u(\omega) \| \leq \Phi_0 \). Then

\[
\left\| \frac{\partial W(\theta_0)}{\partial \theta} \right\| \leq 2\Phi_0 \int_{-\infty}^{\infty} \| H(-i \omega) \| \| \hat{G}^n(i \omega) \| d\omega \\
\leq 2\Phi_0 \sqrt{\int_{-\infty}^{\infty} \| H(-i \omega) \|^2 d\omega} \sqrt{\int_{-\infty}^{\infty} \| \hat{G}^n(i \omega) \|^2 d\omega} \tag{7.23}
\]

where Schwarz’ inequality [15] was used in the last step.

The worst situation is considered in all cases above, that is, the input is assumed to be bounded in amplitude, or have a bounded variance or spectrum. This implies that the Hessian, \( W'' \), is bounded from above, cf (7.18). Consequently, the inverse of the Hessian will be bounded from below. From (7.8) it is observed that \( (W'')^{-1} \) acts as a proportional constant on the parameter bias \( \dot{\theta} \). The inverse of the Hessian is bounded from below, but it is also required to be bounded from above, or the contribution to the bias will be large. This corresponds to the demand that the Hessian is non-singular. For this reason, it is a necessary assumption that the input signal is not too small in amplitude, in which case its spectrum would cause the Hessian to be too small, cf (7.18). Also, use of persistently exciting input signals is necessary for the existence of a unique solution to the identification problem, [5]. The requirement on persistently exciting input signals gives a condition on the frequency content of the input.

The Hessian, either expressed in the time domain, (7.16), or in the frequency domain, (7.18), is independent of the model order. Hence, the parameter bias, \( \dot{\theta} \), is essentially characterized by quantities like

\[
\int_{0}^{\infty} \| \hat{G}^n(i \omega) \| d\omega, \quad \max_{\omega} \| \hat{G}^n(i \omega) \|, \quad \int_{0}^{\infty} \| \hat{g}^n(t) \| dt
\]
etc. There are certain relations between the measures above. It can be noted that

$$\tilde{G}^n(i \omega) = \int_0^\infty \tilde{g}^n(t)e^{-i\omega t} dt \quad (7.24)$$

Then

$$\left| \tilde{G}^n(i \omega) \right| \leq \int_0^\infty |\tilde{g}^n(t)e^{-i\omega t}| dt$$

$$\leq \int_0^\infty |\tilde{g}^n(t)| dt \quad (7.25)$$

In the following chapters, the influence of the model order on the parameter bias will be theoretically examined for the heat diffusion equation. It will be shown that the errors in the estimated parameters, characterized as above, are small when the model order \( n \to \infty \). The gradient will be used for finding a quantitative measure of the bias contribution \( || \theta - \theta_0 || \).
8 Approximation errors of auxiliary parameters

Chapter 3 described the heat diffusion model and its system dynamics. In Chapter 4, two alternative methods by which the diffusion can be modeled were given. Certain parameters, coupled to the material constants, $\alpha_k$, $\beta_{1,k}$ and $\beta_{2,k}$ and their approximates, have a large influence of how well the system can be described by the chosen model. There will be a need for comparison between similar expressions in these parameters in the analysis in chapters to come. Therefore, some quantitative measures on the deviation between the parameters are given here. Also, some auxiliary lemmas are stated which will be used in Chapter 9. The majority of the lemmas stated in this chapter will be used in the proof of Lemma 9.1, and indirectly in the proof of Theorem 9.2.

It should be noted that the lemmas below are tied to the specific application that was introduced in Chapter 3. Also, the notation of the parameters used here was fully defined in Chapters 3 and 4.

This chapter should be regarded as auxiliary, and used in order to look up results, if needed, while reading other parts of the text.

First, the parameters $\alpha_k$, $\beta_{i,k}$, $\tilde{\alpha}_k$ and $\tilde{\beta}_{i,k}, i = \{1, 2\}$ are summarized and rewritten, for convenience.

\[
\alpha_k = \frac{\pi^2(2k - 1)^2}{4} \quad \text{(8.1)}
\]
\[
\tilde{\alpha}_k = 2(n + 1)^2 \left(1 - \cos \left(\frac{\pi(2k - 1)}{2n + 1}\right)\right)
= (n + 1)^2 \left(\frac{\pi^2(2k - 1)^2}{(2n + 1)^2} - \frac{\pi^4(2k - 1)^4}{12(2n + 1)^4} + O\left(\frac{k^6}{n^6}\right)\right) \quad \text{(8.2)}
\]
\[
\beta_{1,k} = 2 \quad \text{(8.3)}
\]
\[
\beta_{1,k} = \frac{4(n + 1)}{2n + 1} \cos^2 \left(\frac{\pi(2k - 1)}{2(2n + 1)}\right)
= \frac{2(n + 1)}{2n + 1} \left(\cos \left(\frac{\pi(2k - 1)}{2n + 1}\right) + 1\right)
= \frac{2(n + 1)}{2n + 1} \left(1 - \frac{\pi^2(2k - 1)^2}{2(2n + 1)^2} + O\left(\frac{k^4}{n^4}\right) + 1\right)
= \frac{4(n + 1)}{2n + 1} \left(1 - \frac{\pi^2(2k - 1)^2}{4(2n + 1)^2}\right) + O\left(\frac{k^4}{n^4}\right) \quad \text{(8.4)}
\]
\[
\beta_{2,k} = \pi(2k - 1)(-1)^{k+1} \quad \text{(8.5)}
\]
\[
\beta_{2,k} = \frac{4(n + 1)^2(-1)^{k+1}}{2n + 1} \sin \left( \frac{\pi(2k - 1)}{2n + 1} \right) \cos \left( \frac{\pi(2k - 1)}{2(2n + 1)} \right) \\
= \frac{4(n + 1)^2(-1)^{k+1}}{2n + 1} \left( \frac{\pi(2k - 1)}{2n + 1} \right) - \frac{7\pi^3(2k - 1)^3}{24(2n + 1)^3} + O \left( \frac{k^5}{n^6} \right) \\
= \frac{4\pi(-1)^{k+1}(n + 1)^2(2k - 1)}{(2n + 1)^2} \left( 1 - \frac{7\pi^2(2k - 1)^2}{24(2n + 1)^2} + O \left( \frac{k^5}{n^6} \right) \right)
\]

Different relations between the parameters \( \alpha_k, \beta_{i,k} \), and the approximated parameters \( \hat{\alpha}_k, \hat{\beta}_{i,k} \) are given next. Series expansions will be used for this purpose, in order to approximate the trigonometric functions. The expression (8.7) can be compared to Lemma 8.12.

\[
\beta_{1,k} - \hat{\beta}_{1,k} = 2 - \frac{4(n + 1)}{2n + 1} \left( 1 - \frac{\pi^2(2k - 1)^2}{4(2n + 1)^2} \right) + O \left( \frac{k^4}{n^4} \right) \\
= 2 \left( 1 - \frac{2(n + 1)}{2n + 1} + \frac{\pi^2(n + 1)(2k - 1)^2}{2(2n + 1)^3} \right) + O \left( \frac{k^4}{n^4} \right) \\
= \left( -\frac{2}{2n + 1} + \frac{\pi^2(n + 1)(2k - 1)^2}{2(2n + 1)^3} \right) + O \left( \frac{k^4}{n^4} \right) 
\]

\[
\beta_{2,k} - \hat{\beta}_{2,k} = (-1)^{k+1} \left[ \frac{\pi(2k - 1)}{2n + 1} - \frac{4\pi(n + 1)^2(2k - 1)}{(2n + 1)^2} \right] \\
\times \left( 1 - \frac{7\pi^2(2k - 1)^2}{24(2n + 1)^2} + O \left( \frac{k^4}{n^4} \right) \right) \\
= (-1)^{k+1} \left[ -\frac{\pi(2k - 1)(4n + 3)}{(2n + 1)^2} + O \left( \frac{k^3}{n^4} \right) \right] 
\]

\[
\hat{\alpha}_k \beta_{1,k} - \alpha_k \hat{\beta}_{1,k} = 2 \left[ \frac{\pi^2(n + 1)^2(2k - 1)^2}{(2n + 1)^2} + O \left( \frac{k^4}{n^2} \right) \right] \\
- \frac{\pi^2(2k - 1)^2}{4} \left[ \frac{4(n + 1)}{2n + 1} \left( 1 - \frac{\pi^2(2k - 1)^2}{4(2n + 1)^2} \right) + O \left( \frac{k^4}{n^4} \right) \right] \\
= \left[ \frac{2\pi^2(n + 1)^2(2k - 1)^2}{(2n + 1)^2} - \frac{\pi^2(n + 1)(2k - 1)^2}{2n + 1} \right] + O \left( \frac{k^4}{n^2} \right) \\
= \frac{\pi^2(n + 1)(2k - 1)^2}{2n + 1} \left( \frac{2(n + 1)}{2n + 1} - 1 \right) + O \left( \frac{k^4}{n^2} \right) \\
= \frac{\pi^2(n + 1)(2k - 1)^2}{(2n + 1)^2} + O \left( \frac{k^4}{n^2} \right)
\]
\[ \tilde{\alpha}_{k}\tilde{\beta}_{2,k} - \alpha_{k}\beta_{2,k} = (n+1)^2 \left( \frac{\pi^2(2k-1)^2}{(2n+1)^2} - \frac{\pi^4(2k-1)^4}{12(2n+1)^4} + O \left( \frac{k^6}{n^6} \right) \right) \]
\[ \times \pi(2k-1)(-1)^{k+1} - \frac{\pi^3(-1)^{k+1}(n+1)^2(2k-1)^3}{(2n+1)^2} \]
\[ \times \left( 1 - \frac{7\pi^2(2k-1)^2}{24(2n+1)^2} + O \left( \frac{k^4}{n^4} \right) \right) \]
\[ = \frac{\pi^3(n+1)^2(2k-1)^3}{(2n+1)^2} (-1)^{k+1} \left[ 1 - \frac{\pi^2(2k-1)^2}{12(2n+1)^2} \right] \]
\[ + O \left( \frac{k^4}{n^4} \right) - 1 + \frac{7\pi^2(2k-1)^2}{24(2n+1)^2} + O \left( \frac{k^4}{n^4} \right) \]
\[ = \frac{\pi^3(n+1)^2(2k-1)^3}{(2n+1)^2} (-1)^{k+1} \left( \frac{5\pi^2(2k-1)^2}{24(2n+1)^2} + O \left( \frac{k^4}{n^4} \right) \right) \]
\[ = \frac{5\pi^5(n+1)^2(2k-1)^5}{24(2n+1)^4} (-1)^{k+1} \left( 1 + O \left( \frac{k^2}{n^2} \right) \right) \] (8.10)

These results will be used in the lemmas to follow.

A vector of frequencies
\[ \omega_k \triangleq \frac{\pi(2k-1)}{2n+1}, \quad k = 1, \ldots, n \] (8.11)
is defined for notational reasons. The expression will be used extensively henceforth in order to simplify the writing.

The auxiliary variables \( \beta_1,k, \beta_2,k \) and \( \tilde{\alpha}_k \) are defined for convenience, and will be widely used in the following chapters.
\[ \tilde{\alpha}_k \triangleq \alpha_k - \alpha_k \] (8.12)
\[ \beta_1,k \triangleq \beta_1,k - \beta_1,k \] (8.13)
\[ \beta_2,k \triangleq \beta_2,k - \beta_2,k \] (8.14)
The results in Lemma 8.1–8.3 will be used in the proof of Lemma 9.1.

**Lemma 8.1.** For specific, fixed, indices \( k \) and \( n \), the function
\[ f_{k,n}(t) \triangleq \alpha_k \beta_1,k t e^{-\frac{2\pi t}{\tau}} - \alpha_k \beta_1,k t e^{-\frac{\pi t}{\tau}} \] (8.15)
changes sign when, and only when
\[ t = t_{k,n} \triangleq \frac{\tau}{\alpha_k - \alpha_k} \ln \left( \frac{\alpha_k \beta_1,k}{\alpha_k \beta_1,k} \right) \] (8.16)
For a few indices \( k \) (and \( n \)), \( t_{k,n} < 0 \). But (8.16) is considered for non-negative times only, \( t \geq 0 \). Therefore, whenever \( t_{k,n} < 0 \), it will be set to zero. \( \square \)

**Proof** See Appendix C.
An example of the characteristics of \( t_{k,n} \) is found in Figure 8.1. Since only non-negative times are considered, negative values of \( t_{k,n} \) are transferred to zero. Note that the graph is plotted in logarithmic scale, thus resulting in that values of zero give non-existing points in the figure.

![Graph showing the normed parameter \( t_{k,n}/\tau \) as defined in Lemma 8.1. In this example, the model order was chosen as \( n = 100 \).](image)

**Figure 8.1:** The normed parameter \( t_{k,n}/\tau \) as defined in Lemma 8.1. In this example, the model order was chosen as \( n = 100 \).

**Lemma 8.2.** The following relations hold for large \( n \).

\[
\frac{\alpha_k t_{k,n}}{\tau} \approx \begin{cases} 
\frac{3}{\pi^2} + O\left(\frac{1}{n}\right) & \text{k small} \\
\frac{2\pi^2}{n^{\frac{3}{2}}} \ln\left(\frac{n}{n-k+1}\right) & \text{k large (k \approx n)}
\end{cases}
\]  

(8.17)

\[
\frac{\alpha_k t_{k,n}}{\tau} \approx \begin{cases} 
\frac{\alpha_k t_{k,n}}{\tau} & \text{k small} \\
\frac{4}{\pi^2} \frac{\alpha_k t_{k,n}}{\tau} \approx \frac{8}{\pi^2 - 4} \ln\left(\frac{n}{n-k+1}\right) & \text{k large (k \approx n)}
\end{cases}
\]  

(8.18)

**Proof** See Appendix C.

The following result has a correspondence to Lemma 8.1.
Lemma 8.3. For a specific indices $k$ and $n$, the function
\[ f_{k,n}(t) \triangleq \alpha_k \beta_{2,k} t e^{-\frac{\alpha_k t}{\beta_{2,k}}} - \alpha_k \beta_{2,k} t e^{-\frac{\alpha_k t}{\beta_{3,k}}} \] (8.19)
changes sign when
\[ t = t_{k,n} \triangleq \frac{\tau}{\alpha_k - \alpha_k} \ln \left( \frac{\alpha_k \beta_{2,k}}{\alpha_k \beta_{3,k}} \right) \] (8.20)

Proof See Appendix C.

For $\alpha_k$ defined as in (8.12), the sign is positive for small indices $k$, and negative for large indices. The following result will be used extensively in the upcoming lemmas.

Lemma 8.4. The parameter $\alpha_k$ changes sign once, for one particular index $k$. This occurs at the closest integer value greater than
\[ k_{\alpha} \triangleq \frac{2\sqrt{3}}{\pi} \sqrt{n + 1} + \frac{1}{2} \] (8.21)
under the assumption that $n$ is large.

Proof See Appendix C.

The expression (8.21) is approximate, since the asymptotic case $n \to \infty$ was considered. The error which is introduced by this is, however, small and will in most cases not affect the result since the index $k$ in $\alpha_k$ is an integer while the expression (8.21) is not. This provides a possibility for the calculated $k$ to vary somewhat, without crossing the bound to the next integer value.

The next five results will be used in the proof of Lemma 8.10.

Lemma 8.5. For large $n$, $t \geq \tau$ and $1 \leq k \leq k_{\alpha} - 1$,
\[ \beta_{1,k} e^{-\frac{\alpha_k t}{\beta_{1,k}}} - \beta_{1,k} e^{-\frac{\alpha_k t}{\beta_{3,k}}} \geq 0 \] (8.22)

Proof See Appendix C.

The following lemma is used also in the proof of Lemma 8.7.
Lemma 8.6. For large but bounded $n$, and for the index $k_\alpha$ defined as in (8.21),
\[ k_\alpha = \frac{2\sqrt{3}}{\pi} \sqrt{n + 1} + \frac{1}{2} \] (8.23)
the following holds,
\[ \sum_{k=1}^{k_\alpha-1} \left( \beta_{1,k} - \tilde{\beta}_{1,k} \right) > 0 \] (8.24)
Proof See Appendix C.

Lemma 8.7. For large $n$ and for the index $k_\alpha$ defined as in Lemma 8.4, the following holds,
\[ \sum_{k=1}^{k_\alpha-1} \left( \beta_{1,k} e^{-\frac{\alpha k}{n}} - \tilde{\beta}_{1,k} e^{-\frac{\alpha k}{n}} \right) > 0 \] (8.25)
Proof See Appendix C.

Lemma 8.8. The following holds for large $n$,
\[ \sum_{k=1}^{n} \left( \beta_{1,k} - \tilde{\beta}_{1,k} \right) = n - 1 \] (8.26)
Proof See Appendix C.

Lemma 8.9. For a fixed $k$, $\beta_{1,k} e^{-\frac{\alpha k}{n}} - \tilde{\beta}_{1,k} e^{-\frac{\alpha k}{n}}$ changes sign for
\[ t = t_c = \frac{\tau}{\alpha k - \alpha \tilde{k}} \ln \left( \frac{\beta_{1,k}}{\tilde{\beta}_{1,k}} \right) \] (8.27)
Proof See Appendix C.

The result presented next will be used in the proofs of Lemma 9.1 and Theorem 9.2.
Lemma 8.10. The following relation holds for large $n$ and all $t \geq 0$,
\[ \sum_{k=1}^{n} \left( \beta_{1,k}e^{-\frac{n}{n+1}t} - \tilde{\beta}_{1,k}e^{-\frac{n}{n+1}t} \right) \geq 0 \quad (8.28) \]

Proof See Appendix C.

The next result will be used in the proof of Lemma 9.1. Even if a complete proof is not available, the result is chosen to be presented as a lemma.

Lemma 8.11. For large $n$,
\[ \sum_{k=1}^{n} \left( \beta_{2,k}e^{-\frac{n}{n+1}t} - \tilde{\beta}_{2,k}e^{-\frac{n}{n+1}t} \right) \]
changes sign at the most two times in the time interval $0 \leq t \leq \tau$. \hfill \qed

Proof See Appendix C.

Lemma 8.12. The upper bound in (8.30) holds for all large model orders $n$ and all $k \in [1, n]$. The expression of $\tilde{\beta}_{1,k}$ is taken from (4.21) in Lemma 4.2 where a large model order was assumed.
\[ |\beta_{1,k} - \tilde{\beta}_{1,k}| \leq \left( \frac{2}{2n+1} + \frac{\pi^2(n+1)(2k-1)^2}{(2n+1)^3} \right) \quad (8.30) \]

Proof See Appendix C.

The next three lemmas will be used in the proof of Lemma 9.1. Lemma 8.13 will also be used in Lemma 8.14.

Lemma 8.13. The following relation holds for all large model orders $n$ and all indices $k \in [1, n]$.
\[ \frac{\beta_{1,k}}{\alpha_k} - \frac{\tilde{\beta}_{1,k}}{\alpha_k} > 0 \quad (8.31) \]

Proof See Appendix C.

The following result is also used in the proof of Theorem 9.2.
Lemma 8.14. The following result holds for large model orders $n$,

$$\sum_{k=1}^{n} \left( \frac{\beta_{1,k}}{\alpha_k} - \frac{\bar{\beta}_{1,k}}{\alpha_k} \right) \sim O \left( \frac{1}{n} \right) \quad (8.32)$$

Proof See Appendix C.

Lemma 8.15. The following result holds for large model orders $n$,

$$\sum_{k=1}^{n} \left( \frac{\beta_{2,k}}{\alpha_k} - \frac{\bar{\beta}_{2,k}}{\alpha_k} \right) \sim O \left( \frac{1}{n} \right) \quad (8.33)$$

Proof See Appendix C.
9 A theoretical analysis of the parameter bias

This chapter is devoted to a theoretical study on the parameter bias, and how it is affected by the model order. Analysis will be performed both in the frequency domain and in the time domain. The system dynamics is modeled with finite order using difference approximation. The results hence apply to Methods 1 and 2 in Chapter 5.

In Chapter 7, it was shown that the bias contribution can be modeled by the transfer functions or weighting functions, cf (7.8)–(7.10). Equation (7.8) gives that the bias depends on the Hessian as well as on the gradient. It can be noted that the Hessian is approximately independent of the model order, cf (7.10) and Lemma 9.1. Utilizing this and (7.8), (7.9), it is clear that when the influence of the model order on the bias is of primarily interest, it is sufficient to study the behavior of $G^n_i(i\omega; \theta_0)$, $i = \{1, 2\}$ alone.

The bias considered in this chapter is caused by underparameterization, that is, the model structure has too few parameters to describe the system correctly. The problem with variance on the estimates, caused by for example noisy measurements, was studied numerically in Chapter 6, but is not treated here.

Results from Chapter 8 will be used extensively in the proofs of the forthcoming statements.

9.1 Estimating the bias in the frequency domain

For estimation of the error in the transfer functions, the integral over all frequencies of interest is preferably computed, cf (7.22)–(7.23). Then the following bound is of interest,

$$|G^n_i(i\omega)| \leq \frac{C(\omega)}{nP}, \quad i = \{1, 2\} \quad (9.1)$$

where

$$\int_0^\infty C(\omega)d\omega < \infty \quad (9.2)$$

and where $p \geq 0$. From the results in Chapter 6, see e.g. Figure 6.12 where the slopes of the curves are $-1$, it can be expected that $p \approx 1$. It turns out that the conditions (9.1), (9.2) are too strong to be used. This is illustrated in the following example.
Example 9.1. Consider the approximation error \( \tilde{G}_1^n(s) \). The expressions for \( G_1(s) \) and \( \tilde{G}_1^n(s) \) are found in (3.21) and (4.18).

\[
\tilde{G}_1^n(s) = G_1(s) - \tilde{G}_1^n(s)
\]

\[
= \left( R D_1 - R D_1 \right) + R \sum_{k=1}^{\infty} \frac{\beta_{1,k}}{s^r + \alpha_k} + R \sum_{k=n+1}^{\infty} \frac{\beta_{1,k}}{s^r + \alpha_k} + R \sum_{k=n+1}^{\infty} \frac{\beta_{1,k}}{s^r + \alpha_k}
\]

\[
= -R D_1 + R \sum_{k=1}^{n} \frac{s^r(\beta_{1,k} - \tilde{\beta}_{1,k})}{(s^r + \alpha_k) (s^r + \alpha_k)} + R \sum_{k=1}^{n} \frac{\alpha_k \beta_{1,k} - \alpha_k \beta_{1,k}}{(s^r + \alpha_k) (s^r + \alpha_k)}
\]

\[
+ R \sum_{k=n+1}^{\infty} \frac{\beta_{1,k}}{s^r + \alpha_k}
\]

(9.3)

For (9.1) and (9.2) to hold, they are required to hold for each term in (9.3). Consider the part denoted by \( S_1 \). The expression for \( D_1 \) is obtained from (4.23). Then

\[
|S_1| = \frac{R}{n+1}
\]

(9.4)

gives that

\[
C(\omega) \to R
\]

(9.5)

as \( \omega \to \infty \). But

\[
\int_0^\infty R \, d\omega \to \infty
\]

(9.6)

which apparently is not bounded. Since the conditions (9.2) and (9.6) are contradictory, the integral over all frequencies cannot be used as a rightful measure of how the parameter bias is influenced by the model order.

It can be observed that the result of (9.6) is caused by frequencies ranging up to infinity. In practice, the input signals have such a frequency content that this never happens. On the contrary, the input signals for the heat diffusion are rather slowly varying. The main frequency lies around 1/24 [h\(^{-1}\)], of Section 6.2. Therefore, the expressions (7.22) and (7.23) are unnecessarily conservative. A limited frequency band can instead be used for the integration. If

\[
\int_0^{\omega_{\text{max}}} |\tilde{G}_1^n(i \omega)| \, d\omega, \quad i = \{1, 2\}
\]

(9.7)

is computed, where \( \omega_{\text{max}} \) is the largest possible frequency that can be found in the input signals, then (9.7) is limited.
As an alternative to (9.1) and (9.2), the approximation error at a single frequency point can be regarded, cf (7.21). The result of the influence of the model order on \( \hat{G}^n(\omega) \) is presented in the theorem below.

**Theorem 9.1.** Assume that the model order \( n \) is large. Then the error in the approximated transfer functions, for each frequency point \( \omega \), decreases as \( 1/n \),

\[
|\hat{G}_1^n(\omega)| = |G_1(\omega) - \hat{G}_1^n(\omega)| \sim O \left( \frac{1}{n} \right) \quad (9.8)
\]
\[
|\hat{G}_2^n(\omega)| = |G_2(\omega) - \hat{G}_2^n(\omega)| \sim O \left( \frac{1}{n} \right) \quad (9.9)
\]

**Proof** See Appendix D.

The findings in Theorem 9.1 are illustrated in Chapter 10. The bias will be compared for a selection of single frequency points and for different model orders.

### 9.2 Estimating the bias in the time domain

The bias of the parameter vector is estimated in the time domain using (7.8). It is shown that the result is similar to what was obtained in the frequency domain.

It is first noted that the model order \( n \) does not have a significant influence on the gradient

\[
\frac{\partial \hat{g}^n(t)}{\partial \theta} \quad (9.10)
\]

see Lemma 9.1 below. Hence, the Hessian is approximately independent of the model order, cf (7.10), and acts as a proportional constant. Therefore it is sufficient to study the quantity \( \int_0^\infty \| \hat{g}^n(t) \| dt \) in order to obtain a good estimation of the dependence between the parameter bias and the model order, cf (7.15) and (7.16).

**Lemma 9.1.** The following holds in the time domain,

\[
\int_0^\infty \left| \frac{\partial \hat{g}_{i}^n(t)}{\partial \theta} \right| dt = \int_0^\infty \left| \frac{\partial g_i(t)}{\partial \theta} - \frac{\partial \hat{g}_{i}^n(t)}{\partial \theta} \right| dt \to 0 \quad (9.11)
\]

as \( O(1/n) \) when \( n \to \infty \), for \( i = \{1, 2\} \) and \( \theta = (R^T \tau)^T \).

**Proof** See Appendix D.

Lemma 9.1 indicates that the approximations in e.g. (7.9), (7.10) are valid. The integral of the approximation errors \( \hat{g}_1^n(t) \) and \( \hat{g}_2^n(t) \) are studied in the theorem below.
Theorem 9.2. Assume \( n \) is large. Then the integral of the error in the approximated weighting function in time domain decreases as \( 1/n \),

\[
\int_0^\infty |\tilde{g}_1^n(t)| dt \sim O \left( \frac{1}{n} \right) \quad (9.12)
\]

\[
\int_0^\infty |\tilde{g}_2^n(t)| dt \sim O \left( \frac{1}{n} \right) \quad (9.13)
\]

\[\square\]

**Proof** See Appendix D.

The results are illustrated in the next chapter.
10 Numerical illustrations of the bias contribution

The following holds for the illustrations in Chapter 10 if nothing else is said. The true transfer functions are modeled using (3.13). The expressions (4.8), (4.9) were used for modeling the approximate transfer functions. The frequency points \{\omega\} were chosen in a logarithmic scale in an interval from \(10^{-6}\) to \(10^6\) [rad/h]. The whole interval may, however, not need to be shown in the figures, since a narrower frequency band might illustrate all information of interest. The time constant \(\tau = 1\) [h] and the thermal resistance \(R = 1\ \text{[}^\circ\text{Cm}^2/\text{W}]\) were used.

In order to get an idea of how well the transfer functions are approximated, the errors are plotted as functions of the frequency for two different model orders. Figure 10.1 displays the approximation error \(|\tilde{G}_1^m(i\omega)|\) for the model orders \(n = 10\) (stars) and \(n = 100\) (diamonds). The magnitude of the true transfer function \(|G_1(i\omega)|\) is plotted with a dashed line for comparison. The approximation error is quite small for low frequencies. It increases until \(\omega \approx \pi^2/(4\tau)\) [rad/h] is reached, after which the error is approximately constant. Note that the ‘break point’ corresponds to the dominant pole, cf (3.17). By comparing the two solid curves in the graph, it can be observed that the approximation error decreases

![Figure 10.1: The approximation error \(|\tilde{G}_1^m(i\omega)|\) as a function of the frequency is displayed. In the computation of the approximated model, the model orders \(n = 10\) (stars) and \(n = 100\) (diamonds) were used. The magnitude of the true transfer function \(|G_1(i\omega)|\) is plotted as a reference with a dashed line. The parameters are \(R = 1\ \text{[}^\circ\text{Cm}^2/\text{W}]\), \(\tau = 1\) [h]. The notation \(\Delta G(i\omega) = G(i\omega) - \tilde{G}_1^m(i\omega)\) is used.](image-url)
with the model order as $O(1/n)$ for all frequencies displayed. The corresponding 
for $|G^n(i\omega)|$ is shown in Figure 10.2. The error is largest for frequencies around 
$\omega \approx \pi^2/(4\tau)$ [rad/h]. Important to note is the fact that the approximation errors 
decrease rapidly for large frequencies, unlike for $|G^n(i\omega)|$. The error decreases 
as $O(1/n)$ for low frequencies, but a lot faster for high frequencies. This can 
be observed by comparing the solid lines in Figure 10.2. In this application, 
however, the system will not be stimulated for frequencies much higher than 
approximately $0.26$ [rad/h].

In Chapter 7 it was observed that different measures can be used in order to 
evaluate the parameter bias $\theta$, see Cases 1–3. If for example the input signal $u(t)$ 
is a sinusoid with a single frequency, it is sufficient to study the transfer functions 
at that specific frequency, cf (7.21). The approximation errors of the transfer 
functions for a fixed frequency $\omega_0 = 1/\tau$ [rad/h] are illustrated in Figure 10.3. 
The result of Figure 10.3 corresponds to Theorem 9.1. The curves have slopes 
of $\lambda$, which is equivalent to that the magnitude of the approximation error 
decreases as $O(1/n)$ for both transfer functions. The curves in Figure 10.3 nearly 
coincide since the approximation errors are similar for the chosen frequency $\omega_0$, cf 
Figures 10.1 and 10.2. The corresponding approximation errors $|G^n(i\omega_0)|$, $i = 
\{1, 2\}$ for the fixed frequencies $\omega_0 = 10/\tau$ [rad/h] and $\omega_0 = 100/\tau$ [rad/h],

![Figure 10.2](image)

**Figure 10.2:** The approximation error $|\tilde{G}^n(i\omega)|$ as a function of the frequency is dis-
played. In the computation of the approximated model, the model orders $n = 10$ (stars) 
and $n = 100$ (diamonds) were used. The magnitude of the true transfer function $G(i\omega)$ 
is plotted as a reference with a dashed line. The parameters are $R = 1$ [Cm$^2$/W], 
$\tau = 1$ [h]. The notation $\Delta G(i\omega) = G(i\omega) - \tilde{G}^n(i\omega)$ is used.
Figure 10.3: The approximation errors of the transfer functions as a function of the model order is plotted. The fixed frequency $\omega_0 = 1/\tau$ [rad/h] was used. The solid line shows $|\hat{G}'(i\omega_0)|$ and the dashed line shows $|\hat{G}''(i\omega_0)|$. The notation $\Delta\hat{G}(i\omega) = \hat{G}(i\omega) - \hat{G}^n(i\omega)$ is used. The parameters are $R = 1 \ [\text{C/m}\cdot\text{W}]$, $\tau = 1 \ [\text{h}]$.

respectively, are displayed in Figure 10.4. Plot (a) shows the errors for $\omega_0 = 10/\tau$ [rad/h]. The curves have a slope of $-1$, which is consistent with the findings in Figure 10.3. Plot (b) shows the errors for $\omega_0 = 100/\tau$ [rad/h]. The slopes are $-1$ also in this case which indicates that the errors in the approximated transfer functions decrease as $O(1/n)$ in accordance with Figures 10.1 and 10.2.

Alternatively, the frequency point $\omega$ corresponding to the largest approximation errors can be chosen, instead of a fixed frequency. The values of $\omega$ are then dependent on $n$. Figure 10.5 displays the approximation errors $\max_{\omega}|\hat{G}'(i\omega)|$ and $\max_{\omega}|\hat{G}''(i\omega)|$. The curves have slopes of $-1$. Hence, a doubling of the model order results in a halved approximation error, which is a similar result as was obtained for a fixed (single) frequency.

If the input signal has energy in a large part of the frequency band, it might be better to consider (7.22) or (7.23) instead of (7.21). The expressions (7.8), (7.17) and (7.18) give that the transfer functions contribute to the gradient and Hessian, and consequently to the bias, for those frequencies where $\Phi_n(\omega)$ has energy. Since the Hessian is approximately independent of the model order, cf (7.18) and Lemma 9.1, it is sufficient to study the error of the approximated transfer functions in the frequency region where $u(t)$ has large enough energy, cf (7.22), (7.23). If the input signal is assumed to have energy in the whole spectrum, i.e., $u(t)$ is for example white noise, cf (7.22), the following integral
Figure 10.4: The approximation errors of the transfer functions as a function of the model order is shown. The fixed frequencies $\omega_0 = 10/\tau$ [rad/h] (plot (a)) and $\omega_0 = 100/\tau$ [rad/h] (plot (b)) were used. The solid lines show $|\tilde{G}_1(i\omega_0)|$ and the dashed lines show $|\tilde{G}_2(i\omega_0)|$. The parameters are $R = 1 \, \text{Cm}^2/\text{W}$, $\tau = 1 \, \text{h}$.

Figure 10.5: The approximation error of the transfer functions as a function of the model order. The solid line shows $\max_{\omega} |\tilde{G}_1(i\omega)|$ and the dashed line shows $\max_{\omega} |\tilde{G}_2(i\omega)|$. The parameters are $R = 1 \, \text{Cm}^2/\text{W}$, $\tau = 1 \, \text{h}$.
should be examined,

$$\int_{0}^{\infty} |\tilde{G}_i(\omega)| d\omega, \quad i = \{1, 2\} \quad (10.1)$$

As was discussed in Example 9.1, (10.1) for \( i = 1 \) is unbounded. But, since input signals with such frequency content are unrealistic, (10.1) can be studied with a somewhat limited frequency band,

$$\int_{\omega_{\text{low}}}^{\omega_{\text{high}}} |\tilde{G}_i(\omega)| d\omega, \quad i = \{1, 2\} \quad (10.2)$$

The bounds \( \omega_{\text{low}} \) and \( \omega_{\text{high}} \) affect the size of the error. But for a specific frequency band, the dependency of the model order can be studied.

The frequency \( \omega \) in the transfer functions \( G_1(i\omega), G_2(i\omega) \) is continuous in the interval zero to infinity. In the computation of (10.2), a limited number of discrete samples of \( \omega \) was used. The choice of \( \omega_{\text{low}} \) and \( \omega_{\text{high}} \) influences primarily the level of (10.2) for \( i = 1 \) since the error does not decrease for high frequencies, cf Figure 10.1. The integral of the approximation error \( \tilde{G}_2(i\omega) \) is not significantly affected by the truncated frequency band, as long as \( \omega_{\text{high}} \) is chosen large enough. The reason for this is that the approximation error decreases rapidly for high frequencies, cf Figure 10.2.

A comparison between the approaches of taking the integral of the error and choosing the frequency point giving the maximum error, is found in Figure 10.6. The frequency bounds were chosen as \( \omega_{\text{low}} = 10^{-6} \) [rad/h] and \( \omega_{\text{high}} = 10^6 \) [rad/h]. Taking the integral over the chosen frequency interval results in a larger value of the measure than if a single frequency is considered since the approximation error is bounded for all (single) frequencies, and since the error is small for frequencies close to \( \omega = 0 \). The latter claim was illustrated by Figures 10.1 and 10.2. Both methods give that the bias decreases with the model order as \( O(1/n) \), which is illustrated by the curves having slopes of \(-1\).

The approximation errors are also studied in the time domain. Figure 10.7 displays the error in the first weighting function, \( \tilde{g}_1(t) \), for the model orders \( n = 10 \) and \( n = 100 \), respectively. Plots (a) and (b) show different resolutions in time. The infinite sums for the true weighting functions are truncated at \( n = 1000 \). This affects the result significantly only for small \( t \approx 0 \), where a smaller value of the error is obtained. It can be observed that for \( t \to 0 \), the error increases unlimited even if this is not shown in the figure, cf plot (b). However, taking the integral of the approximation error results in a bounded solution, cf Theorem 9.2. For larger \( t \), \( |\tilde{g}_1(t)| \) decreases approximately to \( 1/10 \)th every hour. By comparing the curves in plot (a), it can be noted that the approximation error decreases with the model order as \( O(1/n) \), at least after some initial time.
Figure 10.6: The approximation error in the frequency domain as a function of the model order. Plot (a) displays the result of $|G_1^1(i\omega)|$ and plot (b) shows the result of $|G_2^2(i\omega)|$. The solid lines show $\max_{\omega} |G^\omega(i\omega)|$ and the dashed lines show $\int_{-\infty}^{\infty} |G^\omega(i\omega)|\,d\omega$. The parameters are $R = 1 \, \text{Cm}^2/\text{W}$, $\tau = 1 \, \text{h}$.

Figure 10.7: The approximation error $|\hat{y}_1(t)|$ for $n = 10$ (stars) and $n = 100$ (diamonds) is illustrated. The parameters are $R = 1 \, \text{Cm}^2/\text{W}$, $\tau = 1 \, \text{h}$. The notation $\Delta y(t) = y(t) - \hat{y}^n(t)$ is used.
Figure 10.8 shows the approximation error for the second weighting function, \( \tilde{g}_2^n(t) \), for \( n = 10 \) and \( n = 100 \). Plots (a) and (b) show different resolutions in time. After some initial settling time, \( |\tilde{g}_2^n(t)| \) decreases approximately to 1/10th per hour. Comparing the curves in plot (a), shows that the approximation error decreases as \( O(1/n) \) after some initial time. The curves in plot (b) look a bit peculiar for times around \( 10^{-4} \) [h]. The levels of the curves in the time interval \( t \in (10^{-6} \ 10^{-2}) \) [h] approximately, are however so small that errors can occur for example due to finite precision in the computer.

Finally, the result of Theorem 9.2 is illustrated in Figure 10.9. The weighting functions (3.28), (3.29), (4.25) and (4.26) were used. The integration is performed from \( t_{\text{low}} = 10^{-6} \) [h] to \( t_{\text{high}} = 10^{10} \) [h]. The error which is introduced by integrating over the limited time interval is negligible. For \( t > 10^{10} \) [h], \( \tilde{g}_2^n(t) \) is very small. For \( t < 10^{-6} \) [h], the integral would give only a small contribution. The slopes of the curves are roughly \( -1 \). Hence, the approximation errors decrease as \( O(1/n) \). It can be noted that the approximation error

\[
\int_0^\infty |\tilde{g}_i^n(t)| \, dt \geq \max |\tilde{G}_i^n(i\omega)|, \quad i = \{1, \ 2\}
\]

by comparing Figures 10.5 and 10.9. This is in accordance with (7.25).

![approximation error graphs](image)

**Figure 10.8**: The approximation error \( |\tilde{g}_2^n(t)| \) for \( n = 10 \) (stars) and \( n = 100 \) (diamonds) is illustrated. The parameters are \( R = 1 \ [\circ Cm^2/W] \), \( \tau = 1 \) [h]. The notation \( \Delta g(t) = g(t) - \tilde{g}^n(t) \) is used.
Figure 10.9: The approximation error in the time domain as a function of the model order. The solid line shows $\int_{t_{\text{low}}}^{t_{\text{high}}} |y_1(t)| dt$ and the dashed line shows $\int_{t_{\text{low}}}^{t_{\text{high}}} |y_2(t)| dt$. The parameters are $R = 1 \, \text{[\Omega \, \text{cm}^2 / \text{W}]}$, $\tau = 1 \, \text{[h]}$. The notation $\Delta g(t) = g(t) - \hat{g}(t)$ is used.

Concluding remark
Several alternative ways to study the approximation error of the transfer functions or the corresponding weighting functions have been carried out. The emphasis was on investigating how the model order influences the approximation error of the transfer functions and weighting functions. These in turn, affect the parameter bias $\hat{\theta}$, see Chapter 7. The result obtained was that the approximation error decreases with the model order as $O(1/n)$. This could be observed in the figures since the curves, plotted in logarithmic scale, are almost straight lines with a slope of $-1$. 
11 Conclusions

This thesis deals with the diffusion process and its properties. The motivation for the work is to estimate system parameters using various techniques from the field of system identification, and to gain knowledge about the accuracy of the obtained estimates.

In Chapter 2, a brief review of estimation algorithms for general systems is given. Chapter 3 describes the heat diffusion model. The diffusion equation is modeled by a PDE of infinite order. Laplace transformation made it possible to describe the heat equation as a transfer function model, relating the heat flux and the exterior temperature as inputs to the interior temperature as output. The approximated models used are often of finite order, since this is required for some approaches for parameter estimation. The approximation of the diffusion model by finite difference approximation is described in Chapter 4.

Chapter 5 deals with different approaches for parameter estimation for a general system. Five methods are presented, including both approaches in the time domain and in the frequency domain. In Chapter 6, these methods are applied to the heat diffusion problem. It is of primary interest to find out how the bias of the parameter estimates is affected by the model order, and if prefiltering of the output has any positive impact on the estimates. The methods derived in the thesis can essentially be divided into two groups, direct approaches and indirect approaches, cf Sections 5.1–5.3. The methods in the two groups have similar behavior. In the direct approaches (Methods 1–2), a model output is fitted to the given output measurement. In the indirect approaches (Methods 3–5), the transfer function of the system is first estimated using some standard black-box technique. In a second step the system parameters are extracted from the estimated transfer function. Of particular concern in the study is the effects of approximating the dynamics, which is of infinite order, by a finite-order model. The parameter bias as obtained from the five methods have been evaluated numerically.

Several conclusions can be drawn from the evaluations. A model of large order describes the system dynamics well, and hence improves the estimates, but some drawbacks may occur. Using large model orders results in increased computational load, especially for Methods 1 and 2. If complexity is not considered as a problem, a large model order may be used for the direct approaches since the estimation errors due to a finite order model decrease. For the indirect approaches, however, utilizing a black-box model, numerical problems might occur for larger model orders, resulting in bad parameter estimates. This is a serious problem, but incorrect estimates can be sorted out through the use of a scheme for model validation, see Section 6.6.

It can also be observed that the direct approaches require larger model orders than the indirect approaches in order to obtain approximately equal accuracy on the estimates. The indirect approaches are more sensitive to noise on the output than the direct approaches. The indirect approaches give, however, even
for noisy data, a bias of only a few percent when the data are prefiltered with a low pass filter prior to the parameter estimation.

Since the true dynamics is more complex than the model structure used in the approximation, the estimated parameters will be biased. It was observed that the bias of the estimates is larger than the standard deviation. Therefore, the emphasis in the last chapters of the thesis has been to theoretically evaluate the bias, and describe how it is affected by the model order. This is dealt with in a general framework in Chapter 7. It turns out that the bias depends on the Hessian and the gradient of the chosen loss function. The Hessian is approximately independent of the model order and acts hence merely as an amplifier. This results in that the influence of the model order on the bias can be evaluated by studying the gradient of the loss function.

Chapter 8 is of auxiliary character, stating some results that are used in Chapter 9. Chapter 9 contains a theoretical analysis of the parameter bias for the heat equation. The analysis shows that the bias depends on the selected model order as $O(1/n)$, where $n$ is the model order. Finally, Chapter 10 illustrates the findings in the previous chapter.

**Future research**

During the progress of the research that resulted in this thesis, several ideas on open problems for future research have come up.

The suggested approaches for parameter estimation have been applied entirely to simulated data. It would be valuable to test them on real data, which would indicate whether they are useful in a practical situation.

One thing to consider when real data is used is the validity of the heat equation (3.1), (3.2). It has in this thesis been assumed to describe the heat diffusion perfectly. If this is not entirely true, the built-in error in (3.1), (3.2) will affect the result of the parameter estimates. It is of course interesting to find out if the possible equation error exists. If it does but is smaller than the error caused by the use of a finite model order, it can be disregarded. Otherwise, the size of this model error should also be investigated.

Another idea is to develop the estimation technique to recursive algorithms, opening for the possibility to estimate temperature dependent material constants, cf Remark 1 in Section 3.1.

So far, only difference approximation has been studied for approximating the dynamics of the system. Thermal networks were only briefly treated. There are other possibilities for model reduction. One potential approach is the collocation method [18], which is expected to require smaller model orders for equivalent results compared to the difference approximation approach.

Concerning difference approximation, an uneven sampling in the $x$ direction is an approach which would certainly result in smaller parameter bias. However, the $n$ dependence is expected to remain the same.

Since the bias turned out to dominate over the standard deviation of the parameter estimates, all effort has been put on analyzing the bias. A natural
continuation is to study the standard deviation for different estimation methods.

In this thesis, the simplest possible geometry was chosen, an homogeneous wall. More complicated structures are easily found, e.g. the transfer of some substance or heat in fluid. Knowledge of transfer of pollution in still waters can be expected to be of increasing public interest. An expansion of the analysis to more complicated physical structures could be an interesting extension of this work.

If the homogeneous wall seems somewhat simplistic, a wall with segments of different materials can be considered, see Figure 11.1. Some work on parameter estimation of a three-layer wall is presented in [17].

In order to find a solution of the heat equation close to the boundary, the Green’s function [4, 20, 24, 25] can be used. The Green’s function is interpreted as the temperature caused by a single source in the substance, subject to some boundary conditions. It is often expressed as a function with the heat flux as input, and where the impact of \( T_i(t) \) is neglected. This corresponds to an half plane, \( x \geq 0 \). It could be of practical interest to develop an expression that also takes \( T_i(t) \) into consideration, and to study the behaviour close to the boundary \( x = 0, t = 0 \).

The bias studied in Chapter 9 is coupled to the direct parameter estimation approaches, see Sections 5.1 and 5.2. A theoretical analysis of the bias for the indirect approaches, where the system was modeled with an \( n \)th order ARX process, is not given here. Some results regarding the distribution of the bias when the dimension of the system exceeds the model order are given in e.g. [26], but an exact description of the approximation properties is not yet available. This could be a topic for future research.

\[
\begin{array}{c|c|c|c|c}
T(x_1, t) & T(x_2, t) & T(x_n, t) \\
\hline
T_i & \zeta_1 & \zeta_2 & \ldots & \zeta_{n+1} & T_e \\
q_i & \kappa_1 & \kappa_2 & \ldots & \kappa_{n+1} & \\
\end{array}
\]

**Figure 11.1:** Heat diffusion through a wall with several homogeneous segments.
Also, for the direct approaches, it remains to evaluate the exact dependence of the model order, or, an upper bound of the bias for each model order.

The list is by no means exhaustive, but includes some ideas that might be issues of future work. Research will always result in answers but also in new questions.
A Proofs of results in Chapter 3

A.1 Proof of Lemma 3.1

First, consider the transfer function $G_2(s)$. It is in its original form written as

$$G_2(s) = \frac{1}{\cosh(\sqrt{s\tau})} \quad \text{(A.1)}$$

Using partial decomposition, (A.1) may also be written

$$G_2(s) = \sum_{k=1}^{\infty} \frac{\beta_{2,k}}{s\tau + \alpha_k} + D_2 \quad \text{(A.2)}$$

The problem is now to determine $\alpha_k$, $\beta_{2,k}$ and $D_2$. The poles of the system are $\{-\alpha_k/\tau\}$, and are given by

$$\cosh(\sqrt{s\tau}) = 0 \Rightarrow e^{\sqrt{s\tau}} + e^{-\sqrt{s\tau}} = 0 \Rightarrow e^{2\sqrt{s\tau}} = -1$$

leading to

$$s\tau = -\frac{\pi^2}{4}(2k - 1)^2$$

and

$$p_k = -\frac{\pi^2}{4\tau}(2k - 1)^2$$

The parameters $\{\alpha_k\}$ are hence given by

$$\alpha_k = \frac{\pi^2(2k - 1)^2}{4}, \quad k = 1, 2, \ldots \quad \text{(A.3)}$$

In order to determine $\beta_{2,k}$, multiply (A.2) with $(s\tau + \alpha_j)$, where $j$ is some index larger than or equal to 1. This gives

$$G_2(s)(s\tau + \alpha_j) = \sum_{k=1}^{\infty} \frac{\beta_{2,k}}{s\tau + \alpha_k} + D_2(s\tau + \alpha_j)$$

$$= \beta_{2,j} + \sum_{k=1}^{\infty} \frac{\beta_{2,k} s\tau + \alpha_j}{s\tau + \alpha_k} + D_2(s\tau + \alpha_j) \quad \text{(A.4)}$$

Now let $s\tau \rightarrow -\alpha_j$, resulting the right hand side of (A.4) to go towards $\beta_{2,j}$. The left hand side of (A.4) behaves as

$$\lim_{s\tau \rightarrow -\alpha_j} [G_2(s)(s\tau + \alpha_j)] \quad \text{(A.5)}$$
When $s \tau \to -\alpha_j$. It can be noted that $\cosh(\sqrt{s\tau})|_{s\tau=-\alpha_j}=0$ since $-\alpha_j/\tau$ is a pole of $G_2(s)$. Using l'Hospital’s rule and (A.1), (A.5) becomes

$$\lim_{s\tau \to -\alpha_j} [G_2(s)(s\tau + \alpha_j)] = \frac{\beta_{2,j}(s\tau + \alpha_j)}{s \cosh(\sqrt{s\tau})|_{s\tau=-\alpha_j}} = \frac{2\sqrt{-\alpha_j}}{\sinh(-\alpha_j)} \tag{A.6}$$

This gives that the parameters $\beta_{2,j}$ equal (A.6). This expression can be simplified by insertion of $\alpha_k$. Developing the square root and noting that $\sin(\pi(k - \frac{1}{2})) = (-1)^{k+1}$ gives

$$\beta_{2,k} = \frac{2\sqrt{-\alpha_k}}{\sinh(-\alpha_k)} = \frac{i\pi(2k-1)}{\sinh\left(\frac{i\pi}{2}(2k-1)\right)}$$

$$= \frac{i\pi(2k-1)}{i\sin\left(\frac{i\pi}{2}(2k-1)\right)} = \pi(2k-1)(-1)^{k+1} \tag{A.7}$$

Finally $D_2$ is determined by considering the static gain of (A.1), that is, let $s = 0$. It is known that the static gain of $G_2(s)$ equals 1, since $G_2(0) = (\cosh(0))^{-1} = 1$. Therefore

$$G_2(0) = 1 = \sum_{k=1}^{\infty} \frac{\beta_{2,k}}{\alpha_k} + D_2 \tag{A.8}$$

resulting in an expression for $D_2$

$$D_2 = 1 - \sum_{k=1}^{\infty} \frac{\beta_{2,k}}{\alpha_k} \tag{A.9}$$

The actual value of $D_2$ can be evaluated by insertion of expressions of $\alpha_k$ (A.3) and $\beta_{2,k}$ (A.7) in (A.9). This results in the following expression for $D_2$.

$$D_2 = 1 - \sum_{k=1}^{\infty} \frac{4\pi(2k-1)(-1)^{k+1}}{\pi^2(2k-1)^2}$$

$$= 1 - \sum_{k=1}^{\infty} \frac{4(-1)^{k+1}}{\pi(2k-1)}$$

$$= 1 - \frac{4}{\pi} \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \right)$$

$$= 1 - \frac{4}{\pi} \arctan(1)$$

$$= 0 \tag{A.10}$$

So far (3.22), (3.23), (3.25) and (3.27) are proved. Next consider the transfer function

$$G_1(s) = R \frac{\tanh(\sqrt{s\tau})}{\sqrt{s\tau}} = \frac{R \sinh(\sqrt{s\tau})}{\sqrt{s\tau} \cosh(\sqrt{s\tau})} \tag{A.11}$$
A.1 Proof of Lemma 3.1

which is also expressed as a sum using partial decomposition,

\[
G_1(s) = R \sum_{k=1}^{\infty} \frac{\beta_{1,k}}{s^\tau + \alpha_k} + RD_1
\]  
(A.12)

The poles of \(G_1(s)\) equals the poles of \(G_2(s)\), which gives that \(\alpha_k\) is equal in both transfer functions. The parameters \(\beta_{1,k}\) and \(D_1\) differ from what was obtained for \(G_2(s)\), but the procedure to obtain them is the same as above, resulting in

\[
\beta_{1,k} = \frac{2\sqrt{-\alpha_k} \sinh(\sqrt{-\alpha_k})}{\cosh(\sqrt{-\alpha_k}) + \sqrt{-\alpha_k} \sinh(\sqrt{-\alpha_k})}
\]

\[
= \frac{-2\pi (k - \frac{1}{2}) \sin(\pi (k - \frac{1}{2}))}{\cos(\pi (k - \frac{1}{2})) - \pi (k - \frac{1}{2}) \sin(\pi (k - \frac{1}{2}))}
\]

\[
= 2
\]  
(A.13)

where the expression for \(\alpha_k\) was used. Further,

\[
D_1 = 1 - \sum_{k=1}^{\infty} \frac{\beta_{1,k}}{\alpha_k}
\]

\[
= 1 - \sum_{k=1}^{\infty} \frac{2}{\pi^2 (k - \frac{1}{2})^2}
\]

\[
= 1 - \frac{2}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(k - \frac{1}{2})^2}
\]

\[
= 1 - \frac{8}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k + 1)^2}
\]

\[
= 1 - \frac{8}{\pi^2} \frac{\pi^2}{8}
\]

\[
= 0
\]  
(A.14)

where it was observed that \(\frac{1}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k + 1)^2}\) is a Bernoulli number equal to 1/6, [15]. This concludes the proof of Lemma 3.1.
B Proofs of results in Chapter 4

B.1 Proof of Lemma 4.1

The approximate transfer functions will have the following structure,

\[
\tilde{G}_1^n(s) = R \sum_{k=1}^{n} \frac{\tilde{\beta}_{1,k}}{s^{\tau} + \tilde{\alpha}_k} + RD_1
\]

\[
\tilde{G}_2^n(s) = \sum_{k=1}^{n} \frac{\tilde{\beta}_{2,k}}{s^{\tau} + \tilde{\alpha}_k} + \tilde{D}_2
\]

(B.1)

(B.2)

If the model (4.3)-(4.7) is diagonalized [27] it is easy to obtain the wanted parameters in (B.1) and (B.2).

Introduce a new state vector \( \tilde{x} \triangleq Tx \) where \( T \) is a transformation matrix. Then the model (written using ‘standard notation’) \( \dot{x} = Ax + Bu \) \( y = Cx + Du \) is transformed as

\[
\dot{\tilde{x}} = T \dot{x} = TAx + TBu = TAT^{-1}\tilde{x} + TBu
\]

\[
y = Cx + Du = CT^{-1}\tilde{x} + Du
\]

into

\[
\dot{\tilde{x}} = TAT^{-1}\tilde{x} + TBu \triangleq \tilde{A}\tilde{x} + \tilde{B}u
\]

\[
y = CT^{-1}\tilde{x} + Du \triangleq \tilde{C}\tilde{x} + Du
\]

(B.3)

(B.4)

where \( x \) and \( \tilde{x} \) here are the old respective the new state vector and where the matrix \( TAT^{-1} \) is diagonal. Since the system matrix \( \tilde{A} \) is diagonal, the transfer function associated to model (B.3)-(B.4) can conveniently be rewritten as

\[
\tilde{G}_2^n(s) = \sum_{j=1}^{n} \frac{\tilde{C}_j\tilde{B}_j}{s - \tilde{A}_{jj}} + \tilde{D}
\]

(B.5)

The parameters \( \tilde{\alpha}_k \), \( \tilde{\beta}_{1,k} \), \( \tilde{\beta}_{2,k} \), \( \tilde{D}_1 \) and \( \tilde{D}_2 \) are obtained from (B.5). The problem of finding the structure (B.1), (B.2) is thus reduced to determine how \( \tilde{C} \), \( \tilde{B} \) and \( \tilde{A} \) depend on the eigenvalues and eigenvectors, and then to determine these. Note that for the diffusion model, \( \tilde{B} \) and \( \tilde{D} \) are not vectors but matrices, since there are two input signals.

The matrices \( A \) and \( \tilde{A} = TAT^{-1} \) are similar, which gives that \( \lambda(A) = \lambda(\tilde{A}) \), where \( \lambda \) is an eigenvalue. The poles of the system equal the eigenvalues of the system matrix, [27]. Comparing (B.1), (B.2) with (B.5), then gives that

\[
\tilde{\alpha}_k = -\lambda_k(\tilde{A}) \tau
\]

(B.6)
The other parameters are identified similarly.

The transformation matrix \( T^{-1} \) is built up from the eigenvectors in the columns. The eigenvalues are distinct giving that the eigenvectors are linearly independent. Hence, \( T^{-1} \) is a non-singular matrix.

In the following, the elements of a specific eigenvector will be denoted \( x_{j,k} \), where the index \( j \) gives the position in the \( k \)th eigenvector. The quantity \( x_j \) denotes the \( j \)th element in an arbitrary eigenvector. The transformation matrix is thus constructed in the following way,

\[
T^{-1} = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n,1} & x_{n,2} & \cdots & x_{n,n} \end{pmatrix} \quad (B.7)
\]

The matrix \( A \) is symmetric and the eigenvectors can be chosen as real valued and with norms equal to one. Then \( T \) becomes orthogonal and \( T^{-1} = T^T \). Hence

\[
T = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n,1} & x_{n,2} & \cdots & x_{n,n} \end{pmatrix} \quad (B.8)
\]

Now use the transformation matrix to rewrite the original model (4.3)-(4.7)

\[
\begin{cases}
\dot{x} &= \mu Ax + (\mu \gamma e_1 \mu e_n) \begin{pmatrix} q_i \\ T_e \end{pmatrix} \\
y &= T_i = e_i^T x + \gamma q_i
\end{cases} \quad (B.9)
\]

to the diagonalized model

\[
\begin{cases}
\dot{\bar{x}} &= \mu T A T^{-1} \bar{x} + T (\mu \gamma e_1 \mu e_n) \begin{pmatrix} q_i \\ T_e \end{pmatrix} \\
\tilde{y} &= e_i^T T^{-1} \bar{x} + \gamma q_i
\end{cases} \quad (B.10)
\]

The sparse structure of \( c_1 \) and \( e_n \) in (4.7) are utilized in order to determine the components of the diagonal form (B.10).

\[
\mu T A T^{-1} = \mu \text{diag}(\lambda_k), \quad k = 1, \ldots, n \quad (B.11)
\]

\[
T \mu \gamma e_1 = \mu \gamma \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\ x_{1,2} & x_{2,2} & \cdots & x_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1,n} & x_{2,n} & \cdots & x_{n,n} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \mu \gamma \begin{pmatrix} x_{1,1} \\ x_{1,2} \\ \vdots \\ x_{1,n} \end{pmatrix} \quad (B.12)
\]
\begin{equation}
T \mu_{n} = \mu \begin{pmatrix}
x_{1,1} & x_{2,1} & \cdots & x_{n,1} \\
x_{1,2} & x_{2,2} & \cdots & x_{n,2} \\
\vdots & \vdots & \ddots & \vdots \\
x_{1,n} & x_{2,n} & \cdots & x_{n,n}
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
\vdots \\
1
\end{pmatrix}
= \mu \begin{pmatrix}
x_{n,1} \\
x_{n,2} \\
\vdots \\
x_{n,n}
\end{pmatrix}
\tag{B.13}
\end{equation}

\begin{equation}
P^{-1}_1 = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}
= \begin{pmatrix}
x_{1,1} & x_{1,2} & \cdots & x_{1,n}
\end{pmatrix}
\tag{B.14}
\end{equation}

The diagonalized model thus has the following structure,
\begin{equation}
\begin{cases}
\tilde{\xi} = \mu \begin{pmatrix}
\lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_n
\end{pmatrix} \tilde{\xi} + 
\begin{pmatrix}
\mu \gamma x_{1,1} \\
\mu \gamma x_{1,2} \\
\vdots \\
\mu \gamma x_{1,n}
\end{pmatrix} \begin{pmatrix}
q_i \\
q_i \\
\vdots \\
q_i
\end{pmatrix} T_e \\
y = \begin{pmatrix}
x_{1,1} & x_{1,2} & \cdots & x_{1,n}
\end{pmatrix} \tilde{\xi} + \begin{pmatrix}
\gamma \\
0
\end{pmatrix} \begin{pmatrix}
q_i \\
q_i \\
\vdots \\
q_i
\end{pmatrix} T_e
\end{cases}
\tag{B.15}
\end{equation}

where \( \mu \) and \( \gamma \) are obtained from (4.2). The associated transfer functions, (B.1) and (B.2), may now be expressed as
\begin{equation}
G_1^H(s) = \mu \gamma \sum_{k=1}^{n} \frac{x_{1,k}^2}{s t - \mu \lambda_k t} + \gamma 
\tag{B.16}
\end{equation}
\begin{equation}
G_2^H(s) = \mu \sum_{k=1}^{n} \frac{x_{n,k} \bar{x}_{1,k} t}{s t - \mu \lambda_k t} 
\tag{B.17}
\end{equation}

Now there is a model in diagonal form. In order to express (B.16) and (B.17) explicitly, the eigenvalues and eigenvectors are needed. These are determined next. It is noted that
\begin{equation}
eig(\mu A) = \mu \eig(A) 
\tag{B.18}
\end{equation}

and that the eigenvalues are obtained from the characteristic equation
\begin{equation}
eig(A) \Leftrightarrow |A - \lambda I| = 0 
\tag{B.19}
\end{equation}

where
\begin{equation}
A = \begin{pmatrix}
-1 & 1 & 0 \\
1 & -2 & 1 \\
0 & -2 & 1 \\
1 & -2
\end{pmatrix}
\tag{B.20}
\end{equation}
is the $n \times n$ system matrix from the difference approximation model, (4.6), and where $\lambda$ is an eigenvalue of the matrix $A$. The eigenvalues and eigenvectors are determined using the standard procedure, $(A - \lambda I)x = 0$, where $x \neq 0$ is an eigenvector corresponding to $\lambda$.

Since the matrix $A$ is sparse and has a particular tridiagonal structure that can be utilized, the following system of equations

$$(A - \lambda I)x = 0$$

implies

$$x_{j-2} + (-2 - \lambda)x_{j-1} + x_j = 0, \quad j = 3, \ldots, n$$

(B.21)

with the boundary conditions

$$(-1 - \lambda)x_1 + x_2 = 0$$

(B.22)

$$x_{n-1} - (2 + \lambda)x_n = 0$$

(B.23)

The characteristic equation of the difference equation (B.21) equals

$$z^2 - (2 + \lambda)z + 1 = 0$$

(B.24)

and has the solutions

$$z_i = \frac{2 + \lambda}{2} \pm \frac{\sqrt{4(\lambda + 4)}}{2}, \quad i = 1, 2$$

(B.25)

It can be noted that $z_2 = z_1^{-1}$. The solution to (B.21) has the form

$$x_j = c_1 z_1^j + c_2 z_2^j, \quad j = 1, \ldots, n$$

(B.26)

for some constants $c_1$ and $c_2$. Inserting (B.26) into the boundary conditions (B.22)–(B.23), results in

$$\begin{pmatrix}
-(1 + \lambda)z_1 + z_1^2 & -(1 + \lambda)z_2 + z_2^2 \\
\frac{z_1}{z_1^{n-1} - (2 + \lambda)z_1^{n-1}} & \frac{z_2}{z_2^{n-1} - (2 + \lambda)z_2^{n-1}}
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}$$

(B.27)

Utilizing the fact that $z_2 = z_1^{-1}$ simplifies (B.27) to

$$\begin{pmatrix}
-(1 + \lambda)z_1 + z_1^2 & -(1 + \lambda)z_1^{-1} + z_1^{-2} \\
\frac{z_1}{z_1^{n-1} - (2 + \lambda)z_1^{n-1}} & \frac{z_1^{-1}}{z_1^{-n+1} - (2 + \lambda)z_1^{-n}}
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}$$

(B.28)

Denote the $(2 \times 2)$ matrix in (B.28) by $Z$. As $(c_1 \ c_2)^T$ is nonzero it must hold that $\det(Z) = 0$. This implies

$$[z_1^2 - (1 + \lambda)z_1][z_1^{-n+1} - (2 + \lambda)z_1^{-n}]$$

$$- [z_1^2 - (1 + \lambda)z_1^{-1}][z_1^{n-1} - (2 + \lambda)z_1] = 0$$

(B.29)
Utilizing (B.24) and \( z = z_1 z_2^{-1}, z z^{-1} = 1 \), makes it possible to eliminate \( \lambda \),

\[
\lambda = z + z^{-1} - 2 
\]  

(B.30)

Use of equation (B.30) now simplifies (B.29) to

\[
z^n - z^{-n} - z^{n+1} + z^{-(n+1)} = 0 
\]  

(B.31)

Let \( z = e^{i \omega} \). Equation (B.31) then becomes

\[
[e^{i \omega n} - e^{-i \omega n}] - [e^{i \omega (n+1)} - e^{-i \omega (n+1)}] = 0 
\]  

(B.32)

which can be simplified to

\[
2 \sin \left( \frac{\omega n - \omega n - \omega}{2} \right) \cos \left( \frac{\omega n + (\omega n + \omega)}{2} \right) = 0 
\]  

(B.33)

\[
-2 \sin \left( \frac{\omega}{2} \right) \cos \left( \frac{(2n+1)\omega}{2} \right) = 0 
\]  

(B.33)

Equation (B.33) holds if

\[
\frac{(2n+1)\omega}{2} = \frac{\pi}{2} + k\pi, \; k = 1, \ldots, n 
\]  

(B.34)

This gives the frequencies of interest as

\[
\omega_k = \frac{\pi(2k - 1)}{2n + 1}, \; k = 1, \ldots, n 
\]  

(B.35)

It follows that \( z_k = e^{i \omega_k} = \cos(\omega_k) + i \sin(\omega_k) \). Now the eigenvalues of the

\( A \) matrix (B.20) can be evaluated using (B.30) and (B.35). There will be one
eigenvalue \( \lambda_k \) for each frequency point in \( \{ \omega_k \} \).

\[
\lambda_k = z_k + z_k^{-1} - 2 
\]  

\[
e^{i \omega_k} + e^{-i \omega_k} - 2 
\]  

\[
= 2 \left( \cos \left( \frac{\pi(2k - 1)}{2n + 1} \right) - 1 \right), \; k = 1, \ldots, n 
\]  

(B.36)

Hence, the quantity \( \tilde{\alpha}_k \) (B.6) is given by

\[
\tilde{\alpha}_k = -\mu \lambda_k \tau 
\]  

\[
= 2 \mu \left( 1 - \cos \left( \frac{\pi(2k - 1)}{2n + 1} \right) \right) \tau 
\]  

\[
= 2(n + 1)^2 \left( 1 - \cos \left( \frac{\pi(2k - 1)}{2n + 1} \right) \right) 
\]  

(B.37)
Next the parameters $\tilde{\beta}_{1,k}$ and $\tilde{\beta}_{2,k}$ associated with $G_1^1(s)$ and $G_2^2(s)$ are determined. The elements in the eigenvector $\{x_j\}$ are obtained from (B.26),

$$
x_j = c_1 z^j + c_2 z^{-j}
$$

where $z \triangleq z_1 = z_2^{-1}$ is utilized. The constants $c_1$ and $c_2$ are determined from (B.22), (B.30) and (B.38), giving the relation

$$
zc_1 = c_2
$$

Inserting (B.39) into (B.38) gives the expression for the elements in the eigenvector

$$
x_j = c_1 z^j + z c_1 z^{-j} = c_1 (z^j + z^{-j}), \quad j = 1, \ldots, n
$$

The constant $c_1$ may be chosen arbitrarily. If $c_1$ is chosen as $z^{-1/2}$, the elements in the eigenvectors become real valued. Insert $z = e^{i\omega_k}$ into (B.40) where $\omega_k$ is obtained from (B.35). The eigenvectors then get the following expression.

$$
x_{j,k} = z^{-1/2} (z^j + z^{-j}) = z^{(1/2-j)} + z^{1/2-j} = e^{-i\omega_k (1/2-j)} + e^{i\omega_k (1/2-j)} = 2 \cos (\omega_k (j - 1/2)) = 2 \cos \left( \frac{\pi (2k-1)(j-\frac{1}{2})}{2n+1} \right), \quad k = 1, \ldots, n, \quad j = 1, \ldots, n
$$

where (B.35) was used in the last step. The eigenvector (B.41) is next modified so that $\|x_j\| = 1$.

$$
x_{j,k} = \frac{2 \cos \left( \frac{\pi (2k-1)(j-\frac{1}{2})}{2n+1} \right)}{\sqrt{\sum_{k=1}^{n} 2 \cos^2 \left( \frac{\pi (2k-1)(j-\frac{1}{2})}{2n+1} \right)}} = \frac{\cos \left( \frac{\pi (2k-1)(j-\frac{1}{2})}{2n+1} \right)}{\sqrt{\sum_{k=1}^{n} \cos^2 \left( \frac{\pi (2k-1)(j-\frac{1}{2})}{2n+1} \right)}}
$$

Comparing (B.1)-(B.2) and (B.16)-(B.17), the wanted parameters may now be
identified.

\[
\tilde{\alpha}_k = -\mu \lambda_k \tau = 2(n + 1)^2 \left( 1 - \cos \left( \frac{\pi(2k - 1)}{2n + 1} \right) \right) \quad (B.43)
\]

\[
\tilde{\beta}_{1,k} = \mu \gamma \tau x_{1,k}^2 = (n + 1) \frac{\cos^2 \left( \frac{\pi(2k - 1)}{2(2n+1)} \right)}{\sum_{k=1}^{n} \cos^2 \left( \frac{\pi(2k - 1)}{2(2n+1)} \right)} \quad (B.44)
\]

\[
\tilde{\beta}_{2,k} = \mu \tau x_{n,k} x_{1,k} \left( n + 1 \right)^2 \frac{\cos \left( \frac{\pi(2k - 1)(n - 1)}{2n+1} \right) \cos \left( \frac{\pi(2k - 1)}{2(2n+1)} \right)}{\sqrt{\sum_{k=1}^{n} \cos^2 \left( \frac{\pi(2k - 1)(n - 1)}{2n+1} \right) \sqrt{\sum_{k=1}^{n} \cos^2 \left( \frac{\pi(2k - 1)}{2(2n+1)} \right)}}} \quad (B.45)
\]

\[
\tilde{D}_1 = \frac{\gamma}{\tau} = \frac{1}{n + 1} \quad (B.46)
\]

\[
\tilde{D}_2 = 0 \quad (B.47)
\]

This concludes the proof of Lemma 4.1.
B.2 Proof of Lemma 4.2

Consider the first normalization factor in $\beta_{1,k}$ (4.15), and let it be denoted by $N_1$. In order to simplify the expressions below, let

$$u_{1,n} = \frac{\pi(2n - 1)}{2n + 1} \quad (B.48)$$

Then

$$N_1^2 = \sum_{k=1}^{n} \cos^2 \left( \left(k - \frac{1}{2}\right) u_{1,n} \right) = \sum_{k=1}^{n} \left( \frac{1}{2} \cos((2k - 1)u_{1,n}) + \frac{1}{2} \right) = \frac{n}{2} + \frac{1}{2} \sum_{k=1}^{n} \cos((2k - 1)u_{1,n}) \quad (B.49)$$

The sum in (B.49) is computed under the assumption that $n \to \infty$. The following known sums of trigonometric functions [15] will be used,

$$\sum_{k=1}^{n} \sin(kx) = \frac{\sin \left( \frac{nx}{2} \right) \sin \left( \frac{(n+1)x}{2} \right)}{\sin \left( \frac{x}{2} \right)} \quad (B.50)$$

$$\sum_{k=0}^{n} \cos(kx) = \frac{\cos \left( \frac{nx}{2} \right) \sin \left( \frac{(n+1)x}{2} \right)}{\sin \left( \frac{x}{2} \right)} \quad (B.51)$$

Then

$$\sum_{k=1}^{n} \cos \left( (2k - 1)u_{1,n} \right) = \sum_{k=1}^{n} \cos(2u_{1,n}k - u_{1,n})$$

$$= \cos(u_{1,n}) \sum_{k=1}^{n} \cos(2u_{1,n}k) + \sin(u_{1,n}) \sum_{k=1}^{n} \sin(2u_{1,n}k)$$

$$= \cos(u_{1,n}) \sum_{k=0}^{n} \cos(2u_{1,n}k) - \cos(u_{1,n}) +$$

$$+ \sin(u_{1,n}) \sum_{k=1}^{n} \sin(2u_{1,n}k)$$

$$= \cos(u_{1,n}) \left( \frac{\cos(mu_{1,n}) \sin((n + 1)u_{1,n})}{\sin(u_{1,n})} - 1 \right) +$$

$$+ \sin(u_{1,n}) \frac{\sin(mu_{1,n}) \sin((n + 1)u_{1,n})}{\sin(u_{1,n})} \quad (B.52)$$
Further
\[
\sum_{k=1}^{n} \cos((2k-1)u_{1,n}) = \cos(u_{1,n}) \left( \frac{\cos(nu_{1,n}) \sin((n+1)u_{1,n})}{\sin(u_{1,n})} - 1 \right)
+ \sin(nu_{1,n}) \sin((n+1)u_{1,n}) \tag{B.53}
\]

The different components in (B.53) are studied where \( n \to \infty \) is assumed to hold.

\[
\cos(u_{1,n}) = \cos \left( \frac{\pi(2n-1)}{2n+1} \right) = \cos \left( \frac{\pi \left( \frac{2}{n} - \frac{1}{n} \right)}{2 + \frac{1}{n}} \right) \to -1 \tag{B.54}
\]

\[
\cos(nu_{1,n}) = \cos \left( \frac{\pi(2n-1)n}{2n+1} \right) = \cos \left( \pi n - \frac{2\pi n}{2n+1} \right)
= \cos(\pi n) \cos \left( \frac{2\pi n}{2n+1} \right) + \sin(\pi n) \sin \left( \frac{2\pi n}{2n+1} \right)
= (-1)^n \cos \left( \frac{2\pi n}{2n+1} \right) = (-1)^n \cos \left( \frac{2\pi}{2 \left( 1 + \frac{1}{n} \right)} \right) \approx (-1)^{n+1} \tag{B.55}
\]

\[
\sin(u_{1,n}) \to \sin(\pi) = 0 \tag{B.56}
\]

\[
\sin(nu_{1,n}) \to \sin(n\pi) = 0 \tag{B.57}
\]

\[
\sin((n+1)u_{1,n}) = \sin(nu_{1,n}) \cos(u_{1,n}) + \cos(nu_{1,n}) \sin(u_{1,n}) \to 0 \tag{B.58}
\]

The problem with the expression
\[
\frac{\sin((n+1)u_{1,n})}{\sin(u_{1,n})}
\]
in (B.53) is that both the numerator and the denominator approach zero when \( n \to \infty \). It is not possible to use l'Hôpital's rule [28], since derivation gives a zero in the denominator. Instead trigonometric rules are used to provide useful approximations of the sinus functions,

\[
\frac{\sin((n+1)u_{1,n})}{\sin(u_{1,n})} = \frac{\sin(nu_{1,n}) \cos(u_{1,n}) + \cos(nu_{1,n}) \sin(u_{1,n})}{\sin(u_{1,n})} = \frac{\sin(nu_{1,n}) \cos(u_{1,n})}{\sin(u_{1,n})} + \cos(nu_{1,n})
\approx \frac{\sin(nu_{1,n})}{\sin(u_{1,n})}(-1) + (-1)^{n+1} \tag{B.59}
\]

where (B.54) and (B.55) were used. The arguments of \( \sin(nu_{1,n}) \) and \( \sin(u_{1,n}) \) are close to \( \pi \) and \( n\pi \), respectively, cf (B.48). The sinus functions are rewritten so that their arguments are small. In that case the arguments themselves
approximate the functions well. [15]. Let
\[
\sin(u_{1,n}) = \sin \left( \frac{\pi(2n - 1)}{2n + 1} \right) \\
= \sin \left( \pi - \frac{2\pi}{2n + 1} \right) \\
= \sin(\pi) \cos \left( \frac{2\pi}{2n + 1} \right) - \cos(\pi) \sin \left( \frac{2\pi}{2n + 1} \right) \\
= \sin \left( \frac{2\pi}{2n + 1} \right) \\
\approx \frac{2\pi}{2n + 1} \quad \text{(B.60)}
\]
since \( n \) is assumed to be large. Next proceed with \( \sin(mu_{1,n}) \). Then
\[
\sin(mu_{1,n}) = \sin \left( n\pi - \frac{2\pi n}{2n + 1} \right) \\
= \sin(n\pi) \cos \left( \frac{2\pi n}{2n + 1} \right) - \cos(n\pi) \sin \left( \frac{2\pi n}{2n + 1} \right) \\
= (-1)^{n+1} \sin \left( \frac{2\pi n}{2n + 1} \right) \\
= (-1)^{n+1} \left[ -\sin \left( \frac{2\pi n}{2n + 1} - \pi \right) \right] \\
= (-1)^{n+1} \left[ -\sin \left( -\frac{\pi}{2n + 1} \right) \right] \\
= (-1)^{n+1} \sin \left( \frac{\pi}{2n + 1} \right) \\
\approx (-1)^{n+1} \frac{\pi}{2n + 1} \quad \text{(B.61)}
\]
Now (B.60) and (B.61) are inserted into (B.59) resulting in
\[
\lim_{n \to \infty} \frac{\sin((n + 1)u_{1,n})}{\sin(u_{1,n})} = \lim_{n \to \infty} \left( \frac{(-1)^{n+1} - \frac{\pi}{2n+1}}{2\pi} \right) \left( -1 + (-1)^{n+1} \right) \\
= \lim_{n \to \infty} \left( \frac{1}{2} (-1)^{n+2} + (-1)^{n+1} \right) \\
= \frac{1}{2} (-1)^{n+2} + (-1)^{n+1} \\
= \frac{1}{2} (-1)^n - (-1)^n \\
= \frac{1}{2} (-1)^{n+1} \quad \text{(B.62)}
\]
Finally, the expression of (B.53) is obtained as

\[ \sum_{k=1}^{n} \cos((2k-1)u_{1,n}) \approx -\left((-1)^{n+1} \times \frac{1}{2}(-1)^{n+1} - 1\right) = \frac{1}{2} \quad (B.63) \]

When \( n \to \infty \), the following value of \( N_1^2 \) is then obtained, cf. (B.49).

\[ N_1^2 = \frac{n}{2} + \frac{1}{4} \]

giving the normalization factor

\[ N_1 = \sqrt{\frac{n}{2} + \frac{1}{4}} \quad (B.64) \]

Next consider the second normalization factor in (B.45). (It appears also in (B.44)). Let

\[ u_{2,n} = \frac{\pi(2k - 1)}{2n + 1} \quad (B.65) \]

Then

\[ N_2^2 = \sum_{k=1}^{n} \cos^2\left(\frac{\pi(2k - 1)}{2(2n + 1)}\right) \]

\[ = \sum_{k=1}^{n} \cos^2\left(\frac{u_{2,n}}{2}\right) \]

\[ = \sum_{k=1}^{n} \left(\frac{1}{2} \cos(u_{2,n}) + \frac{1}{2}\right) \]

\[ = \frac{n}{2} + \frac{1}{2} \sum_{k=1}^{n} \cos(u_{2,n}) \quad (B.66) \]

The sum in (B.66) can be rewritten as a geometric series, which can be utilized for the proof of the lemma.
\[
2 \sum_{k=1}^{n} \cos(u_{2,n}) = \sum_{k=1}^{n} (e^{iu_{2,n}} + e^{-iu_{2,n}})
= \sum_{k=1}^{n} e^{\frac{i \pi (2k-1)}{2n+1}} + \sum_{k=1}^{n} e^{-\frac{i \pi (2k-1)}{2n+1}}
= e^{-\frac{i \pi}{2n+1}} \left( \sum_{k=0}^{n-1} e^{\frac{i \pi (2k+1)}{2n+1}} - 1 + e^{\frac{i \pi}{2n+1}} \right)
+ e^{\frac{i \pi}{2n+1}} \left( \sum_{k=0}^{n-1} e^{-\frac{i \pi (2k+1)}{2n+1}} - 1 + e^{-\frac{i \pi}{2n+1}} \right)
\]

The sums of the geometric series in (B.67) are found in e.g. [15]. Then
\[
2 \sum_{k=1}^{n} \cos(u_{2,n}) = e^{-\frac{i \pi}{2n+1}} \left( \frac{1 - e^{\frac{i \pi}{2n+1}}}{1 - e^{\frac{i \pi}{2n+1}}} - 1 + e^{\frac{i \pi}{2n+1}} \right)
+ e^{\frac{i \pi}{2n+1}} \left( \frac{1 - e^{\frac{i \pi}{2n+1}}}{1 - e^{\frac{i \pi}{2n+1}}} - 1 + e^{-\frac{i \pi}{2n+1}} \right)
= \frac{e^{-\frac{i \pi}{2n+1}} - e^{\frac{i \pi}{2n+1}}}{1 - e^{\frac{i \pi}{2n+1}}} - e^{-\frac{i \pi}{2n+1}} + e^{\frac{i \pi}{2n+1}}
+ \frac{e^{\frac{i \pi}{2n+1}} - e^{-\frac{i \pi}{2n+1}}}{1 - e^{-\frac{i \pi}{2n+1}}} - e^{\frac{i \pi}{2n+1}} + e^{-\frac{i \pi}{2n+1}}
= - \left( e^{-\frac{i \pi}{2n+1}} + e^{\frac{i \pi}{2n+1}} \right) + e^{\frac{i \pi}{2n+1}} + e^{-\frac{i \pi}{2n+1}}
+ \left( e^{-\frac{i \pi}{2n+1}} + e^{\frac{i \pi}{2n+1}} \right) + e^{\frac{i \pi}{2n+1}} - e^{-\frac{i \pi}{2n+1}}
\]

Further, rewriting the exponentials as cosine functions gives
\[
2 \sum_{k=1}^{n} \cos(u_{2,n}) = -2 \cos \left( \frac{\pi}{2n+1} \right) + 2 \cos \left( \frac{\pi(2n-1)}{2n+1} \right)
+ \frac{e^{-\frac{i \pi}{2n+1}} - e^{\frac{i \pi}{2n+1}}}{2 - e^{-\frac{i \pi}{2n+1}}} - e^{-\frac{i \pi}{2n+1}} + e^{\frac{i \pi}{2n+1}}
+ \frac{e^{\frac{i \pi}{2n+1}} - e^{-\frac{i \pi}{2n+1}}}{2 - e^{\frac{i \pi}{2n+1}}} - e^{\frac{i \pi}{2n+1}} + e^{-\frac{i \pi}{2n+1}}
\]
which equals

\[ 2 \sum_{k=1}^{n} \cos(u_{2,n}) = -2 \cos \left( \frac{\pi}{2n+1} \right) + 2 \cos \left( \frac{\pi(2n-1)}{2n+1} \right) \]

\[ + \frac{2 \cos \left( \frac{\pi}{2n+1} \right) - 2 \cos \left( \frac{\pi(2n-1)}{2n+1} \right)}{2 - 2 \cos \left( \frac{2\pi}{2n+1} \right)} \]

\[ + \frac{-2 \cos \left( \frac{3\pi}{2n+1} \right) + 2 \cos \left( \frac{\pi(2n-3)}{2n+1} \right)}{2 - 2 \cos \left( \frac{2\pi}{2n+1} \right)} \]  \hspace{1cm} (B.70)

This expression can be simplified by rewriting the terms in (B.70). The following relation is used,

\[ \sin(x) \sin(y) = \frac{1}{2} \left[ \cos(x - y) - \cos(x + y) \right] \]  \hspace{1cm} (B.71)

Using (B.71), terms in (B.70) are combined in pairs and simplified as follows,

\[ 2 \cos \left( \frac{\pi}{2n+1} \right) - 2 \cos \left( \frac{3\pi}{2n+1} \right) = 2 \cos \left( \frac{2\pi}{2n+1} \right) - 2 \cos \left( \frac{2\pi}{2n+1} \right) \]

\[ = 4 \sin \left( \frac{2\pi}{2n+1} \right) \sin \left( \frac{\pi}{2n+1} \right) \]  \hspace{1cm} (B.72)

Further,

\[ -2 \cos \left( \frac{\pi(2n-1)}{2n+1} \right) + 2 \cos \left( \frac{\pi(2n-3)}{2n+1} \right) = \]

\[ = -2 \left[ \cos \left( \frac{\pi(2n-2)+1}{2n+1} \right) - \cos \left( \frac{\pi(2n-2)}{2n+1} \right) \right] \]

\[ = -2 \left[ 2 \sin \left( \frac{\pi(n-1)}{2n+1} \right) \sin \left( -\frac{\pi}{2n+1} \right) \right] \]

\[ = 4 \sin \left( \frac{2\pi(n-1)}{2n+1} \right) \sin \left( \frac{\pi}{2n+1} \right) \]  \hspace{1cm} (B.73)

and

\[ 2 - 2 \cos \left( \frac{2\pi}{2n+1} \right) = 4 \sin^2 \left( \frac{\pi}{2n+1} \right) \]  \hspace{1cm} (B.74)
Finally,

\[
-2 \cos \left( \frac{\pi}{2n+1} \right) + 2 \cos \left( \frac{\pi(2n-1)}{2n+1} \right) = \\
= -2 \left( \cos \left( \frac{\pi(n-1-n)}{2n+1} \right) - \cos \left( \frac{\pi(n-1+n)}{2n+1} \right) \right) \\
= -4 \sin \left( \frac{\pi(n-1)}{2n+1} \right) \sin \left( \frac{\pi n}{2n+1} \right)
\]  

(B.75)

Inserting (B.72)–(B.75) into (B.70) gives

\[
2 \sum_{k=1}^{n} \cos(u_{2,n}) = -4 \sin \left( \frac{\pi(n-1)}{2n+1} \right) \sin \left( \frac{\pi n}{2n+1} \right) + 4 \sin \left( \frac{2\pi(n-1)}{2n+1} \right) \sin \left( \frac{\pi n}{2n+1} \right) \\
= -4 \sin \left( \frac{\pi(n-1)}{2n+1} \right) \sin \left( \frac{\pi n}{2n+1} \right) + \frac{\sin \left( \frac{2\pi}{2n+1} \right) + \sin \left( \frac{2\pi(n-1)}{2n+1} \right)}{\sin \left( \frac{\pi}{2n+1} \right)} \\
= \frac{p_2}{p_2}
\]

(B.76)

In order to simplify (B.76) further, introduce the auxiliary variables

\[
\alpha_n \triangleq \frac{\pi n}{2n+1} \quad \text{ (B.77)}
\]

\[
\beta_n \triangleq \frac{\pi}{2n+1} \quad \text{ (B.78)}
\]

and rewrite the part of (B.76) denoted by $p_2$,

\[
p_2 = \frac{\sin(2\beta_n) + \sin(2(\alpha_n - \beta_n))}{\sin(\beta_n)} \\
= \frac{2 \sin(\beta_n) \cos(\beta_n) + 2 \sin(\alpha_n - \beta_n) \cos(\alpha_n - \beta_n)}{\sin(\beta_n)} \\
= 2 \cos(\beta_n) + \frac{2 \sin(\alpha_n - \beta_n) \cos(\alpha_n - \beta_n)}{\sin(\beta_n)} \\
\]  

(B.79)
It is noted that the following relation holds,

\[
\sin(\alpha_n - \beta_n) \cos(\alpha_n - \beta_n) = \\
= [\sin(\alpha_n) \cos(\beta_n) - \cos(\alpha_n) \sin(\beta_n)] [\cos(\alpha_n) \cos(\beta_n) + \sin(\alpha_n) \sin(\beta_n)] \\
= \sin(\alpha_n) \cos(\alpha_n) \left( \cos^2(\beta_n) - \sin^2(\beta_n) \right) \\
+ \sin(\beta_n) \cos(\beta_n) \left( \sin^2(\alpha_n) - \cos^2(\alpha_n) \right) \\
= \sin(\alpha_n) \cos(\alpha_n) \cos(2\beta_n) - \sin(\beta_n) \cos(\beta_n) \cos(2\alpha_n) \\
\]  

(B.80)

Inserted into (B.79) gives

\[
p_2 = 2 \cos(\beta_n) + 2 \frac{\sin(\alpha_n) \cos(\alpha_n) \cos(2\beta_n) - \sin(\beta_n) \cos(\beta_n) \cos(2\alpha_n)}{\sin(\beta_n)} \\
= 2 \cos(\beta_n) + \frac{2 \sin(\alpha_n) \cos(\alpha_n) \cos(2\beta_n)}{\sin(\beta_n)} - 2 \cos(\beta_n) \cos(2\alpha_n) \\
= 2 \cos(\beta_n) (1 - \cos(2\alpha_n)) + \frac{2 \sin(\alpha_n) \cos(\alpha_n) \cos(2\beta_n)}{\sin(\beta_n)} \\
\]  

(B.81)

The expression (B.81) inserted into (B.76) gives

\[
2 \sum_{k=1}^{n} \cos(u_{2,n}) = -4 \sin(\alpha_n - \beta_n) \sin(\alpha_n) \\
+ 2 \cos(\beta_n) \left( 1 - \cos(2\alpha_n) \right) + \frac{2 \sin(\alpha_n) \cos(\alpha_n) \cos(2\beta_n)}{\sin(\beta_n)} \\
= -4 \sin(\alpha_n) \left[ \sin(\alpha_n) \cos(\beta_n) - \cos(\alpha_n) \sin(\beta_n) \right] \\
+ 2 \cos(\beta_n) \left( 1 - \cos(2\alpha_n) \right) + \frac{2 \sin(\alpha_n) \cos(\alpha_n) \cos(2\beta_n)}{\sin(\beta_n)} \\
= -4 \sin^2(\alpha_n) \cos(\beta_n) + 4 \sin(\alpha_n) \cos(\alpha_n) \sin(\beta_n) \\
+ 2 \cos(\beta_n) \left( 1 - \cos(2\alpha_n) \right) + \frac{2 \sin(\alpha_n) \cos(\alpha_n) \cos(2\beta_n)}{\sin(\beta_n)} \\
= -4 \sin^2(\alpha_n) \cos(\beta_n) + 4 \sin(\alpha_n) \cos(\alpha_n) \sin(\beta_n) \\
+ 4 \cos(\beta_n) \sin^2(\alpha_n) + \frac{2 \sin(\alpha_n) \cos(\alpha_n)}{\sin(\beta_n)} \left( 1 - 2 \sin^2(\beta_n) \right) \\
= \frac{2 \sin(\alpha_n) \cos(\alpha_n)}{\sin(\beta_n)} \\
= \frac{\sin(2\alpha_n)}{\sin(\beta_n)} \\
\]  

(B.82)
Further,

\[
\frac{\sin(2\alpha_n)}{\sin(\beta_n)} = \frac{\sin \left( \frac{2\pi n}{2n+1} \right)}{\sin \left( \frac{\pi}{2n+1} \right)}
= \frac{\sin \left( \frac{\pi(2n+1-1)}{2n+1} \right)}{\sin \left( \frac{\pi}{2n+1} \right)}
= \frac{\sin \left( \pi - \frac{\pi}{2n+1} \right)}{\sin \left( \frac{\pi}{2n+1} \right)}
= \frac{\sin(\pi) \cos \left( \frac{\pi}{2n+1} \right) - \cos(\pi) \sin \left( \frac{\pi}{2n+1} \right)}{\sin \left( \frac{\pi}{2n+1} \right)}
= 1 \quad (B.83)
\]

Hence, \( (B.66), (B.82) \) and \( (B.83) \) give that

\[
N_2 = \sqrt{\frac{n}{2} + \frac{1}{4}} \quad (B.84)
\]

Finally, one of the cosine functions in the expression for \( \beta_{2,k} \) can be rewritten exactly.

\[
\cos \left( \frac{\pi(2k-1)(n - \frac{1}{2})}{2n+1} \right) = \sin \left( \frac{\pi}{2} - \frac{\pi(2k-1)(n - \frac{1}{2})}{2n+1} \right)
= \sin \left( \pi - \frac{\pi}{2} + \frac{\pi(2k-1)(n - \frac{1}{2})}{2n+1} \right)
= \sin \left( \frac{\pi(2n+1) + 2\pi(2k-1)(n - \frac{1}{2})}{2(2n+1)} \right)
= \sin \left( \frac{\pi(2kn - k + 1)}{2n+1} \right)
= \sin \left( \frac{\pi k(2n + 1) - \pi(2k - 1)}{2n+1} \right)
= \sin \left( \frac{\pi k - \pi(2k - 1)}{2n+1} \right) \quad (B.85)
\]
Further, expanding the sinus function gives

\[
\cos \left( \frac{\pi (2k - 1)(n - \frac{1}{2})}{2n + 1} \right) = \sin(\pi k) \cos \left( \frac{\pi (2k - 1)}{2n + 1} \right) \\
- \cos(\pi k) \sin \left( \frac{\pi (2k - 1)}{2n + 1} \right) \\
= (-1)^{k+1} \sin \left( \frac{\pi (2k - 1)}{2n + 1} \right) \tag{B.86}
\]

To summarize, the normalization factors (B.64) and (B.84), and (B.86) are used in order to obtain the approximated transfer functions in the lemma. This concludes the proof of Lemma 4.2.
C  Proofs of results in Chapter 8

C.1  Proof of Lemma 8.1

In order to determine for what times the function $f_{k,n}(t)$ changes sign, it is evaluated when $f_{k,n}(t)$ equals zero. First $f_{k,n}(t)$ is rewritten in such a way, that all parameters that are known to be positive, are separated from the rest of the function.

$$f_{k,n}(t) = \alpha_k \beta_{1,k} t e^{-\frac{\alpha_k t}{\tau}} - \alpha_k \beta_{1,k} t_0 e^{-\frac{\alpha_k t_0}{\tau}}$$

$$= \alpha_k \beta_{1,k} t e^{-\frac{\alpha_k t}{\tau}} \left( 1 - \frac{\alpha_k \beta_{1,k}}{\alpha_k \beta_{1,k}} e^{-\frac{|\alpha_k - \alpha_k t|}{\tau}} \right) \quad (C.1)$$

which is zero if $t = 0$ or

$$1 - \frac{\alpha_k \beta_{1,k}}{\alpha_k \beta_{1,k}} e^{-\frac{|\alpha_k - \alpha_k t|}{\tau}} = 0 \quad (C.2)$$

This can equivalently be written as

$$e^{-\frac{|\alpha_k - \alpha_k t|}{\tau}} = \frac{\alpha_k \beta_{1,k}}{\alpha_k \beta_{1,k}} \quad (C.3)$$

or

$$\frac{(\alpha_k - \alpha_k) t}{\tau} = \ln \left( \frac{\alpha_k \beta_{1,k}}{\alpha_k \beta_{1,k}} \right)$$

$$= - \ln \left( \frac{\alpha_k \beta_{1,k}}{\alpha_k \beta_{1,k}} \right) \quad (C.4)$$

The time index where the sequence $f_{k,n}(t)$ changes sign is then

$$t = \frac{\tau}{\alpha_k - \alpha_k} \ln \left( \frac{\alpha_k \beta_{1,k}}{\alpha_k \beta_{1,k}} \right) \quad (C.5)$$

This concludes the proof of Lemma 8.1.
C.2 Proof of Lemma 8.2

The following expressions are to be investigated,
\[ \frac{\bar{\alpha}_{k,l,k,n}}{\tau} \] \hspace{1cm} (C.6)
\[ \frac{\bar{\alpha}_{l,k,n}}{\tau} \] \hspace{1cm} (C.7)

where the definition for \( t_{k,n} \) is stated in (8.16). The expression for \( \omega_k \), used below, is found in (8.11). Consider the case of small indices \( k \) first. Observe the following,

\[
\frac{\bar{\alpha}_{k,l,k,1}}{\bar{\alpha}_{k,l,1,k}} = \frac{16(n+1)^3(1 - \cos(\omega_k)) \cos^2(\omega_k/2)}{\pi^2(2k-1)^2(2n+1)}
\]
\[= \frac{32(n+1)^3 \sin^2(\omega_k/2) \cos^2(\omega_k/2)}{\pi^2(2k-1)^2(2n+1)}
\]
\[= \frac{8(n+1)^3 \sin^2(\omega_k)}{\pi^2(2k-1)^2(2n+1)}
\]
\[\approx \frac{8(n+1)^3}{\pi^2(2k-1)^2(2n+1)(2n+1)^2}
\]
\[= \frac{8(n+1)^3}{(2n+1)^3} \] \hspace{1cm} (C.8)

where the relation
\[
\lim_{x \to 0} \frac{\sin(x)}{x} = 1
\]

was used. Further,

\[
\frac{\bar{\alpha}_k}{\alpha_k} = \frac{8(n+1)^2(1 - \cos(\omega_k))}{\pi^2(2k-1)^2}
\]
\[= \frac{16(n+1)^2 \sin^2(\omega_k/2)}{\pi^2(2k-1)^2}
\]
\[\approx \frac{16(n+1)^2}{\pi^2(2k-1)^2} \frac{\pi^2(2k-1)^2}{4(2n+1)^2}
\]
\[= \frac{4(n+1)^2}{(2n+1)^2}
\]
\[= 1 + \frac{4n+3}{(2n+1)^2} \] \hspace{1cm} (C.9)
From (8.16), the expression for $t_{k,n}$ is obtained.

\[
\frac{\alpha_k t_{k,n}}{\tau} = \frac{1}{\alpha_k - 1} \ln \left( \frac{\alpha_k \beta_{1,k}}{\alpha_k \beta_{1,1,k}} \right) \\
\approx \frac{(2n+1)^2}{4n+3} \ln \left( \frac{8(n+1)^3}{(2n+1)^3} \right) \\
= \frac{(2n+1)^2}{4n+3} \ln \left( 1 + \frac{12n^2 + 18n + 7}{(2n + 1)^3} \right) \\
\triangleq \frac{(2n+1)^2}{4n+3} \ln(1 + \delta)
\]

where $\delta$ is introduced as an auxiliary variable. The natural logarithm is next derived by series expansion,

\[
\frac{\alpha_k t_{k,n}}{\tau} \approx \frac{(2n+1)^2}{4n+3} \left( \delta - \frac{\delta^2}{2} + O(\delta^3) \right) \\
= \frac{(2n+1)^2}{4n+3} \delta + O\left( \frac{1}{n} \right) \\
= \frac{(2n+1)^2}{4n+3} \frac{12n^2 + 18n + 7}{(2n + 1)^3} + O\left( \frac{1}{n} \right) \\
= \frac{12n^2 + 18n + 7}{(4n+3)(2n+1)} + O\left( \frac{1}{n} \right) \\
= \frac{(12 + \frac{18}{n} + \frac{7}{n})}{(4 + \frac{3}{n})(2 + \frac{1}{n})} + O\left( \frac{1}{n} \right) \\
= \frac{3}{2} + O\left( \frac{1}{n} \right)
\]

This gives the first part of (8.17).

Next consider the case for large $k$, $k \approx n$. The trigonometric function is rewritten so that the argument of the sine function is small which enables a
reasonable approximation,

\[
\frac{\alpha_k}{\alpha_k} = \frac{16(n + 1)^2 \sin^2(\omega_k/2)}{\pi^2(2k - 1)^2},
\]

\[
= \frac{16(n + 1)^2 (1 - \sin^2(\pi/2 - \omega_k/2))}{\pi^2(2k - 1)^2},
\]

\[
= \frac{16(n + 1)^2 \left(1 - \sin^2\left(\frac{\pi(n-k+1)}{2n+1}\right)\right)}{\pi^2(2k - 1)^2},
\]

\[
\approx \frac{16(n + 1)^2 \left(1 - \frac{\pi^2(n-k+1)^2}{(2n+1)^2}\right)}{\pi^2(2k - 1)^2},
\]

\[
\approx \frac{4}{\pi^2 n^2}.
\] (C.12)

and

\[
\frac{\alpha_k \beta_1, k}{\alpha_k \beta_1, k} = \frac{32(n + 1)^3 \sin^2(\omega_k/2) (1 - \sin^2(\omega_k/2))}{\pi^2(2n + 1)(2k - 1)^2},
\]

\[
= \frac{32(n + 1)^3 (1 - \sin^2(\pi/2 - \omega_k/2)) \sin^2(\pi/2 - \omega_k/2)}{\pi^2(2n + 1)(2k - 1)^2},
\]

\[
\approx \frac{32(n + 1)^3 \left(1 - \left(\frac{\pi}{2} - \frac{\omega_k}{2}\right)^2\right) \left(\frac{\pi}{2} - \frac{\omega_k}{2}\right)^2}{\pi^2(2n + 1)(2k - 1)^2},
\]

\[
\approx \frac{32(n + 1)^3 \left(1 - \left(\frac{\pi(n-k+1)}{2(2n+1)}\right)^2\right) \left(\frac{\pi(n-k+1)}{2(2n+1)}\right)^2}{\pi^2(2n + 1)(2k - 1)^2},
\]

\[
\approx \frac{32(n + 1)^3 (n - k + 1)^2}{(2n + 1)^3(2k - 1)^2},
\]

\[
\approx \frac{4(n - k + 1)^2}{n^2},
\]

\[
\approx \frac{(n - k + 1)^2}{n^2}.
\] (C.13)
This results in
\[
\frac{\alpha_k t_{k,n}}{\tau} = \frac{1}{\alpha_k - 1} \ln \left( \frac{\alpha_k \beta_{1,k}}{\alpha_k \beta_{1,k}} \right) \\
\approx \frac{1}{\frac{\pi^2}{4} - 1} \ln \left( \frac{(n - k + 1)^2}{n} \right) \\
= \frac{2n^2}{\pi^2 - 4} \ln \left( \frac{n}{n - k + 1} \right) \quad (C.14)
\]

Next, \( \frac{\alpha_k t_{k,n}}{\tau} \) is considered. For small indices \( k \) it is approximately equal to (C.11) since \( \alpha_k \approx \alpha_k \), cf (C.9). The following holds for large \( k \) where equation (C.12) gives an approximate relation between \( \alpha_k \) and \( \alpha_k \).
\[
\frac{\alpha_k t_{k,n}}{\tau} = \frac{\alpha_k}{\alpha_k} \times \frac{\alpha_k t_{k,n}}{\tau} \\
\approx \frac{4}{\pi^2} \times \frac{\alpha_k t_{k,n}}{\tau} \quad (C.15)
\]

This concludes the proof of Lemma 8.2.
C.3 Proof of Lemma 8.3

The function to study is

\[ f'_{k,n}(t) = \alpha_k \beta_{2,k} e^{-\frac{\theta t}{\tau}} - \tilde{\alpha}_k \tilde{\beta}_{2,k} e^{-\frac{\tilde{\theta} t}{\tau}} \quad \text{(C.16)} \]

A factor \((-1)^{k+1}\) is separated from the parameters \(\beta_{2,k}\) and \(\tilde{\beta}_{2,k}\). This does not change the \(t\) dependence. The function \(f'_{k,n}(t)\) is then rewritten by putting as many factors as possible outside the brackets.

\[ f'_{k,n}(t) = \alpha_k \beta_{2,k} e^{-\frac{\theta t}{\tau}} - \tilde{\alpha}_k \tilde{\beta}_{2,k} e^{-\frac{\tilde{\theta} t}{\tau}} = \alpha_k |\beta_{2,k}| (-1)^{k+1} e^{-\frac{\theta t}{\tau}} - \tilde{\alpha}_k |\tilde{\beta}_{2,k}| (-1)^{k+1} e^{-\frac{\tilde{\theta} t}{\tau}} = \alpha_k |\beta_{2,k}| e^{-\frac{\theta t}{\tau}} \left(1 - \frac{\tilde{\alpha}_k |\tilde{\beta}_{2,k}|}{\alpha_k |\beta_{2,k}|} e^{-\frac{\tilde{\theta} t}{\tau}}\right) \quad \text{(C.17)} \]

where \(\tilde{\alpha}_k = \alpha_k - \alpha_k\), cf (8.12). It is now evaluated for what values of \(t\) (C.17) changes sign. This is determined by the expression within the parentheses and happens when it equals zero.

\[ 1 - \frac{\tilde{\alpha}_k |\tilde{\beta}_{2,k}|}{\alpha_k |\beta_{2,k}|} e^{-\frac{\tilde{\theta} t}{\tau}} = 0 \quad \text{(C.18)} \]

This gives

\[ e^{-\frac{\tilde{\theta} t}{\tau}} = \frac{\alpha_k |\beta_{2,k}|}{\tilde{\alpha}_k |\tilde{\beta}_{2,k}|} \quad \text{(C.19)} \]

or

\[ \frac{\tilde{\alpha}_k t}{\tau} = \ln \left(\frac{\alpha_k |\beta_{2,k}|}{\tilde{\alpha}_k |\tilde{\beta}_{2,k}|}\right) = - \ln \left(\frac{\tilde{\alpha}_k |\tilde{\beta}_{2,k}|}{\alpha_k |\beta_{2,k}|}\right) \quad \text{(C.20)} \]

The time index indicating alternating signs is

\[ t = \frac{\tau}{\tilde{\alpha}_k} \ln \left(\frac{\alpha_k |\beta_{2,k}|}{\tilde{\alpha}_k |\tilde{\beta}_{2,k}|}\right) = \frac{\tau}{\tilde{\alpha}_k - \alpha_k} \ln \left(\frac{\alpha_k |\beta_{2,k}|}{\tilde{\alpha}_k |\tilde{\beta}_{2,k}|}\right) \quad \text{(C.21)} \]

This means that for a specific \(k\) and \(n\), \(f'_{k,n}(t)\) changes sign when

\[ t = t'_{k,n} = - \frac{\tau}{\tilde{\alpha}_k - \alpha_k} \ln \left(\frac{\alpha_k |\beta_{2,k}|}{\tilde{\alpha}_k |\tilde{\beta}_{2,k}|}\right) \quad \text{(C.22)} \]

This concludes the proof of Lemma 8.3.
C.4 Proof of Lemma 8.4

The parameter error $\hat{\alpha}_k$, defined in (8.12), changes sign once when $\hat{\alpha}_k$ is evaluated for increasing indices $k$. The intersection between $\alpha_k$ and $\hat{\alpha}_k$ is evaluated by determination of the particular $k$ where the parameters are equal. The obtained value of $k$ will generally not be an integer. Therefore, the $k$ giving intersection of $\alpha_k$ and $\hat{\alpha}_k$ is increased until an integer value is reached. That particular value is the wanted parameter. In order to simplify the notation, let

$$x \triangleq 2k - 1$$  \hspace{1cm} (C.23)

It is then of interest to solve

$$\frac{\alpha_k}{\pi^2 x^2} = \frac{\pi^2 x^2}{4} = 2(n + 1)^2 \left( 1 - \cos \left( \frac{\pi x}{2n + 1} \right) \right)$$

$$\frac{\pi^2 x^2}{4} = 4(n + 1)^2 \sin^2 \left( \frac{\pi x}{2(n + 1)} \right)$$

$$x = \frac{4(n + 1)}{\pi} \sin \left( \frac{\pi x}{2(n + 1)} \right)$$  \hspace{1cm} (C.24)

It can be observed that exactly one solution to (C.24) exists. Introduce

$$f(x) \triangleq x - \frac{4(n + 1)}{\pi} \sin \left( \frac{\pi x}{2(n + 1)} \right)$$  \hspace{1cm} (C.25)

which is a function in the positive, real variable $x$. It will be shown that $f(x)$ is negative for small indices $x$ and positive for large $x$. Also, the second derivative, $f''(x)$, is larger than zero for all $x$ which indicates that $f(x)$ is convex. Hence, $f(x)$ crosses the $x$ axis only once, giving that a unique solution of (C.24) exists.

Study first $f(x)$ for small variables $x$. Series expansion of the sine function will be used.

$$f(x) = x - \frac{4(n + 1)}{\pi} \sin \left( \frac{\pi x}{2(n + 1)} \right)$$

$$= x - \frac{4(n + 1)}{\pi} \left( \frac{\pi x}{2(n + 1)} \right) + O \left( \frac{1}{n^2} \right)$$

$$= x \left( 1 - \frac{2(n + 1)}{2n + 1} \right) + O \left( \frac{1}{n^2} \right)$$

$$= x \left( 1 - \frac{2}{2n + 1} \right) + O \left( \frac{1}{n^2} \right)$$

$$< 0$$  \hspace{1cm} (C.26)
for small $x$ and large $n$.

The largest possible $x$ equals $2n - 1$. Therefore $f(2n - 1)$ is considered next.

\[
f(2n - 1) = 2n - 1 - \frac{4(n + 1)}{\pi} \sin \left( \frac{\pi}{2} - \frac{\pi}{2n + 1} \right) \\
= 2n - 1 - \frac{4(n + 1)}{\pi} \cos \left( \frac{\pi}{2n + 1} \right) \\
= 2n - 1 - \frac{4(n + 1)}{\pi} \left( 1 - \frac{\pi^2}{2(2n + 1)^2} + O \left( \frac{1}{n^4} \right) \right) \\
= 2n - 1 - \frac{4(n + 1)}{\pi} \frac{2\pi(n + 1)}{(2n + 1)^2} + O \left( \frac{1}{n^4} \right) \\
= (n + 1) \left( 2 - \frac{4}{\pi} \right) - 3 + O \left( \frac{1}{n} \right) > 0 \quad \text{(C.27)}
\]

for sufficiently large $n$. It remains to study $f''(x)$. It equals

\[
f''(x) = \frac{d^2}{dx^2} \left( x - \frac{4(n + 1)}{\pi} \sin \left( \frac{\pi x}{2(2n + 1)} \right) \right) \\
= \frac{d}{dx} \left( 1 - \frac{2(n + 1)}{2n + 1} \cos \left( \frac{\pi x}{2(2n + 1)} \right) \right) \\
= \frac{\pi(n + 1)}{(2n + 1)^2} \sin \left( \frac{\pi x}{2(2n + 1)} \right) \\
> 0 \quad \text{(C.28)}
\]

since

\[
0 < \frac{\pi x}{2(2n + 1)} < \frac{\pi}{2}
\]

which gives that the sine function is larger than zero. Hence, (C.26)–(C.28) give that there is a unique solution to (C.24).

The solution to (C.24) is derived by developing the sine function using series expansion. It is assumed that the requested value of $k$ is considerably smaller than $n$. Retaining the two first terms in the expansion is therefore expected to introduce an error that is small enough not to cause false solutions.

\[
x \approx \frac{4(n + 1)}{\pi} \left( \frac{\pi x}{2(2n + 1)} - \frac{\pi^3 x^3}{48(2n + 1)^3} \right) \\
= \frac{2(n + 1)x}{2n + 1} - \frac{\pi^2(n + 1)x^3}{12(2n + 1)^3} \quad \text{(C.29)}
\]

Both sides are reduced by $x$ since $2k - 1 = 0$ is not an interesting solution.
Further, the expression containing $x$ are put on the left side.

\[
1 = \frac{2(n + 1)}{2n + 1} - \frac{\pi^2(n + 1)x^2}{12(2n + 1)^3} \\
x^2 = \frac{12(2n + 1)^3}{\pi^2(n + 1)} \left( \frac{2(n + 1)}{2n + 1} - 1 \right) \\
= \frac{12(2n + 1)^3}{\pi^2(n + 1)(2n + 1)} \\
= \frac{12(2n + 1)^2}{\pi^2(n + 1)} \\
x = \frac{\sqrt{12}(2n + 1)}{\pi \sqrt{n + 1}}
\]  

(C.30)

This gives

\[
k = \frac{\sqrt{3}(2n + 1)}{\pi \sqrt{n + 1}} + \frac{1}{2} \\
= \frac{2\sqrt{3}}{\pi} \sqrt{n + 1} + \frac{1}{2} - \frac{\sqrt{3}}{\pi} \frac{1}{\sqrt{n + 1}} \\
= \frac{2\sqrt{3}}{\pi} \sqrt{n + 1} + \frac{1}{2} + O \left( \frac{1}{\sqrt{n}} \right)
\]  

(C.31)

The $k$ for which $\alpha_k$ gets a negative value for the first time is the first integer value larger than or equal to (C.31).

To verify the solution (C.31), $f(x)$ is examined for different values of $x$. Introduce

\[
k^* \triangleq \frac{2\sqrt{3}}{\pi} \sqrt{n + 1} + \frac{1}{2}
\]  

(C.32)

and let, cf (C.23),

\[
x^* \triangleq 2k^* - 1 = \frac{4\sqrt{3}}{\pi} \sqrt{n + 1}
\]  

(C.33)

\[
x^*_+ \triangleq 2(k^* - 1) - 1 = 2k^* - 3 = \frac{4\sqrt{3}}{\pi} \sqrt{n + 1} - 2
\]  

(C.34)

\[
x^*_+ \triangleq 2(k^* + 1) - 1 = 2k^* + 1 = \frac{4\sqrt{3}}{\pi} \sqrt{n + 1} + 2
\]  

(C.35)

The variables $x^*_-$ and $x^*_+$ are smaller respectively larger than $x^*$ corresponding
to $k^*$. Then

$$f(x^*) = \frac{4\sqrt{3}\sqrt{n+1}}{\pi} - \frac{4(n+1)}{\pi} \sin \left( \frac{\pi \cdot 4\sqrt{3}\sqrt{n+1}}{2(2n+1)} \right)$$

$$= \frac{4\sqrt{3}\sqrt{n+1}}{\pi} - \frac{4(n+1)}{\pi} \sin \left( \frac{2\sqrt{3}\sqrt{n+1}}{2n+1} \right)$$

$$= \frac{4\sqrt{3}\sqrt{n+1}}{\pi} - \frac{4(n+1)}{\pi} \left( \frac{2\sqrt{3}\sqrt{n+1}}{2n+1} - \frac{4\sqrt{3}(n+1)^{3/2}}{(2n+1)^3} \right)$$

$$+ \frac{12\sqrt{3}(n+1)^{5/2}}{5(2n+1)^3} + O \left( \frac{1}{n^{5/2}} \right)$$

$$= \frac{4\sqrt{3}\sqrt{n+1}(4n+3)}{\pi(2n+1)^3} - \frac{9.6\sqrt{3}(n+1)^{7/2}}{\pi(2n+1)^5} + O \left( \frac{1}{n^{5/2}} \right)$$

$$> 0 \quad (C.36)$$

for large $n$. Further,

$$f(x_{-}^*) = \frac{4\sqrt{3}\sqrt{n+1}}{\pi} - 2 - \frac{4(n+1)}{\pi} \sin \left( \frac{\pi \cdot 4\sqrt{3}\sqrt{n+1}}{2(2n+1)} - 2 \right)$$

$$= \frac{4\sqrt{3}\sqrt{n+1}}{\pi} - 2 - \frac{4(n+1)}{\pi} \sin \left( \frac{2\sqrt{3}\sqrt{n+1}}{2n+1} - \frac{\pi}{2n+1} \right)$$

$$= \frac{4\sqrt{3}\sqrt{n+1}}{\pi} - 2 - \frac{4(n+1)}{\pi} \left( \frac{2\sqrt{3}\sqrt{n+1}}{2n+1} - \frac{\pi}{2n+1} + O \left( \frac{1}{n^{5/2}} \right) \right)$$

$$= \frac{4\sqrt{3}\sqrt{n+1}}{\pi} - 2 \cdot \frac{8\sqrt{3}(n+1)^{3/2}}{\pi(2n+1)^3} + \frac{4(n+1)}{2n+1} + O \left( \frac{1}{\sqrt{n}} \right)$$

$$= \frac{4\sqrt{3}\sqrt{n+1}}{\pi} \left( 1 - \frac{2(n+1)}{2n+1} \right) + \frac{4(n+1)}{2n+1} - 2 + O \left( \frac{1}{\sqrt{n}} \right)$$

$$= \frac{4\sqrt{3}\sqrt{n+1}}{\pi} \left( -\frac{1}{2n+1} + \frac{2}{2n+1} + O \left( \frac{1}{\sqrt{n}} \right) \right)$$

$$< 0 \quad (C.37)$$

for large $n$. Consider also the function $f(x_{-}^*)$. It is expected to be larger than
$f(x^*)$ and therefore exceed zero.

$$f(x^*_n) = \frac{4\sqrt{3}}{\pi} \sqrt{n + \frac{1}{n}} + 2 - \frac{4(n + 1)}{\pi} \sin \left( \frac{\pi}{2(2n + 1)} \left( \frac{4\sqrt{3}}{\pi} \sqrt{n + \frac{1}{n}} + 2 \right) \right)$$

$$= \frac{4\sqrt{3}}{\pi} \sqrt{n + \frac{1}{n}} + 2 - \frac{4(n + 1)}{\pi} \sin \left( \frac{2\sqrt{3}}{2n + 1} \sqrt{n + \frac{1}{n}} + \frac{\pi}{2n + 1} \right)$$

$$= \frac{4\sqrt{3}}{\pi} \sqrt{n + \frac{1}{n}} + 2 - \frac{4(n + 1)}{\pi} \left[ \frac{2\sqrt{3}}{2n + 1} \sqrt{n + \frac{1}{n}} + \frac{\pi}{2n + 1} \right]$$

$$- \frac{1}{6} \left( \frac{2\sqrt{3}}{2n + 1} + \frac{\pi}{2n + 1} \right)^3 + O \left( \frac{1}{n^{5/2}} \right)$$

$$= \frac{4\sqrt{3}}{\pi} \sqrt{n + \frac{1}{n}} - \frac{1}{2n + 1} + \frac{4(n + 1)^2}{(2n + 1)^3} + \frac{\pi^2(n + 1)}{(2n + 1)^3}$$

$$+ \frac{1}{2n + 1} \left[ \frac{4n + 3}{(2n + 1)^2} + \frac{\pi^2(n + 1)}{(2n + 1)^2} \right] + O \left( \frac{1}{n^{3/2}} \right)$$

Examine the expressions $I_1$ and $I_2$ separately.

$$I_1 = \left( -\frac{1}{2n + 1} + \frac{4(n + 1)^2}{(2n + 1)^3} + \frac{\pi^2(n + 1)}{(2n + 1)^3} \right)$$

$$= \frac{1}{2n + 1} \left[ \frac{4(n + 1)^2}{(2n + 1)^2} - 1 + \frac{\pi^2(n + 1)}{(2n + 1)^2} \right]$$

$$= \frac{1}{2n + 1} \left[ \frac{4n + 3}{(2n + 1)^2} + \frac{\pi^2(n + 1)}{(2n + 1)^2} \right]$$

$$> 0 \quad \text{(C.39)}$$
and

\[
I_2 = \left( -1 + \frac{12(n+1)^2}{(2n+1)^2} + \frac{\pi^2(n+1)}{3(2n+1)^2} \right) \\
= \frac{12(n+1)^2}{(2n+1)^2} - \frac{(2n+1)^2}{(2n+1)^2} + \frac{\pi^2(n+1)}{3(2n+1)^2} \\
= \frac{8n^2 + 20n + 11}{(2n+1)^2} + \frac{\pi^2(n+1)}{3(2n+1)^2} \\
> 0 \quad \text{(C.40)}
\]

Insertion of the result of (C.39) and (C.40) gives that

\[
f(x^*_+) = \frac{4\sqrt{3}\sqrt{n+1}}{\pi}I_2 + \frac{2}{2n+1}I_3 + O\left(\frac{1}{n^{3/2}}\right) \\
> 0 \quad \text{(C.41)}
\]

for sufficiently large \(n\). Hence, \(f(x^*_+) < 0\), \(f(x^*) > 0\) and \(f(x^*_+) > 0\) which verifies the solution (C.31). This concludes the proof of Lemma 8.4.
C.5 Proof of Lemma 8.5

To prove
\[ \beta_{1,ke} \cdot \frac{\omega_k}{\hat{t}_{1,ke}} - \hat{\beta}_{1,ke} \cdot \frac{\omega_k}{\hat{t}_{1,ke}} \geq 0 \]  \hspace{1cm} (C.42)

is equivalent to show that
\[ \frac{\hat{\beta}_{1,k}}{\beta_{1,k}} \leq e^{-\frac{\omega_k \cdot \alpha_k}{\hat{t}_{1,k}}} \]  \hspace{1cm} (C.43)

holds. This corresponds to the requirement
\[ \alpha_k - \alpha_k \geq \ln \left( \frac{\hat{\beta}_{1,k}}{\beta_{1,k}} \right) \]  \hspace{1cm} (C.44)

where the worst case, \( t = \tau \), was used. The function
\[ f(k) \triangleq \alpha_k - \alpha_k - \ln \left( \frac{\hat{\beta}_{1,k}}{\beta_{1,k}} \right) \]  \hspace{1cm} (C.45)

is introduced. In the following, (C.45) will be shown to be larger or equal to zero. The expressions for \( \alpha_k, \alpha_k, \beta_{1,k}, \) and \( \hat{\beta}_{1,k} \) are found in (3.23), (3.24), (4.20) and (4.21). Further, the definition of \( \omega_k \) is found in (8.11).

\[ f(k) = 2(n+1)^2 (1 - \cos(\omega_k)) - \frac{\pi^2(2k-1)^2}{4} - \ln \left( \frac{2(n+1)}{2n+1} \cos^2 \left( \frac{\omega_k}{2} \right) \right) \]
\[ = 4(n+1)^2 \sin^2 \left( \frac{\omega_k}{2} \right) - \frac{\pi^2(2k-1)^2}{4} \]
\[ - \ln \left( \left( 1 + \frac{1}{2n+1} \right) \cos^2 \left( \frac{\omega_k}{2} \right) \right) \]  \hspace{1cm} (C.46)

In order to show that (C.46) is non-negative, it is constrained by a lower bound which, in turn, will be proved to be larger or equal to zero. For this, the following known inequality can be used,
\[ \ln \left( 1 + \frac{b}{a} \right) \leq \frac{b}{a} \]  \hspace{1cm} (C.47)

This relation can be rewritten to fit the logarithm in (C.46).
\[ \ln \left( \frac{a+b}{a} \right) \leq \frac{b}{a} \]  \hspace{1cm} (C.48)

giving
\[ -\ln(a + b) \geq -\ln(a) - \frac{b}{a} \]  \hspace{1cm} (C.49)
Identifying the quantities “a” and “b” in (C.49), the following lower bound of (C.46) is obtained,

\[
f(k) \geq 4(n + 1)^2 \sin^2 \left(\frac{\omega_k}{2}\right) - \frac{\pi^2(2k - 1)^2}{4} - \ln \left(\cos^2 \left(\frac{\omega_k}{2}\right)\right) - \frac{1}{2n+1} \tag{C.50}
\]

In order to evaluate (C.50) further, the inequality (C.47) is again used,

\[
\ln \left(\cos^2 \left(\frac{\omega_k}{2}\right)\right) = \ln \left(1 - \sin^2 \left(\frac{\omega_k}{2}\right)\right)\]

\[
\leq -\sin^2 \left(\frac{\omega_k}{2}\right) \tag{C.51}
\]

which is equivalent to

\[
-\ln \left(\cos^2 \left(\frac{\omega_k}{2}\right)\right) \geq \sin^2 \left(\frac{\omega_k}{2}\right) \tag{C.52}
\]

Inserting (C.52) into (C.50) gives

\[
f(k) \geq (4(n + 1)^2 + 1) \sin^2 \left(\frac{\omega_k}{2}\right) - \frac{\pi^2(2k - 1)^2}{4} - \frac{1}{2n+1} \tag{C.53}
\]

A useful lower bound of the sine function is

\[
\sin^2(x) \geq x^2 - \frac{x^4}{3} \tag{C.54}
\]

where \(x \triangleq \omega_k/2\) is introduced for convenience. This is equivalent to the bound

\[
\cos(2x) \leq 1 - 2x^2 + \frac{2x^4}{3} \tag{C.55}
\]

Next, the relation (C.55) will be shown to hold. It is observed that since \(0 < \omega_k < \pi\), then \(0 < x < \pi/2\). Let

\[
r(x) \triangleq \cos(2x) - 1 + 2x^2 - \frac{2x^4}{3} \tag{C.56}
\]

The function \(r(x)\) will be shown to be negative, hence proving (C.54). Compute the first derivative of \(r(x)\).

\[
\frac{\partial r(x)}{\partial x} = -2 \sin(2x) + 4x - \frac{8x^3}{3}
\]

\[
= -2 \left(\sin(2x) - 2x + \frac{(2x^3)^3}{3!}\right) \tag{C.57}
\]
Series expansion of the sine function will be used. The terms will be arranged in pairs, and it can be observed that all new terms (within parentheses) are positive.

\[
\frac{\partial r(x)}{\partial x} = -2 \left[ 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \ldots - 2x + \frac{(2x)^3}{3!} \right]
\]

\[
= -2 \left[ \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \frac{(2x)^9}{9!} - \frac{(2x)^{11}}{11!} + \ldots \right]
\]

\[
= -2 \left[ \frac{(2x)^5}{5!} \left( 1 - \frac{(2x)^2}{6 \times 7} \right) + \frac{(2x)^9}{9!} \left( 1 - \frac{(2x)^2}{10 \times 11} \right) + \ldots \right]
\]

\[
= -2 \left[ \frac{(2x)^5}{5!} \left( 1 - \frac{(2x)^2}{42} \right) + \frac{(2x)^9}{9!} \left( 1 - \frac{(2x)^2}{110} \right) + \ldots \right] \quad \text{(C.58)}
\]

Since \( x < \pi/2 \), all terms within the inner parentheses are positive. This means that (C.58) is negative and that the maximum value of \( r(x) \) occurs for the smallest value of \( x \). Since \( x > 0 \), \( \max_x r(x) < r(0) \). Using (C.56) gives

\[
r(0) = 1 - 1 = 0 \quad \text{(C.59)}
\]

Consequently \( r(x) < 0 \) for all \( 0 < x < \pi/2 \). The lower bound (C.54) thus holds and can be used in the expression for \( f(k) \) in (C.53).

\[
f(k) \geq (4(n+1)^2+1) \sin^2 \left( \frac{\omega_k}{2} \right) - \frac{\pi^2(2k-1)^2}{4} - \frac{1}{2n+1}
\]

\[
\geq (4(n+1)^2+1) \left( \frac{\omega_k^2}{4} - \frac{\omega_k^4}{48} \right) - \frac{\pi^2(2k-1)^2}{4} - \frac{1}{2n+1} \quad \text{(C.60)}
\]

The expression for \( \omega_k \) (8.11) is inserted, and the following is obtained,

\[
f(k) \geq (4(n+1)^2+1) \left( \frac{\pi^2(2k-1)^2}{4(2n+1)^2} - \frac{\pi^4(2k-1)^4}{48(2n+1)^4} \right) - \frac{\pi^2(2k-1)^2}{4}
\]

\[
= \frac{1}{2n+1} - \frac{\pi^2(2k-1)^2}{4(2n+1)^2} \left( \frac{4(n+1)^2}{2(2n+1)^2} - 1 \right) + \frac{\pi^2(2k-1)^2}{4(2n+1)^2}
\]

\[
- \frac{\pi^4(n+1)^2(2k-1)^4}{12(2n+1)^4} - \frac{\pi^4(2k-1)^4}{48(2n+1)^4} + \frac{1}{2n+1}
\]

\[
= \frac{\pi^2(4n+3)(2k-1)^2}{4(2n+1)^2} - \frac{\pi^2(2k-1)^2}{4(2n+1)^2} - \frac{\pi^4(n+1)^2(2k-1)^4}{12(2n+1)^4}
\]

\[
- \frac{\pi^4(2k-1)^4}{48(2n+1)^4} + \frac{1}{2n+1}
\]

\[\triangleq f_l(k) \quad \text{(C.61)}\]
Recall that the objective was to show that \( f(k) \geq 0 \). In order to do this, the extreme values of \( f(k) \), introduced in (C.61) as a lower bound of \( f(k) \), are derived. Then the values of \( f(k) \) are determined in the extremes and at the end points. It will be shown that neither of the resulting values of \( f(k) \) are negative. Even though \( \{ k \} \) are integer indices, \( f(k) \) will be studied for all real values in the interval \( k \in [1, n] \).

First, the derivative of \( f(k) \) is calculated,

\[
\frac{\partial f(k)}{\partial k} = \frac{4\pi^2(n + 1)(2k - 1)}{(2n + 1)^2} - \frac{2\pi^4(n + 1)^2(2k - 1)^3}{3(2n + 1)^4} - \frac{\pi^4(2k - 1)^3}{6(2n + 1)^4}
\]

\[
= \frac{\pi^2(2k - 1)}{(2n + 1)^2} \left( 4(n + 1) - \frac{2\pi^2(n + 1)^2(2k - 1)^2}{3(2n + 1)^2} - \frac{\pi^2(2k - 1)^2}{6(2n + 1)^2} \right)
\]

\[
= \frac{\pi^2(2k - 1)}{(2n + 1)^2} \left( 4(n + 1) - \frac{\pi^2(2k - 1)^2}{6(2n + 1)^2} (1 + 4(n + 1)^2) \right) \quad (C.62)
\]

The derivative equals zero if \( k \triangleq k_1 = 1/2 \) or if

\[
(2k - 1)^2 = \frac{24(n + 1)(2n + 1)^2}{\pi^2(1 + 4(n + 1)^2)}
\]

\[
\approx \frac{24n}{\pi^2} \quad (C.63)
\]

giving

\[
k \triangleq k_2 \approx \frac{\sqrt{6n}}{\pi} + \frac{1}{2} \quad (C.64)
\]

Both \( k_1 \) and \( k_2 \) are non-integers. Since \( k \geq 1, 1 \) will be used instead of \( k_1 \). Computing \( f(k) \) using the candidate indices reveals if any of them gives a negative value of \( f(k) \).

\[
f(1) = \frac{\pi^2(n + 1)}{(2n + 1)^2} - \frac{\pi^4(n + 1)^2}{12(2n + 1)^4} - \frac{\pi^4}{48(2n + 1)^4} - \frac{1}{2n + 1}
\]

\[
= \frac{1}{2n + 1} \left( \frac{\pi^2(n + 1)}{2n + 1} - 1 \right) - \frac{\pi^4(n + 1)^2}{12(2n + 1)^4} \left( 1 + \frac{1}{4(n + 1)^2} \right)
\]

\[
= \frac{1}{2n + 1} \left( \frac{\pi^2(n + 1)}{2n + 1} - 1 \right) + O \left( \frac{1}{n^2} \right) \quad (C.65)
\]

The first term is positive since

\[
\frac{\pi^2(n + 1)}{2n + 1} - 1 = \frac{\pi^2(n + 1) - 2n - 1}{2n + 1}
\]

\[
= \frac{n(\pi^2 - 2) + \pi^2 - 1}{2n + 1} > 0 \quad (C.66)
\]
Hence, (C.65) exceeds zero for sufficiently large $n$. Further, using the approximate expression of $k_2$ from (C.64),

$$f_1(k_2) = \frac{24n(n+1)}{(2n+1)^2} - \frac{48n^2(n+1)^2}{(2n+1)^4} - \frac{12n^2}{(2n+1)^4} - \frac{1}{2n+1} \approx 3 + O\left(\frac{1}{n}\right)$$

which exceeds zero for sufficiently large $n$. The approximations introduce only negligible errors for large $n$.

Finally, $f_1(k)$ is computed for the largest possible index, $k_\alpha - 1$. Inserting the expression for $k_\alpha - 1$ into (C.61) gives, after some calculations

$$f_1(k_\alpha - 1) = \frac{192\pi(n+1)^{7/2}}{\sqrt{3}(2n+1)^4} + O\left(\frac{1}{n}\right)$$

$$= \frac{4\sqrt{3}\pi}{\sqrt{n}} + O\left(\frac{1}{n}\right)$$

which is positive for large $n$.

This results in that $f_1(k) > 0$ and consequently that $f(k) > 0$, cf (C.61). Then (C.42) holds for large $n$ when $t \geq \tau$ and $1 \leq k \leq k_\alpha - 1$. This concludes the proof of Lemma 8.5.
C.6 Proof of Lemma 8.6

It will be shown that

$$S \geq \sum_{k=1}^{k_n-1} (\beta_{1,k} - \beta_{1,k}) > 0$$  \hspace{1cm} (C.69)

when \(n\) is bounded but large. Series expansion will be used in order to rewrite the cosine function in \(\beta_{1,k}\).

\[
S = \sum_{k=1}^{k_n-1} \left( 2 - \frac{4(n+1)}{2n+1} \cos^2 \left( \frac{\pi(2k-1)}{2(2n+1)} \right) \right)
\]

\[
= 2(k_n - 1) - \frac{4(n+1)}{2n+1} \sum_{k=1}^{k_n-1} \left( 1 - \sin^2 \left( \frac{\pi(2k-1)}{2(2n+1)} \right) \right)
\]

\[
= 2(k_n - 1) - \frac{4(n+1)}{2n+1} (k_n - 1) + \frac{4(n+1)}{2n+1} \sum_{k=1}^{k_n-1} \left( \frac{\pi(2k-1)}{2(2n+1)} - \frac{\pi^3(2k-1)^3}{48(2n+1)^3} + O \left( \frac{k^5}{n^5} \right) \right)^2
\]

\[
= -\frac{\pi^4(n+1)}{12(2n+1)^3} \sum_{k=1}^{k_n-1} (2k-1)^2 + \sum_{k=1}^{k_n-1} O \left( \frac{k^6}{n^5} \right) \hspace{1cm} (C.70)
\]

The sums in (C.70) are computed next.

\[
\sum_{k=1}^{k_n-1} (2k-1)^2 = \sum_{k=1}^{k_n-1} (4k^2 - 4k + 1)
\]

\[
= \frac{4}{3}k_n^3 - 4k_n^2 + \frac{11}{3}k_n - 1 \hspace{1cm} (C.71)
\]

The second sum is computed in a similar way.

\[
\sum_{k=1}^{k_n-1} (2k-1)^4 = \frac{16}{5}k_n^5 - 16k_n^4 + \frac{48}{3}k_n^3 - 24k_n^2 + \frac{127}{15}k_n - 1 \hspace{1cm} (C.72)
\]
Inserting (C.71) and (C.72) into (C.70) gives

\[
S = -(k_\alpha - 1) \frac{2}{2n + 1} + \frac{\pi^2(n + 1)}{(2n + 1)^3} \left( \frac{4}{3} k_\alpha^3 - 4k_\alpha^2 + \frac{11}{3} k_\alpha - 1 \right) \\
- \frac{\pi^4(n + 1)}{12(2n + 1)^5} \left( \frac{16}{5} k_\alpha^5 - 16k_\alpha^4 + \frac{88}{3} k_\alpha^3 - 24k_\alpha^2 + \frac{127}{15} k_\alpha - 1 \right)
\]

\[
+ \sum_{k=1}^{k_\alpha-1} O \left( \frac{k^6}{n^5} \right)
\]

\[
= \frac{2k_\alpha}{2n + 1} \left( -1 + \frac{2\pi^2(n + 1)k_\alpha^2}{3(2n + 1)^2} \right) + \frac{2}{2n + 1} \left( 1 - \frac{2\pi^2(n + 1)k_\alpha^2}{(2n + 1)^2} \right)
\]

\[
+ O \left( \frac{1}{n^{3/2}} \right)
\]

(C.73)

where \( k_\alpha \approx \sqrt{n} \) obtained from (8.23) was utilized. Inserting the expression for \( k_\alpha \) into the first parenthesis of (C.73) gives that it is positive,

\[
-1 + \frac{2\pi^2(n + 1)k_\alpha^2}{3(2n + 1)^2} = -1 + \frac{2\pi^2(n + 1)}{3(2n + 1)^2} \left( \frac{2 \sqrt{3\sqrt{n + 1} + 1}}{\pi} + \frac{1}{2} \right)^2
\]

\[
= -1 + \frac{8(n + 1)^2}{(2n + 1)^2} + \frac{4\pi(n + 1)^{3/2}}{\sqrt{3}(2n + 1)^2} + \frac{\pi^2(n + 1)}{6(2n + 1)^2}
\]

\[
= 1 + \frac{8n + 6}{(2n + 1)^2} + \frac{4\pi(n + 1)^{3/2}}{\sqrt{3}(2n + 1)^2} + \frac{\pi^2(n + 1)}{6(2n + 1)^2}
\]

\[
= 1 + O \left( \frac{1}{\sqrt{n}} \right)
\]

(C.74)

The second parenthesis of (C.73) is \( O(1) \) since \( k_\alpha \sim O(\sqrt{n}) \). Hence,

\[
S = \frac{2k_\alpha}{2n + 1} \left( 1 + O \left( \frac{1}{\sqrt{n}} \right) \right) + O \left( \frac{1}{n} \right)
\]

(C.75)

giving that

\[
\sum_{k=1}^{k_\alpha-1} (\beta_{1,k} - \hat{\beta}_{1,k}) = \frac{2k_\alpha}{2n + 1} + O \left( \frac{1}{n} \right)
\]

(C.76)

which is positive for sufficiently large but bounded \( n \). This concludes the proof of Lemma 8.6.
C.7 Proof of Lemma 8.7

It is known that $\alpha_k$, $\beta_k$, $\beta_{1,k}$ and $\hat{\beta}_{1,k}$ are positive for all $n$ and indices $k$. Also, $\alpha_k$ grows with increasing $k$, (3.23). Then

$$\sum_{k=1}^{n-1} \left( \beta_{1,k} e^{-\frac{\alpha_k}{n}} - \beta_{1,k} e^{-\frac{\alpha_{k+1}}{n}} \right) = \sum_{k=1}^{n-1} e^{-\frac{\alpha_k}{n}} \left( \beta_{1,k} - \beta_{1,k} e^{-\frac{\alpha_k}{n}} e^{-\frac{\alpha_{k+1}}{n}} \right)$$

$$\geq \sum_{k=1}^{n-1} e^{-\frac{\alpha_k}{n}} \left( \beta_{1,k} - \beta_{1,k} e^{-\frac{\alpha_k}{n}} e^{-\frac{\alpha_{k+1}}{n}} \right)$$

$$\geq e^{-\frac{\alpha_k}{n}} \sum_{k=1}^{n-1} \left( \beta_{1,k} - \beta_{1,k} \right) \quad (C.77)$$

Using Lemma 8.6, (C.77) is known to be positive. This concludes the proof of Lemma 8.7.
C.8 Proof of Lemma 8.8

The expressions for $\beta_{1,k}$ and $\hat{\beta}_{1,k}$ are found in (3.24) and (4.21).

\[
\sum_{k=1}^{n} \left( \beta_{1,k} - \hat{\beta}_{1,k} \right) = 2n - \frac{4(n+1)}{2n+1} \sum_{k=1}^{n} \cos^2 \left( \frac{\pi(2k-1)}{2(2n+1)} \right) \quad (C.78)
\]

In the proof of Lemma 4.2, the sum in (C.78) is already computed, see (B.66) and (B.84). Using this result gives

\[
\sum_{k=1}^{n} \left( \beta_{1,k} - \hat{\beta}_{1,k} \right) = 2n - \frac{4(n+1)}{2n+1} \left( \frac{n}{2} + \frac{1}{4} \right) \\
= 2n - \frac{2n(n+1)}{2n+1} - \frac{n+1}{2n+1} \\
= n - 1 \quad (C.79)
\]

This concludes the proof of Lemma 8.8.
C.9 Proof of Lemma 8.9

The following expression is studied,

$$\beta_{1,k}e^{-\frac{\alpha_k t}{\tau}} - \beta_{1,k}e^{-\frac{\alpha_k t}{\tau}}$$  \hspace{1cm} (C.80)

For a fixed $k$, the sequence (C.80) changes sign when

$$\beta_{1,k}e^{-\frac{\alpha_k t}{\tau}} = \beta_{1,k}e^{-\frac{\alpha_k t}{\tau}}$$  \hspace{1cm} (C.81)

Reorganization of the terms gives

$$e^{-\frac{|\alpha_k - \alpha_k| t}{\tau}} = \frac{\beta_{1,k}}{\beta_{1,k}}$$  \hspace{1cm} (C.82)

or

$$-\frac{(\alpha_k - \alpha_k)t}{\tau} = \ln\left(\frac{\beta_{1,k}}{\beta_{1,k}}\right)$$  \hspace{1cm} (C.83)

This gives that the time $t$ for which (C.80) changes sign equals

$$t = \frac{\tau}{\alpha_k - \alpha_k} \ln\left(\frac{\beta_{1,k}}{\beta_{1,k}}\right)$$  \hspace{1cm} (C.84)

This concludes the proof of Lemma 8.9.
C.10 Proof of Lemma 8.10

The following sum is studied,

\[ S(t) \triangleq \sum_{k=1}^{n} \left( \beta_{1,k} e^{-\frac{a_{x,t}}{\tau}} - \bar{\beta}_{1,k} e^{-\frac{a_{x,t}}{\tau}} \right) \]  

(C.85)

It can be observed that each single element in the summation may have negative sign for some indices \( k \) and times \( t \). It will be shown, however, that the sum is non-negative. It is noted that the parameters \( \alpha_k, \bar{\alpha}_k, \beta_{1,k} \) and \( \bar{\beta}_{1,k} \) are positive for all \( n \) and all indices \( k \).

The sum (C.85) is separated into two parts, one for small indices \( k \), and one for large. The index which separates the two sums is the \( k \) defined in (8.21) in Lemma 8.4,

\[ k_n = \frac{2\sqrt{3}}{\pi} \sqrt{n + 1} + \frac{1}{2} \]  

(C.86)

The index \( k_n \) has the property that for \( k < k_n \), then \( \alpha_k < \bar{\alpha}_k \) and vice versa. The quantity \( k_n \) is in general not an integer. The expression (C.86) will persist this be used in the computations to come, since the exact value of the \( k \) giving \( \alpha_k \approx \bar{\alpha}_k \) is unknown in the general case. Now, let

\[ S(t) = \sum_{k=1}^{n} \left( \beta_{1,k} e^{-\frac{a_{x,t}}{\tau}} - \bar{\beta}_{1,k} e^{-\frac{a_{x,t}}{\tau}} \right) = \sum_{k=1}^{k_n} \left( \beta_{1,k} e^{-\frac{a_{x,t}}{\tau}} - \bar{\beta}_{1,k} e^{-\frac{a_{x,t}}{\tau}} \right) + \sum_{k=k_n}^{n} \left( \beta_{1,k} e^{-\frac{a_{x,t}}{\tau}} - \bar{\beta}_{1,k} e^{-\frac{a_{x,t}}{\tau}} \right) \]  

(C.87)

It is first noted that the result of Lemma 8.7 gives that \( S_1(t) > 0 \) for all \( t \). It remains to show that \( S_2(t) \) does not influence the sum (C.87) in such a way that it becomes negative.

Different time intervals will be treated separately. First, consider the case \( t \geq \tau \). An upper bound of \( S_2(t) \) will be generated and compared to \( S_1(t) \).

\[ S_2(t) \leq \sum_{k=k_n}^{n} \beta_{1,k} e^{-\frac{a_{x,t}}{\tau}} + \sum_{k=k_n}^{n} \bar{\beta}_{1,k} e^{-\frac{a_{x,t}}{\tau}} \]  

(C.88)

The terms in \( S_2 \) decrease much faster than a geometric series. Let

\[ S_{21} \triangleq \sum_{k=k_n}^{n} \alpha_1 (2k-1)^2 \]  

(C.89)
where

\[ a_1 = e^{-\frac{2a}{\pi}} \]  \hspace{1cm} (C.90)

Then

\[ S_{21} = a_1^{(2k_a-1)^2} + a_1^{(2k_a+1)^2} + \ldots + a_1^{(2n-1)^2} \leq a_1^{(2k_a-1)^2} \left( 1 + a_1^{8k_a} + a_1^{16k_a} + \ldots \right) = a_1^{(2k_a-1)^2} \frac{1}{1 - a_1^{8k_a}} \]  \hspace{1cm} (C.91)

Equations (C.89)–(C.91) give

\[ S_{21} = 2 \sum_{k=k_a}^{n} e^{-\frac{2\pi}{\pi^2}(2k-1)^2} \leq 2e^{-\frac{\pi^2(2k_a-1)^2}{4\pi^2}} \frac{1}{1 - e^{-\frac{2\pi^2k_a}{\pi}}} \]  \hspace{1cm} (C.92)

When determining an upper bound of \( S_{22} \), it is observed that \( \beta_{1,k} \) (4.21) is larger than two for small \( k \), but that it under no circumstances exceeds three. Also, since \( k_a \leq k \leq n \), \( 0 < \omega_k < \pi/2 \) follows, where the definition of \( \omega_k \) is found in (8.11). Then

\[ \sin \omega_k > \frac{2}{\pi} \omega_k \]  \hspace{1cm} (C.93)

and

\[ \sin^2 \omega_k > \frac{4}{\pi^2} \omega_k^2 \]  \hspace{1cm} (C.94)

This gives a lower bound on \( \alpha_k \).

\[ \alpha_k = 2(n+1)^2 \left( 1 - \cos \left( \frac{\pi(2k-1)}{2n+1} \right) \right) \]
\[ = 4(n+1)^2 \sin^2 \left( \frac{\pi{2k-1}}{2(2n+1)} \right) \]
\[ > 4(n+1)^2 \frac{\pi^2(2k-1)^2}{4(2n+1)^2} \]
\[ = \frac{4(n+1)^2}{(2n+1)^2}(2k-1)^2 \]
\[ > (2k-1)^2 \]  \hspace{1cm} (C.95)
Utilizing this result

\[ S_{22} = \sum_{k=k_a}^{n} \beta_{1,k} e^{-\frac{2k}{\tau}} \]

\[ \leq 3 \sum_{k=k_a}^{n} e^{-\frac{2(k-1)}{\tau}} \] (C.96)

The terms in (C.96) decrease faster than a geometric series. Similar calculations as were performed for \( S_{21} \), are used next in order to determine an upper bound of \( S_{22} \). Let

\[ S_{22} \triangleq \sum_{k=k_a}^{n} a_2^{(2k-1)^2} \] (C.97)

where

\[ a_2 = e^{-\frac{1}{\tau}} \] (C.98)

Then

\[ S_{22} = a_2^{(2k_a-1)^2} + a_2^{(2k_a+1)^2} + \ldots + a_2^{(2n-1)^2} \]

\[ \leq a_2^{(2k_a-1)^2} \left( 1 + a_2^{8k_a} + a_2^{16k_a} + \ldots \right) \]

\[ = a_2^{(2k_a-1)^2} \frac{1}{1 - a_2^{8k_a}} \] (C.99)

which gives

\[ S_{22} \leq 3 \sum_{k=k_a}^{n} e^{-\frac{2(k-1)}{\tau}} \]

\[ \leq \frac{3e^{-\frac{(2k_a-1)^2}{\tau}}}{1 - e^{-\frac{8k_a}{\tau}}} \] (C.100)

Equations (C.92) and (C.100) give an upper bound of \( S_2 \). For \( t > \tau \), the bound is small since \( n \) is large.

It remains to show that \( S_1 \) is larger than the sum of (C.92) and (C.100). Recall that \( t > \tau \) is assumed. It is noted that all terms in \( S_1(t) \geq 0 \) when \( t > \tau \), see Lemma 8.5. Then the following lower bound can be used,

\[ S_1(t) = \sum_{k=1}^{k_a-1} \left( \beta_{1,k} e^{-\frac{2k}{\tau}} - \beta_{1,k} e^{-\frac{8k}{\tau}} \right) \]

\[ \geq \beta_{1,1} e^{-\frac{2}{\tau}} - \beta_{1,1} e^{-\frac{8}{\tau}} \]

\[ = 2e^{-\frac{2}{\tau}} - \frac{4(n+1)}{2n+1} \cos^2 \left( \frac{\pi}{2(n+1)} \right) e^{-4(n+1)\frac{\pi}{(2n+1)}} \sin^2 \left( \frac{\pi n + \pi}{2(n+1)} \right) \] (C.101)
Series expansion of the trigonometric terms are used next.

\[ S_1(t) \geq 2e^{-\frac{\pi^2}{8\pi^2}} - \frac{4(n + 1)}{2n + 1} \left[ 1 - \frac{\pi^2}{4(2n + 1)^2} + O\left(\frac{1}{n^4}\right) \right] \]
\[ \times e^{-\frac{\pi^2}{8\pi^2}} \left( \frac{1}{\sqrt{2\pi n + \pi}} + O\left(\frac{1}{n^{3/2}}\right) \right) \]
\[ = 2e^{-\frac{\pi^2}{8\pi^2}} - \left( 2 + \frac{2}{2n + 1} \right) e^{-\frac{\pi^2}{8\pi^2}} \left( \frac{4(n + 1)^2}{(2n + 1)^2} + O\left(\frac{1}{n^2}\right) \right) \]
\[ + \frac{\pi^2 (n + 1)}{(2n + 1)^3} e^{-\frac{\pi^2}{8\pi^2}} \left( \frac{4(n + 1)^2}{(2n + 1)^2} + O\left(\frac{1}{n^2}\right) \right) \]
\[ = 2e^{-\frac{\pi^2}{8\pi^2}} \left[ 1 - e^{-\frac{\pi^2}{8\pi^2}} + O\left(\frac{1}{n^2}\right) \right] + \frac{\pi^2}{8n^2} e^{-\frac{\pi^2}{8\pi^2}} e^{-\frac{\pi^2}{8\pi^2}} \]
\[ \approx 2e^{-\frac{\pi^2}{8\pi^2}} \left[ 1 - e^{-\frac{\pi^2}{8\pi^2}} - \frac{1}{2n + 1} e^{-\frac{\pi^2}{8\pi^2}} \right] + \frac{\pi^2}{8n^2} e^{-\frac{\pi^2}{8\pi^2}} e^{-\frac{\pi^2}{8\pi^2}} \tag{C.102} \]

for large \( n \). The expression within parenthesis is positive if

\[ \left( 1 + \frac{1}{2n + 1} \right) e^{-\frac{\pi^2}{8\pi^2}} < 1 \]

This corresponds to the requirement

\[ \frac{\pi^2 t}{4\pi n} > \ln \left( 1 + \frac{1}{2n + 1} \right) \tag{C.103} \]

A harder condition than (C.103) is

\[ \frac{\pi^2 t}{4\pi n} > \frac{1}{2n + 1} \tag{C.104} \]

resulting in

\[ t > \frac{4\pi n}{\pi^2 (2n + 1)} \approx \frac{2\pi}{\pi^2} \]

which holds since \( t > \tau \) is assumed in this part of the proof. The last term within the parenthesis in (C.102) can be disregarded, since it is small compared to the other ones. Compensating by removing the constant two in front of the expression in (C.102) gives the simplified expression,

\[ S_1(t) \geq e^{-\frac{\pi^2}{8\pi^2}} \left( 1 - e^{-\frac{\pi^2}{8\pi^2}} \right) \tag{C.105} \]

which is larger than \( e^{-nt/\tau} \).
Inserting (C.92), (C.100) into (C.88) gives an upper bound of $S_2(t)$. Further, (C.105) gives a lower bound of $S_1(t)$. Then, since

$$S_1(t) \geq e^{-\frac{\tau^2}{2\alpha_k}} \left( 1 - e^{-\frac{\tau^2}{8\alpha_k}} \right) > \frac{2e^{-\frac{\tau^2}{2\alpha_k}} - 1}{1 - e^{-\frac{\tau^2}{8\alpha_k}}} \geq S_2(t) \quad \text{(C.106)}$$

it holds that (C.85) exceeds zero for $t > \tau$. The exponentials to the right in (C.106) go towards zero faster than $\exp(-\frac{\tau^2}{4\alpha_k}) \to 1$. Hence, (8.28) holds for $t > \tau$.

Next consider small times, $0 \leq t \leq \tau$. It is first noted that

$$S(0) = \sum_{k=1}^{n} (\beta_{1,k} - \bar{\beta}_{1,k}) = n - 1 > 0 \quad \text{(C.107)}$$

This was shown in Lemma 8.8. Using (C.107) and (C.76) in the proof of Lemma 8.6 give

$$S_2(0) = \sum_{k=1}^{n} (\beta_{1,k} - \bar{\beta}_{1,k}) - \sum_{k=1}^{k_{\text{max}}-1} (\beta_{1,k} - \bar{\beta}_{1,k}) = (n - 1) - \frac{2k_{\alpha}}{2n + 1} + O \left( \frac{1}{n} \right) > 0 \quad \text{(C.108)}$$

for sufficiently large $n$. It is worth repeating that $k_{\alpha} \sim \sqrt{n}$. Hence, both $S_1(t)$ and $S_2(t)$ exceed zero when $t = 0$.

It is known from Lemma 8.7 that $S_1(t) > 0$ for all $t$. It is interesting to study the behaviour of $S_2(t)$. The terms in the sum $S_2(t)$ change sign for

$$t_e \triangleq \frac{\tau}{\alpha_k} \ln \left( \frac{\beta_{1,k}}{\bar{\beta}_{1,k}} \right) \quad \text{(C.109)}$$

according to Lemma 8.9. Figure C.1 shows a plot of $t_e/\tau$ for increasing indices $k$. The model order was chosen as $n = 100$. The smallest index $k$ shown is $k = 12$, which corresponds to the first integer value larger than $k_{\alpha}$, cf (8.21). The smallest respectively largest values of $t_e$ are $t_{e_{\text{min}}} \approx 8.7e^{-5}\tau$ [h], and $t_{e_{\text{max}}} \approx 2.5e^{-2}\tau$ [h]. This means that for $t < t_{e_{\text{min}}}$, the terms in $S_2(t)$ will have constant sign. According to (C.108) they are positive. This gives that (C.85) exceeds zero for $t < t_{e_{\text{min}}}$.

For $t > t_{e_{\text{max}}}$, all terms in $S_2(t)$ will, on the other hand, be negative. Next, it is shown that the contribution from $S_1(t)$ exceeds the contribution from $S_2(t)$ for $t_{e_{\text{max}}} \leq t \leq \tau$. Instead of computing the sum $S_2(t)$, an upper bound will be
utilized.

\[
|S_2(t)| = \left| \sum_{k=k_a}^{n} \left( \beta_{1,k} e^{-\frac{\alpha_k t}{\tau}} - \bar{\beta}_{1,k} e^{-\frac{\alpha_k t}{\tau}} \right) \right|
\]

\[
\leq (n - k_{\alpha} + 1) \max_{k_{\alpha} \leq k \leq n} \left| \frac{\beta_{1,k} e^{-\frac{\alpha_k t}{\tau}} - \bar{\beta}_{1,k} e^{-\frac{\alpha_k t}{\tau}}}{\beta_{1,k} e^{-\frac{\alpha_k t}{\tau}} + \bar{\beta}_{1,k} e^{-\frac{\alpha_k t}{\tau}}} \right|
\]

This approximation is, without exaggeration, very coarse since the terms in \( S_2(t) \) are known to decrease fast with \( k \). In the interval \( t_{\text{max}} \leq t \leq \tau \), the quantity (C.110) is maximized for \( t = t_{\text{max}} \) and \( k = k_{\alpha} \). The exponents in (C.110) are then

\[
-\frac{\alpha_k t_{\text{max}}}{\tau} = -\frac{\alpha_k}{\tau} \frac{\tau}{\alpha_k} + \frac{\tau}{\alpha_k} \ln \left( \frac{\beta_{1,k_{\alpha}}}{\beta_{1,k_{\alpha}}} \right)
\]

\[
= -\frac{1}{1 - \frac{\alpha_k}{\alpha_{k_{\alpha}}} \ln \left( \frac{\beta_{1,k_{\alpha}}}{\beta_{1,k_{\alpha}}} \right)}
\]

\[
-\frac{\alpha_k t_{\text{max}}}{\tau} = -\frac{1}{1 - \frac{\alpha_k}{\alpha_{k_{\alpha}}} \ln \left( \frac{\beta_{1,k_{\alpha}}}{\beta_{1,k_{\alpha}}} \right)}
\]

Since \( \alpha_{k_{\alpha}} \approx \alpha_k \), the exponents have rather large negative values. Hence, (C.110) is small despite the factor \( n - k_{\alpha} + 1 \), and will be smaller than \( S_1(t) \) for
small \( t \leq \tau \),

\[
|S_2(t)| \leq (n - k_\alpha + 1) \left( \beta_{1,k_\alpha} e^{-\frac{\alpha_\beta t_{\mathrm{max}}}{\tau}} + \beta_{1,k_\alpha} e^{-\frac{\alpha_\beta t_{\mathrm{max}}}{\tau}} \right)
\]

\[
\leq S_1(t)
\]

(C.113)

It remains to study the time interval \( t_{\mathrm{min}} \leq t \leq t_{\mathrm{max}} \). As stated earlier, \( S_1(t) > 0 \) for all \( t \). Also, all terms in \( S_2(t) \) are positive for \( t < t_{\mathrm{min}} \). Increasing \( t \) results in that more and more terms in \( S_2(t) \) adopt negative signs, but increasing \( t \) also gives smaller terms. Both the positive part and the possible negative part decrease in size. In the extreme (in this interval), \( t = t_{\mathrm{max}} \), all terms in \( S_2(t) \) are negative, but the contribution to the sum \( S_2(t) \) is close to zero for large \( n \). The quantity \( |S_2(t)| \) decreases faster than \( S_1(t) \) for increasing \( t \), since the exponents have larger negative values in \( S_2(t) \). This gives that the impact of \( S_2(t) \) after some short time will be smaller than the impact of \( S_1(t) \). Consequently, when \( t \) is large enough to make \( S_2(t) \) negative, its absolute value has decreased beyond the value of \( S_1(t) \). Note that when \( t \) increases towards \( t_{\mathrm{max}} \), the terms in \( S_2(t) \) which remain positive the longest, are the ones corresponding to the smallest indices \( k \), cf Figure C.1. These terms are consequently the largest of all terms. The indices \( k \) of size approximately \( k_\alpha \) contribute more to \( S_2(t) \) than larger \( k \), since

\[
\beta_{1,k_\alpha} e^{-\frac{\alpha_\beta t_{\mathrm{max}}}{\tau}} - \beta_{1,k_\alpha} e^{-\frac{\alpha_\beta t_{\mathrm{max}}}{\tau}}
\]
declines with increasing \( k \) for a fixed \( t \).

Figure C.2 shows a comparison of the sums \( S_1(t) \) and \( S_2(t) \). Plot (a) illustrates \( S_1(t) \), \(|S_2(t)| \) and the upper bound (C.110) in the time interval \( [t_{\mathrm{min}}, \tau] \). In this example the model order \( n \) equals 100 and \( \tau = 1 \) [h]. It can be noted that \( S_1(t) \) exceeds the size of the upper bound. Plot (b) displays the two sums in the interval \( t \in [t_{\mathrm{min}}, t_{\mathrm{max}}] \). Also \( S_2(t) \) is positive, except for when \( t \to t_{\mathrm{min}} \), but then its absolute value is smaller than \( S_1(t) \). Since the curves are plotted in logarithmic scale, the negative values of \( S_2(t) \) are not shown.

Hence \( S(t) = S_1(t) + S_2(t) \geq 0 \). This concludes the proof of Lemma 8.10.
Figure C.2: An example of $S_1(t)$ and $S_2(t)$ for the time intervals $t \in [t_{\text{min}}, t]$ (plot (a)) and $t \in [t_{\text{min}}, t_{\text{max}}]$ (plot (b)). The model order was chosen as $n = 100$ and the time constant $\tau = 1 \text{[h]}$. 
C.11 Proof of Lemma 8.11

N. B. As mentioned in Section 8, the proof is not fully completed.

Let

\[ g_k(t) \triangleq \beta_{2,k} e^{-\frac{\alpha_k t}{\tau}} - \tilde{\beta}_{2,k} e^{-\frac{\tilde{\alpha}_k t}{\tau}} \]  \hspace{1cm} (C.114)

The objective is to show that \( \sum_{k=1}^{n} g_k(t) \) changes sign at most two times. First note that \( \sum_{k=1}^{n} g_k(t) \) can be divided into several parts, each either strictly increasing or strictly decreasing.

\[
\sum_{k=1}^{n} g_k(t) = \sum_{k=1}^{n} \left( \beta_{2,k} e^{-\frac{\alpha_k t}{\tau}} - \tilde{\beta}_{2,k} e^{-\frac{\tilde{\alpha}_k t}{\tau}} \right)
= \sum_{k=1 \text{ odd}}^{n} \beta_{2,k} e^{-\frac{\alpha_k t}{\tau}} + \sum_{k=2 \text{ even}}^{n} \beta_{2,k} e^{-\frac{\alpha_k t}{\tau}} - \sum_{k=3 \text{ odd}}^{n} \tilde{\beta}_{2,k} e^{-\frac{\tilde{\alpha}_k t}{\tau}}
- \sum_{k=4 \text{ even}}^{n} \tilde{\beta}_{2,k} e^{-\frac{\tilde{\alpha}_k t}{\tau}}
= \sum_{k=1 \text{ odd}}^{n} \beta_{2,k} e^{-\frac{\alpha_k t}{\tau}} - \sum_{k=2 \text{ even}}^{n} |\beta_{2,k}| e^{-\frac{\alpha_k t}{\tau}} - \sum_{k=3 \text{ odd}}^{n} \tilde{\beta}_{2,k} e^{-\frac{\tilde{\alpha}_k t}{\tau}}
+ \sum_{k=4 \text{ even}}^{n} |\tilde{\beta}_{2,k}| e^{-\frac{\tilde{\alpha}_k t}{\tau}} \hspace{1cm} (C.115)
\]

cf the definition of \( \beta_{2,k} \) and \( \tilde{\beta}_{2,k} \) from (3.25) and (4.22). All terms in the sums of (C.115) are positive.

Now show that the sums with positive signs are i) positive, and ii) strictly decreasing for all \( t \). Let

\[ f_{pos}(t) \triangleq \sum_{k=1 \text{ odd}}^{n} \beta_{2,k} e^{-\frac{\alpha_k t}{\tau}} + \sum_{k=2 \text{ even}}^{n} |\beta_{2,k}| e^{-\frac{\alpha_k t}{\tau}} \hspace{1cm} (C.116)\]

i) \( f_{pos}(t) \) is positive since \( \beta_{2,k} > 0 \) for odd indices \( k \).

ii) The derivative of \( f_{pos}(t) \) equals

\[
\frac{\partial f_{pos}(t)}{\partial t} = - \sum_{k=1 \text{ odd}}^{n} \frac{\alpha_k \beta_{2,k}}{\tau} e^{-\frac{\alpha_k t}{\tau}} - \sum_{k=2 \text{ even}}^{n} \tilde{\beta}_{2,k} |e^{-\frac{\tilde{\alpha}_k t}{\tau}}| - \sum_{k=2 \text{ even}}^{n} \frac{\tilde{\alpha}_k |\tilde{\beta}_{2,k}|}{\tau} e^{-\frac{\tilde{\alpha}_k t}{\tau}} < 0 \hspace{1cm} (C.117)
\]

which gives that \( f_{pos}(t) \) is strictly decreasing.
The same procedure is performed for the sums with negative signs in (C.115). Let

$$f_{neg}(t) \triangleq - \sum_{k=1}^{n} \beta_{2,k} e^{-\frac{\alpha_k t}{\tau}} - \sum_{k=1}^{n} \frac{\alpha_k}{\tau} \beta_{2,k} e^{-\frac{\alpha_k t}{\tau}}$$

(C.118)

and show that $f_{neg}(t)$ is iii) negative, and iv) strictly increasing for all $t$.

iii) This assumption holds since $\beta_{2,k}$ is positive for odd indices $k$.

iv) The derivative of $f_{neg}(t)$ equals

$$\frac{\partial f_{neg}(t)}{\partial t} = \sum_{k=1}^{n} \frac{\alpha_k}{\tau} \beta_{2,k} e^{-\frac{\alpha_k t}{\tau}} + \sum_{k=1}^{n} \frac{\alpha_k^2}{\tau^2} \beta_{2,k} e^{-\frac{\alpha_k t}{\tau}}$$

$$> 0$$

(C.119)

which gives that $f_{neg}(t)$ is strictly increasing.

Some information of $f_{pos}(t)$ and $f_{neg}(t)$ is now known. It remains to evaluate a more specific appearance of the curves, e.g. the number of bends that can be expected. This is done using the second derivatives.

$$\frac{\partial^2 f_{pos}(t)}{\partial t^2} = \sum_{k=1}^{n} \frac{\alpha_k^2}{\tau^2} \beta_{2,k} e^{-\frac{\alpha_k t}{\tau}} + \sum_{k=1}^{n} \frac{\alpha_k}{\tau} \beta_{2,k} e^{-\frac{\alpha_k t}{\tau}}$$

$$> 0$$

(C.120)

which gives that $f_{pos}(t)$ is convex for all $t$. Further

$$\frac{\partial^2 f_{neg}(t)}{\partial t^2} = - \sum_{k=1}^{n} \frac{\alpha_k^2}{\tau^2} \beta_{2,k} e^{-\frac{\alpha_k t}{\tau}} - \sum_{k=1}^{n} \frac{\alpha_k}{\tau} \beta_{2,k} e^{-\frac{\alpha_k t}{\tau}}$$

$$< 0$$

(C.121)

which gives that $f_{neg}(t)$ is concave for all $t$. Both $f_{pos}(t)$ and $f_{neg}(t)$ converge smoothly towards the $t$ axis. At $t = 0$, the sign of (C.115) depends on the sign of $\beta_{2,n}$ since $|\beta_{2,n}|$ grows with $k$ while $|\beta_{2,k}| \to 0$ for large indices $k$, cf Figure 4.5. Hence

$$\text{sgn} \sum_{k=1}^{n} (\beta_{2,k} - \hat{\beta}_{2,k}) = (-1)^{n+1}$$

(C.122)

For small $t \approx 0$,

$$\sum_{k=1}^{n} (\beta_{2,k} e^{-\frac{\alpha_k t}{\tau}} - \hat{\beta}_{2,k} e^{-\frac{\alpha_k t}{\tau}})$$

(C.123)
Figure C.3: The sum $\sum_{k=-n}^{n} g(t)$ is shown for the even model orders $n = 10, 30, 50, 70$ and $100$. Plot (a) displays the sums for the time interval $t \in [10^{-6}, 10^{7}]$ [h]. In plot (b), the interesting details in the sums are highlighted. The time constant $\tau = 1$ [h] in this example.

can be assumed to slowly converge towards zero for increasing $t$. As $t$ grows towards $\tau$, (C.123) can be expected to change signs only a limited number of times. To find out if this assumption can be regarded as correct, (C.123) is displayed for different values of $n$. In Figure C.3, the model order is even, and in Figure C.4, $n$ is odd. The model orders $n = 10, 30, 50, 70, 100$ respectively $n = 11, 31, 51, 71, 101$ are used. In both figures, plot (b) displays an enlargement of the interesting details. It can be noted that for even model orders, the sum (C.123) changes sign one time and for odd $n$ it changes sign two times. From this, the lemma can be believed to hold for any large $n$. This concludes the proof of Lemma 8.11.
Figure C.4: The sum $\sum_{k=1}^{n} g_k(t)$ is shown for the odd model orders $n = 11, 31, 51, 71$ and 101.
C.12 Proof of Lemma 8.12

From (8.11) the expression for $\omega_k$ is obtained. It is noted that

$$\omega_k = \frac{\pi(2k-1)}{2n+1} \in (0, \pi)$$  \hspace{1cm} (C.124)

for all $k \in [1, n]$. The error in $\tilde{\beta}_{1,k}$ is

$$|\beta_{1,k} - \tilde{\beta}_{1,k}| = \left| 2 - \frac{4(n+1)}{(2n+1)} \cos^2 \left( \frac{\pi(2k-1)}{2(2n+1)} \right) \right|$$

$$= \left| 2 - \frac{4(n+1)}{(2n+1)} \cos^2 \left( \frac{\omega_k}{2} \right) \right|$$

$$= \left| 2 \left( 1 - \frac{(n+1)}{(2n+1)} (\cos(\omega_k) + 1) \right) \right|$$

$$= \frac{2}{(2n+1)} \left| n - (n+1) \cos(\omega_k) \right|$$  \hspace{1cm} (C.125)

Show that

$$\frac{2}{(2n+1)} \left| n - (n+1) \cos(\omega_k) \right| \leq \frac{2}{(2n+1)} \left( 1 + (n+1)\frac{\omega_k^2}{2} \right)$$  \hspace{1cm} (C.126)

holds for all $\omega_k \in (0, \pi)$, independent of the choice of $n$. This result implies (8.30). The relation in (C.126) corresponds to

$$-1 - \frac{(n+1)\omega_k^2}{2} \leq n - (n+1) \cos(\omega_k) \leq 1 + (n+1)\frac{\omega_k^2}{2}$$

$$\iff \quad n - (n+1) \frac{\omega_k^2}{2} \leq (n+1) \cos(\omega_k) \leq n + 1 + (n+1)\frac{\omega_k^2}{2}$$

$$\iff \quad \frac{n-1}{n+1} - \frac{\omega_k^2}{2} \leq \cos(\omega_k) \leq 1 + \frac{\omega_k^2}{2}$$  \hspace{1cm} (C.127)

The rightmost inequality holds for all $\omega_k$ since the cosine function is bounded to one. In order to prove the left inequality, note that

$$\frac{n-1}{n+1} \leq 1$$

and that

$$\cos(\omega) \geq 1 - \frac{\omega^2}{2}$$

for a continuous variable $\omega \in [0, \pi]$ (cf. the series expansion of the cosine function). Hence, (C.125) and (C.126) give (8.30). This concludes the proof of Lemma 8.12.
C.13 Proof of Lemma 8.13

It is known that

$$\frac{\tan(\omega)}{\omega} \geq 1 \quad \text{for} \quad 0 \leq \omega < \frac{\pi}{2} \quad (C.128)$$

holds. This result can be used in order to prove (8.31).

$$\frac{\bar{\beta}_{1,k}}{\alpha_k} = \frac{\beta_{1,k}}{\alpha_k} = \frac{8}{\pi^2(2k-1)^2} - \frac{2\cos^2\left(\frac{\omega_k}{2}\right)}{(n+1)(2n+1)(1 - \cos(\omega_k))}$$

$$= \frac{8}{\pi^2(2k-1)^2} - \frac{\cos^2\left(\frac{\omega_k}{2}\right)}{(n+1)(2n+1)\sin^2\left(\frac{\omega_k}{2}\right)} \quad (C.129)$$

where the definition of $\omega_k$ is found in (8.11). For (C.129) to be larger or equal to zero, the following equivalent relation must hold,

$$\tan^2\left(\frac{\omega_k}{2}\right) \geq \frac{\pi^2(2k-1)^2}{8(n+1)(2n+1)} = \frac{(2n+1)\omega_k^2}{8(n+1)} \quad (C.130)$$

This corresponds to

$$\tan^2\left(\frac{\omega_k}{2}\right) \geq \frac{2n+1}{2n+2} \left(\frac{\omega_k}{2}\right)^2 \quad (C.131)$$

Using (C.128) and the fact that $(2n+1)/(2n+2) < 1$, give that (C.131) holds for all $n$ and all $k \in [1, n]$. This concludes the proof of Lemma 8.13.
C.14 Proof of Lemma 8.14

Since \( n \) is assumed to be large, Lemma 4.2 can be used. The expressions for the parameters are obtained from (3.23), (3.24), (4.20) and (4.21).

\[
\sum_{k=1}^{n} \left( \frac{\beta_{1,k}}{\alpha_k} - \frac{\hat{\beta}_{1,k}}{\hat{\alpha}_k} \right) = \sum_{k=1}^{n} \left( \frac{8}{\pi^2 (2k - 1)^2} - \frac{4(n + 1) \cos^2(\omega_k/2)}{2(2n + 1)(n + 1)^2 (1 - \cos(\omega_k))} \right)
\]
\[
= \sum_{k=1}^{n} \left( \frac{8}{\pi^2 (2k - 1)^2} - \frac{\cos^2(\omega_k/2)}{(2n + 1)(n + 1) \sin^2(\omega_k/2)} \right)
\]
\[
= \frac{8}{\pi^2} \sum_{k=1}^{n} \frac{1}{(2k - 1)^2} - \frac{1}{(2n + 1)(n + 1)} \sum_{k=1}^{n} \frac{\cos^2(\omega_k/2)}{\sin^2(\omega_k/2)}
\]
\[
= \frac{8}{\pi^2} \sum_{k=1}^{n} \frac{1}{(2k - 1)^2}
\]
\[
- \frac{1}{(2n + 1)(n + 1)} \left( \sum_{k=1}^{n} \frac{1}{\sin^2(\omega_k/2)} - n \right) \tag{C.132}
\]

The relation \( \sin(\omega_k/2) \leq \omega_k/2 \) is used, giving

\[
- \frac{1}{\sin^2(\omega_k/2)} \leq - \frac{1}{(\omega_k/2)^2} \tag{C.133}
\]

From Lemma 8.13 it is known that all terms in (C.132) are positive. This result and (C.133) motivates the following upper bound,

\[
\sum_{k=1}^{n} \left( \frac{\beta_{1,k}}{\alpha_k} - \frac{\hat{\beta}_{1,k}}{\hat{\alpha}_k} \right) \leq \frac{8}{\pi^2} \sum_{k=1}^{n} \frac{1}{(2k - 1)^2}
\]
\[
- \frac{1}{(2n + 1)(n + 1)} \left( \sum_{k=1}^{n} \frac{1}{(\omega_k/2)^2} - n \right)
\]
\[
= \frac{8}{\pi^2} \sum_{k=1}^{n} \frac{1}{(2k - 1)^2}
\]
\[
- \frac{1}{(2n + 1)(n + 1)} \left( \frac{4(2n + 1)^2}{\pi^2} \sum_{k=1}^{n} \frac{1}{(2k - 1)^2} - n \right)
\]
\[
= \frac{8}{\pi^2} \sum_{k=1}^{n} \frac{1}{(2k - 1)^2} - \frac{4(2n + 1)}{\pi^2 (n + 1)} \sum_{k=1}^{n} \frac{1}{(2k - 1)^2}
\]
\[
+ \frac{n}{(2n + 1)(n + 1)} \tag{C.134}
\]
Rearranging the terms gives
\[
\sum_{k=1}^{n} \left( \frac{\beta_{1,k}}{\alpha_k} - \frac{\beta_{1,k}}{\alpha_k} \right) \leq \frac{4}{\pi^2} \left( 2 - \frac{(2n + 1)}{(n + 1)} \right) \sum_{k=1}^{n} \frac{1}{(2k - 1)^2} + \frac{n}{(2n + 1)(n + 1)}
\]
\[
= \frac{4}{\pi^2(n + 1)} \sum_{k=1}^{n} \frac{1}{(2k - 1)^2} + \frac{n}{(2n + 1)(n + 1)}
\]
\[
\leq \frac{4}{\pi^2(n + 1)} \sum_{k=1}^{\infty} \frac{1}{(2k - 1)^2} + \frac{n}{(2n + 1)(n + 1)}
\]
\[
= \frac{4}{\pi^2(n + 1)} \sum_{k=1}^{\infty} \frac{1}{(2k - 1)^2} + \frac{n}{(2n + 1)(n + 1)}
\]
\[
\sim O \left( \frac{1}{n} \right) \quad (C.135)
\]
where Bernoulli numbers \[15\] were used in order to compute the infinite sum. This concludes the proof of Lemma 8.14.
C.15 Proof of Lemma 8.15

First simplify the terms in the summation using (8.1), (8.2) and (8.10).

\[
\frac{\beta_{2,k}}{\alpha_k} - \frac{\beta_{2,k}}{\alpha_k} = \frac{\alpha_k\beta_{2,k} - \alpha_k\beta_{2,k}}{\alpha_k\alpha_k}
\]

\[
= \frac{5\pi(2k-1)(-1)^{k+1}}{6(2n+1)^2} \left(1 + O\left(\frac{k^2}{n^2}\right)\right)
\]

It can be noted that (C.136) shifts sign for increasing \( k \). An upper bound of the absolute value is obtained by the following simplification.

\[
\left| \sum_{k=1}^{n} \left( \frac{\beta_{2,k}}{\alpha_k} - \frac{\beta_{2,k}}{\alpha_k} \right) \right| = \left| \sum_{k=1}^{n} \frac{5\pi(2k-1)(-1)^{k+1}}{6(2n+1)^2} \left(1 + O\left(\frac{k^2}{n^2}\right)\right) \right|
\]

\[
\leq \frac{5\pi}{6(2n+1)^2} \left| \sum_{k=1}^{n} (2k-1)(-1)^{k+1} \right|
\]

\[
+ \left| O\left(\frac{1}{n^4}\right) \sum_{k=1}^{n} O(k^3)(-1)^{k+1} \right|
\]

\[
= \frac{5\pi}{6(2n+1)^2} \left| \sum_{k=1}^{n} [(2k-1) - (2k + 1)] \right| + O\left(\frac{1}{n}\right)
\]

\[
= \frac{5\pi}{6(2n+1)^2} \sum_{k=1}^{n} | - 2 | + O\left(\frac{1}{n}\right)
\]

\[
= \frac{5\pi n}{6(2n+1)^2} + O\left(\frac{1}{n}\right)
\]

\[
\sim O\left(\frac{1}{n}\right) \quad \text{(C.137)}
\]

This concludes the proof of Lemma 8.15.
D Proofs of results in Chapter 9

D.1 Proof of Theorem 9.1

Expressions for \( \hat{G}_i^n(s) \), \( i = \{1, 2\} \) are obtained from (3.21), (3.22), (4.18) and (4.19).

\[
\hat{G}_1^n(s) = G_1(s) - G_1^n(s) \\
= \left(RD_1 - RD_1\right) + R \sum_{k=1}^{n} \left( \frac{\beta_{1,k}}{s^{\tau} + \alpha_k} - \frac{\bar{\beta}_{1,k}}{s^{\tau} + \alpha_k} \right) + R \sum_{k=n+1}^{\infty} \frac{\beta_{1,k}}{s^{\tau} + \alpha_k} \\
= \left(-RD_1\right) + R \sum_{k=1}^{n} \frac{s^{\tau}(\beta_{1,k} - \bar{\beta}_{1,k})}{(s^{\tau} + \alpha_k)(s^{\tau} + \alpha_k)} + R \sum_{k=1}^{\infty} \frac{\bar{\alpha}_k\beta_{1,k} - \alpha_k\bar{\beta}_{1,k}}{(s^{\tau} + \alpha_k)(s^{\tau} + \alpha_k)} \\
+ R \sum_{k=n+1}^{\infty} \frac{\beta_{1,k}}{s^{\tau} + \alpha_k}
\]

(D.1)

\[
\hat{G}_2^n(s) = G_2(s) - G_2^n(s) \\
= \left(D_2 - \bar{D}_2\right) + \sum_{k=1}^{n} \left( \frac{\beta_{2,k}}{s^{\tau} + \alpha_k} - \frac{\bar{\beta}_{2,k}}{s^{\tau} + \alpha_k} \right) + \sum_{k=n+1}^{\infty} \frac{\beta_{2,k}}{s^{\tau} + \alpha_k} \\
= \sum_{k=1}^{n} \frac{s^{\tau}(\beta_{2,k} - \bar{\beta}_{2,k})}{(s^{\tau} + \alpha_k)(s^{\tau} + \alpha_k)} + \sum_{k=1}^{\infty} \frac{\bar{\alpha}_k\beta_{2,k} - \alpha_k\bar{\beta}_{2,k}}{(s^{\tau} + \alpha_k)(s^{\tau} + \alpha_k)} \\
+ \sum_{k=n+1}^{\infty} \frac{\beta_{2,k}}{s^{\tau} + \alpha_k}
\]

(D.2)

The quantities \( S_1 - S_7 \) are analyzed separately. Each term is required to converge at least as \( O(1/n) \) when \( n \to \infty \) for arbitrary frequency points \( \omega \).

The expressions (8.7)–(8.10) will be used in the following in order to compute the relations between the errors in the approximated transfer functions and the model order. First consider \( S_1 - S_4 \), the error contributions to \( \hat{G}_1^n(s) \), see (D.1).

\[
S_1 = -RD_1 = -\frac{R}{n+1} \sim O\left(\frac{1}{n}\right)
\]

(D.3)
Further,

\[
S_2 = R \sum_{k=1}^{n} \frac{s \tau (\beta_{1,k} - \bar{\beta}_{1,k})}{(s \tau + \alpha_k)(s \tau + \alpha_k)}
\]

\[
= R \sum_{k=1}^{n} \frac{s \tau (\beta_{1,k} - \bar{\beta}_{1,k})}{(s \tau + \alpha_k)^2}
\]

\[
= R \sum_{k=1}^{n} \frac{s \tau (\beta_{1,k} - \bar{\beta}_{1,k})}{(s \tau + \alpha_k)^2} \left(1 + \frac{\alpha_k - \bar{\alpha}_k}{s \tau + \alpha_k}\right)
\]

\[
\triangleq R \sum_{k=1}^{n} \frac{s \tau (\beta_{1,k} - \bar{\beta}_{1,k})}{(s \tau + \alpha_k)^2} \left(1 + \varepsilon_{\alpha}(n, k, s)\right) \quad (D.4)
\]

Before the investigation of \( S_2 \) continues, the influence of the model order on \( \varepsilon_{\alpha}(n, k, s) \) in (D.4) is determined. If this term behaves as \( O(1) \), then it can be disregarded. Otherwise, the approximation \( s \tau + \alpha_k \approx s \tau + \bar{\alpha}_k \) cannot be used.

The quantity \( \omega_k \), as defined by (8.11), will be used in the computations.

\[
\varepsilon_{\alpha}(n, k, s) = \frac{\alpha_k - \bar{\alpha}_k}{s \tau + \alpha_k}
\]

\[
= \frac{\pi^2(2k-1)^2 - 2(n+1)^2(1 - \cos(\omega_k))}{s \tau + 2(n+1)^2(1 - \cos(\omega_k))}
\]

\[
= \frac{\pi^2(2k-1)^2}{4} - 4(n+1)^2 \sin^2(\omega_k/2)
\]

\[
= \frac{2(n+1)^2 \sin^2(\omega_k/2)}{4(n+1)^2 \sin^2(\omega_k/2)}
\]

\[
= \frac{[(n+1)^2 - (n+\frac{3}{2})] \omega_k^2 - 4(n+1)^2 \sin^2(\omega_k/2)}{s \tau + 4(n+1)^2 \sin^2(\omega_k/2)}
\]

\[
= \frac{(n+1)^2 \omega_k^2 - 4 \sin^2(\omega_k/2) - (n+\frac{3}{2}) \omega_k^2}{s \tau + 4(n+1)^2 \sin^2(\omega_k/2)} \quad (D.5)
\]

An upper bound for \( \varepsilon_{\alpha}(n, k, s) \) is now expressed as (recall \( s = i \omega \) is imaginary)

\[
|\varepsilon_{\alpha}(n, k, s)| \leq \frac{|(n+1)^2[\omega_k^2 - 4 \sin^2(\omega_k/2)] - (n+\frac{3}{2}) \omega_k^2|}{4(n+1)^2 \sin^2(\omega_k/2)}
\]

\[
= \frac{\omega_k^2}{4 \sin^2(\omega_k/2)} - 1 - \frac{(n+\frac{3}{2}) \omega_k^2}{4(n+1)^2 \sin^2(\omega_k/2)}
\]

\[
\leq \frac{\omega_k^2}{4 \sin^2(\omega_k/2)} + 1 + \frac{(n+\frac{3}{2}) \omega_k^2}{4(n+1)^2 \sin^2(\omega_k/2)} \quad (D.6)
\]
Since $0 < \omega_k / 2 < \pi / 2$, the relation $\sin(x) > 2x / \pi$ can be used. This gives

$$|\varepsilon_a(n, k, s)| \leq \frac{\omega_k^2}{4 \left( \frac{\omega_k}{2\pi} \right)^2} + 1 + \frac{(n + \frac{3}{4})\omega_k^2}{4(n + 1)^2 \left( \frac{\omega_k}{2} \right)^2}$$

$$= \frac{\pi^2}{4} + 1 + \frac{\pi^2 (n + \frac{3}{4})}{4(n + 1)^2}$$

$$\sim O(1) \quad (D.7)$$

The result of (D.7) gives that the contribution from $\varepsilon_a(n, k, s)$ is similar (as a function of $k$ and $n$) to the 1 within the last set of parentheses in (D.4). Hence, since $|\varepsilon_a(n, k, s)| \sim O(1)$, then $|1 + \varepsilon_a(n, k, s)| \sim O(1)$. Using (8.7), the term $S_2$ may then be expressed as

$$S_2 = R \sum_{k=1}^{n} \frac{s\tau (\beta_{1,k} - \tilde{\beta}_{1,k})}{(s\tau + \alpha_k)^2} (1 + \varepsilon_a(n, k, s))$$

$$= R \sum_{k=1}^{n} \frac{s\tau \left( -\frac{2}{2n+1} + \frac{\pi^2 (n+1)(2k-1)^2}{(2n+1)^3} \right) + O \left( \frac{k^4}{n^4} \right)}{(s\tau + \frac{\pi^2 (2k-1)^2}{4})^2} (1 + \varepsilon_a(n, k, s))$$

$$= R \sum_{k=1}^{n} \frac{s\tau \left( -\frac{2}{2n+1} + \frac{\pi^2 (n+1)(2k-1)^2}{(2n+1)^3} \right) + O \left( \frac{k^4}{n^4} \right)}{(s\tau + \pi^2 \left( k - \frac{1}{2} \right)^2)^2} (1 + \varepsilon_a(n, k, s))$$

$$= -R \sum_{k=1}^{n} \frac{s\tau}{(s\tau + \pi^2 \left( k - \frac{1}{2} \right)^2)^2} \frac{2}{2n+1} (1 + \varepsilon_a(n, k, s))$$

$$+R \sum_{k=1}^{n} \frac{s\tau \pi^2 (2k-1)^2}{(s\tau + \pi^2 \left( k - \frac{1}{2} \right)^2)^2} \frac{n+1}{(2n+1)^3} (1 + \varepsilon_a(n, k, s)) + O \left( \frac{1}{n^3} \right)$$

$$\sim O \left( \frac{1}{n} \right) \quad (D.8)$$

since $\varepsilon_a(n, k, s)$ is $O(1)$. Regarding $S_3$ below, the same approximation $s\tau + \alpha_k \approx s\tau + \tilde{\alpha}_k$ as was utilized for $S_2$ is used.

$$S_3 = R \sum_{k=1}^{n} \frac{\tilde{\alpha}_k \beta_{1,k} - \alpha_k \tilde{\beta}_{1,k}}{(s\tau + \alpha_k)(s\tau + \tilde{\alpha}_k)}$$

$$= R \sum_{k=1}^{n} \frac{\tilde{\alpha}_k \beta_{1,k} - \alpha_k \tilde{\beta}_{1,k}}{(s\tau + \alpha_k)^2} (s\tau + \alpha_k)$$

$$= R \sum_{k=1}^{n} \frac{\tilde{\alpha}_k \beta_{1,k} - \alpha_k \tilde{\beta}_{1,k}}{(s\tau + \alpha_k)^2} (1 + \varepsilon_a(n, k, s)) \quad (D.9)$$
The quantity \( \varepsilon_\alpha(n, k, s) \) is the same as was used in \( S_2 \). The expression (D.9) can be simplified using (8.9). Then

\[
S_3 = R \sum_{k=1}^{n} \frac{\pi^2(n+1)(2k-1)^2}{(2n+1)^2} \left( \frac{k^4}{n^4} \right)^2 \left( 1 + \varepsilon_\alpha(n, k, s) \right)
\]

\[
= R \sum_{k=1}^{n} \frac{\pi^2(2k-1)^2 n + 1}{(2n+1)^2} \left( 1 + \varepsilon_\alpha(n, k, s) \right) + \sum_{k=1}^{n} O \left( \frac{1}{n^2} \right)
\]

\[
\sim O \left( \frac{1}{n} \right) \tag{D.10}
\]

The last part of (D.1) is analyzed next.

\[
|S_3| = \left| R \sum_{k=n+1}^{\infty} \frac{\beta_{1,k}}{s^2 + \alpha_k} \right|
\]

\[
\leq R \sum_{k=n+1}^{\infty} \left| \frac{\beta_{1,k}}{s^2 + \alpha_k} \right|
\]

\[
\leq R \sum_{k=n+1}^{\infty} \frac{\beta_{1,k}}{\alpha_k}
\]

\[
= \frac{8R}{\pi^2} \sum_{k=n+1}^{\infty} \frac{1}{(2k-1)^2}
\]

\[
\leq \frac{8R}{\pi^2} \int_{n}^{\infty} \frac{1}{(2x-1)^2} dx
\]

\[
= \frac{4R}{\pi^2} \left( \frac{1}{2n-1} \right)
\]

\[
\sim O \left( \frac{1}{n} \right) \tag{D.11}
\]

To obtain the final expression in (D.11), integral estimation was utilized in order to approximate the sum.

To summarize so far, the expressions (D.3)–(D.11) give that

\[
|\hat{G}_1^n(s)| \leq |S_1| + |S_2| + |S_3| + |S_4|
\]

\[
\sim O \left( \frac{1}{n} \right) \tag{D.12}
\]

Next continue the investigation of \( |\hat{G}_2^n(s)| \), see (D.2). The computations are similar to those of \( |\hat{G}_1^n(s)| \), cf especially the use of the quantity \( \varepsilon_\alpha(n, k, s) \).
Consider the sum $S_0$.

\[ S_0 = \sum_{k=1}^{n} \frac{s \tau (\tilde{\beta}_{2,k} - \tilde{\beta}_{2,k})}{(s \tau + \alpha_k)(s \tau + \alpha_k^2)} \]

\[ = \sum_{k=1}^{n} \frac{s \tau (\tilde{\beta}_{2,k} - \tilde{\beta}_{2,k})}{(s \tau + \alpha_k^2)^2} (s \tau + \alpha_k) \]

\[ = \sum_{k=1}^{n} \frac{s \tau (\tilde{\beta}_{2,k} - \tilde{\beta}_{2,k})}{(s \tau + \alpha_k^2)^2} (1 + \varepsilon_a(n, k, s)) \]

\[ = \sum_{k=1}^{n} \frac{s \tau (-1)^{k+1}}{(s \tau + \frac{n^2 (2k-1)^2}{4})^2} \frac{1}{n^2} \right) (1 + \varepsilon_a(n, k, s)) \text{ (D.13)} \]

where (8.8) was utilized. Further,

\[ S_0 = \sum_{k=1}^{n} 16 s \tau (-1)^{k+1} \left[ \frac{- \pi (2k-1)(4n+3)}{(2n+1)^2} + O \left( \frac{k^3}{n^2} \right) \right] (1 + \varepsilon_a(n, k, s)) \]

\[ = - \frac{4n + 3}{(2n + 1)^2} \sum_{k=1}^{n} 16 s \tau (2k-1) \frac{1}{(4s \tau + \pi^2 (2k-1)^2)^2} (-1)^{k+1} (1 + \varepsilon_a(n, k, s)) \]

\[ + \sum_{k=1}^{n} O \left( \frac{1}{n^2 k^2} \right) \]

\[ \sim O \left( \frac{1}{n} \right) \text{ (D.14)} \]

since

\[ \sum_{k=1}^{n} O \left( \frac{1}{k^2} \right) \sim O(1) \]

for $j \geq 2$, and $\varepsilon_a(n, k, s)$ is $O(1)$. The second term in (D.2) is analyzed next.
\[
S_6 = \sum_{k=1}^{n} \frac{\tilde{a}_k \tilde{b}_k - a_k b_k}{(s\tau + a_k)(s\tau + \alpha_k)} \\
= \sum_{k=1}^{n} \frac{\tilde{a}_k \tilde{b}_k - a_k b_k}{(s\tau + \alpha_k)^2} \left(1 + \varepsilon_\alpha(n, k, s)\right) \\
= \sum_{k=1}^{n} \frac{\tilde{a}_k \tilde{b}_k - a_k b_k}{(s\tau + \alpha_k)^2} \left(1 + O\left(\frac{k^2}{n^2}\right)\right) \left(1 + \varepsilon_\alpha(n, k, s)\right) \\
= \frac{5\pi^5(n+1)^2}{24(2n+1)^4} \sum_{k=1}^{n} \frac{(2k-1)^5}{(s\tau + \pi^2(2k-1)^2)^2} \left(1 + O\left(\frac{k^2}{n^2}\right)\right) \left(1 + \varepsilon_\alpha(n, k, s)\right)
\]

where the expression (8.10) was used in the fourth step.

Consider now the expression within the second last set of parentheses in (D.15). The term \(O(k^2/n^2)\) gives a contribution of the same order or smaller to the sum than 1 does, that is, \(O(1)\). Therefore, if it is shown that the sum, excluding the Ordo term, converges at a certain rate, than it is known that the whole expression of (D.15) converges with the same rate. For this reason, and the fact that only the dependence of the model order \(n\) of interest, and not the exact bound, the following approximation of \(S_6\) is allowed.

\[
|S_6| \approx \frac{5\pi^5(n+1)^2}{24(2n+1)^4} \sum_{k=1}^{n} \frac{(2k-1)^5}{(s\tau + \pi^2(2k-1)^2)^2} \left(1 + \varepsilon_\alpha(n, k, s)\right)
\]

The impact of \(\varepsilon_\alpha(n, k, s)\) was studied previously. To continue, observe that (D.16) contains an alternating series. Introduce

\[
p_k \triangleq \frac{(2k-1)^5}{(s\tau + \pi^2(2k-1)^2)^2} \left(1\right)_{k+1}
\]

\[
\text{(D.17)}
\]
and add every two consecutive terms in the sum.

\[
p_k + p_{k+1} = \frac{(-1)^{k+1} \left( \frac{(2k - 1)^5}{\sigma + \pi^2(2k-1)^2} \right)^2 - \left( \frac{(2k + 1)^5}{\sigma + \pi^2(2k+1)^2} \right)^2}{\left( \frac{(2k - 1)^5}{\sigma + \pi^2(2k-1)^2} \right)^2 - \left( \frac{(2k + 1)^5}{\sigma + \pi^2(2k+1)^2} \right)^2} \]

\[
= (-1)^{k+1} \frac{32\pi^4 k^8 + O(k^6)}{\pi^8 k^8 + O(k^6)}
\]

\[
= (-1)^{k+1} \frac{32\pi^4}{\pi^8} + \frac{O \left( \frac{1}{k^2} \right)}{\pi^8 + O \left( \frac{1}{k^2} \right)} \tag{D.18}
\]

Then (D.15)–(D.18) give

\[
|S_6| \approx \frac{5\pi^5 (n+1)^2}{24(2n+1)^4} \sum_{k \text{ odd}}^n (-1)^{k+1} \frac{32\pi^4}{\pi^8} + \frac{O \left( \frac{1}{k^2} \right)}{\pi^8 + O \left( \frac{1}{k^2} \right)}
\]

\[
= \frac{20\pi (n+1)^2}{3(2n+1)^4} \sum_{k \text{ odd}}^n \left( 1 + O \left( \frac{1}{k^2} \right) \right)
\]

\[
\sim O \left( \frac{1}{n} \right) \tag{D.19}
\]

The remaining part of (D.2) is treated next.

\[
S_7 = \sum_{k=n+1}^{\infty} \frac{\beta_{2,k}}{\sigma + \pi k} = \sum_{k=n+1}^{\infty} \frac{\pi(2k-1)(-1)^{k+1}}{\sigma + \pi^2(2k-1)^2}
\]

\[
= 4\pi \sum_{k=n+1}^{\infty} \frac{(2k - 1)(-1)^{k+1}}{4\sigma + \pi^2(2k-1)^2}
\]

\[
\triangleq 4\pi \sum_{k=n+1}^{\infty} q_k \tag{D.20}
\]

where it is noted that the introduced quantity \( \{q_k\}_1^n \) is an alternating series.
This is utilized in the same way as for the computation of \(|S_0|\).

\[
S_7 = 4\pi \sum_{k=0}^{n} \left( q_k + q_{k+1} \right)
\]

\[
\begin{align*}
S_7 &= \frac{2}{\pi} \sum_{k \text{ odd}}^{n+2} \left( q_k + q_{k+1} \right) \\
&= \frac{2}{\pi} \sum_{k \text{ odd}}^{n+2} \left( \frac{k^2 - \frac{1}{4} - \frac{\pi}{\pi^2} k^2 + \frac{\pi^2}{\pi^2}}{2\pi^2} + \frac{1}{10} \right) \\
&= \frac{2}{\pi} \sum_{k \text{ odd}}^{n+2} \frac{1}{k^2} \times O(1) \\
&\sim O\left( \frac{1}{n} \right) \tag{D.22}
\end{align*}
\]

The expressions (D.14), (D.19) and (D.22) finally give that

\[
|\tilde{G}_2(s)| \leq |S_0| + |S_6| + |S_7| \sim O\left( \frac{1}{n} \right) \tag{D.23}
\]

This concludes the proof of Theorem 9.1.
D.2 Proof of Lemma 9.1

It will be shown that

$$\int_0^\infty \left| \frac{\partial g^n_i(t)}{\partial \theta} \right| dt \to 0, \quad i = \{ 1, 2 \} \quad (D.24)$$

when $n \to \infty$. Since $g(t)$ and $\theta$ are $1 \times 2$ and $2 \times 1$ matrices, respectively, (D.24) is of dimension $2 \times 2$. Each component is studied separately.

First the derivatives $\frac{\partial g_i(t)}{\partial \theta}$, $i = \{ 1, 2 \}$, $\theta = \begin{bmatrix} R & \tau \end{bmatrix}^T$ are obtained. Since the model order is assumed to be large, the result of Lemma 4.2 can be applied.

$$\frac{\partial g_1(t)}{\partial R} = \frac{\partial}{\partial R} \frac{R}{\tau} \sum_{k=1}^{\infty} \beta_{1,k} e^{-\frac{a_k}{\tau} t}$$

$$= \frac{1}{\tau} \sum_{k=1}^{\infty} \beta_{1,k} e^{-\frac{a_k}{\tau} t} \quad (D.25)$$

$$\frac{\partial g_1(t)}{\partial \tau} = \frac{\partial}{\partial \tau} \frac{R}{\tau} \sum_{k=1}^{\infty} \beta_{1,k} e^{-\frac{a_k}{\tau} t}$$

$$= \frac{R}{\tau^2} \sum_{k=1}^{\infty} \left( -\beta_{1,k} e^{-\frac{a_k}{\tau} t} + \frac{\alpha_k \beta_{1,k} t}{\tau} e^{-\frac{a_k}{\tau} t} \right) \quad (D.26)$$

$$\frac{\partial g_2(t)}{\partial R} = \frac{\partial}{\partial R} \frac{1}{\tau} \sum_{k=1}^{\infty} \beta_{2,k} e^{-\frac{a_k}{\tau} t} = 0 \quad (D.27)$$

$$\frac{\partial g_2(t)}{\partial \tau} = \frac{\partial}{\partial \tau} \frac{1}{\tau} \sum_{k=1}^{\infty} \beta_{2,k} e^{-\frac{a_k}{\tau} t}$$

$$= \frac{1}{\tau^2} \sum_{k=1}^{\infty} \left( -\beta_{2,k} e^{-\frac{a_k}{\tau} t} + \frac{\alpha_k \beta_{2,k} t}{\tau} e^{-\frac{a_k}{\tau} t} \right) \quad (D.28)$$

The corresponding derivatives are obtained for the approximate model $g^n_i(t)$.

All of these derivatives are not expressed explicitly, because of their resemblance to (D.25)–(D.28). Only the derivative corresponding to (D.25) has an additional term,

$$\frac{\partial g^n_i(t)}{\partial R} = \frac{1}{\tau} \sum_{k=1}^{n} \beta_{1,k} e^{-\frac{a_k}{\tau} t} + \frac{1}{n+1} \delta(t) \quad (D.29)$$

The integrals of the approximation errors are computed next. The parameter values are obtained from Lemma 3.1 and Lemma 4.2. Hence, the model order $n$ is required to be large. The approximation error of (D.25) is the first to be
treated.

\[
\int_0^\infty \left| \frac{\partial \bar{y}_1^m(t)}{\partial R} \right| \, dt = \int_0^\infty \left| \frac{\partial g_1(t)}{\partial R} - \frac{\partial \bar{y}_1^m(t)}{\partial R} \right| \, dt
\]

\[
= \int_0^\infty \left| \frac{1}{\tau} \sum_{k=1}^n \beta_{1,k} e^{-\frac{\alpha_k t}{\tau}} - \frac{1}{\tau} \sum_{k=1}^n \bar{\beta}_{1,k} e^{-\frac{\alpha_k t}{\tau}} - \frac{1}{n+1} \delta(t) \right| \, dt
\]

\[
\leq \int_0^\infty \frac{1}{\tau} \left| \sum_{k=1}^n \left( \beta_{1,k} e^{-\frac{\alpha_k t}{\tau}} - \bar{\beta}_{1,k} e^{-\frac{\alpha_k t}{\tau}} \right) \right| \, dt + \frac{1}{n+1}
\]

(D.30)

From Lemma 8.10 it is known that

\[
\sum_{k=1}^n \left( \beta_{1,k} e^{-\frac{\alpha_k t}{\tau}} - \bar{\beta}_{1,k} e^{-\frac{\alpha_k t}{\tau}} \right) \geq 0
\]

It is also observed that the sequence \( \beta_{1,k} e^{-\frac{\alpha_k t}{\tau}} \geq 0 \) \( \forall \ k, \ t \). Therefore, the absolute value signs in (D.30) may be removed. Hence,

\[
\int_0^\infty \left| \frac{\partial \bar{y}_1^m(t)}{\partial R} \right| \, dt \leq \int_0^\infty \left. \frac{1}{\tau} \sum_{k=1}^n \left( \beta_{1,k} e^{-\frac{\alpha_k t}{\tau}} - \bar{\beta}_{1,k} e^{-\frac{\alpha_k t}{\tau}} \right) \, dt \right|_{S_{11}} + \frac{1}{n+1} \int_0^\infty \left. \sum_{k=n+1}^\infty \beta_{1,k} e^{-\frac{\alpha_k t}{\tau}} \, dt \right|_{S_{12}}
\]

(D.31)

Obviously, the last part of (D.31) is \( O(1/n) \). In order to study the rest of the
expression, each integral in (D.31) is studied separately.

\[
S_{11} = \int_0^\infty \frac{1}{\tau} \sum_{k=1}^n \left( \beta_{1,k} e^{-\frac{\alpha_k t}{\tau}} - \tilde{\beta}_{1,k} e^{-\frac{\tilde{\alpha}_k t}{\tau}} \right) dt
\]

\[
= \frac{1}{\tau} \sum_{k=1}^n \left( \beta_{1,k} e^{-\frac{\alpha_k}{\tau}} - \tilde{\beta}_{1,k} e^{-\frac{\tilde{\alpha}_k}{\tau}} \right) dt
\]

\[
= \sum_{k=1}^n \left( \frac{\beta_{1,k}}{\alpha_k} - \frac{\tilde{\beta}_{1,k}}{\tilde{\alpha}_k} \right)
\]

\[
\sim O \left( \frac{1}{n} \right) \quad \text{(D.32)}
\]

according to Lemma 8.14. The second term in (D.31) is treated next.

\[
S_{12} = \int_0^\infty \frac{1}{\tau} \sum_{k=n+1}^\infty \beta_{1,k} e^{-\frac{\alpha_k t}{\tau}} dt
\]

\[
= \frac{1}{\tau} \sum_{k=n+1}^\infty \beta_{1,k} \int_0^\infty e^{-\frac{\alpha_k t}{\tau}} dt
\]

\[
= \sum_{k=n+1}^\infty \frac{\beta_{1,k}}{\alpha_k}
\]

\[
= \frac{8}{\pi^2} \sum_{k=n+1}^\infty \frac{1}{(2k-1)^2}
\]

\[
\leq \frac{8}{\pi^2} \int_n^\infty \frac{1}{(2x-1)^2} dx
\]

\[
= \frac{4}{\pi^2 (2n-1)}
\]

\[
\sim O \left( \frac{1}{n} \right) \quad \text{(D.33)}
\]

where integral approximation was used in order to determine the sum. Then (D.31), (D.32) and (D.33) give

\[
\int_0^\infty \left| \frac{\partial \tilde{g}^n(t)}{\partial R} \right| dt \leq S_{11} + S_{12} + \frac{1}{n+1}
\]

\[
\sim O \left( \frac{1}{n} \right) \quad \text{(D.34)}
\]
Next the approximation error of (D.26) is considered.

\[
\int_0^\infty \left| \frac{\partial \tilde{g}_1^n(t)}{\partial \tau} \right| dt = \int_0^\infty \left| \frac{\partial g_1(t)}{\partial \tau} - \frac{\partial \tilde{g}_1^n(t)}{\partial \tau} \right| dt \\
= \int_0^\infty \left[ \sum_{k=1}^{\infty} \left( \frac{R \alpha_k \beta_{1,k} t}{\tau^n} e^{-\frac{\alpha_k t}{\tau^n}} - \frac{R \beta_{1,k}}{\tau^n} e^{-\frac{\alpha_k t}{\tau^n}} \right) e^{-\frac{\alpha_k t}{\tau^n}} \right] dt \\
- \sum_{k=1}^{n} \left( \frac{R \alpha_k \beta_{1,k} t}{\tau^n} e^{-\frac{\alpha_k t}{\tau^n}} - \frac{R \beta_{1,k}}{\tau^n} e^{-\frac{\alpha_k t}{\tau^n}} \right) dt \\
= \int_0^\infty \left[ -\frac{R}{\tau^n} \sum_{k=1}^{n} \left( \beta_{1,k} e^{-\frac{\alpha_k t}{\tau^n}} - \beta_{1,k} e^{-\frac{\alpha_k t}{\tau^n}} \right) \right. \\
+ \frac{R}{\tau^n} \sum_{k=1}^{n} \left( \alpha_k \beta_{1,k} t e^{-\frac{\alpha_k t}{\tau^n}} - \alpha_k \beta_{1,k} t e^{-\frac{\alpha_k t}{\tau^n}} \right) \\
- \frac{R}{\tau^n} \sum_{k=n+1}^{\infty} \beta_{1,k} e^{-\frac{\alpha_k t}{\tau^n}} \\
+ \frac{R}{\tau^n} \sum_{k=n+1}^{\infty} \alpha_k \beta_{1,k} t e^{-\frac{\alpha_k t}{\tau^n}} \right] dt \tag{D.35}
\]

An upper bound of (D.35) is computed.

\[
\int_0^\infty \left| \frac{\partial \tilde{g}_1^n(t)}{\partial \tau} \right| dt \leq \int_0^\infty \left[ \frac{R}{\tau^n} \sum_{k=1}^{n} \left( \beta_{1,k} e^{-\frac{\alpha_k t}{\tau^n}} - \beta_{1,k} e^{-\frac{\alpha_k t}{\tau^n}} \right) \right. \\
+ \frac{R}{\tau^n} \sum_{k=1}^{n} \left( \alpha_k \beta_{1,k} t e^{-\frac{\alpha_k t}{\tau^n}} - \alpha_k \beta_{1,k} t e^{-\frac{\alpha_k t}{\tau^n}} \right) \\
- \frac{R}{\tau^n} \sum_{k=n+1}^{\infty} \beta_{1,k} e^{-\frac{\alpha_k t}{\tau^n}} \\
+ \frac{R}{\tau^n} \sum_{k=n+1}^{\infty} \alpha_k \beta_{1,k} t e^{-\frac{\alpha_k t}{\tau^n}} \right] dt \tag{D.36}
\]
The parameters $\beta_{1,k}$, $\alpha_k$ and $t$ are non-negative, resulting in

\[
\int_0^\infty \left| \frac{\partial \gamma(t)}{\partial \tau} \right| dt \leq \underbrace{\frac{R}{\tau^2} \int_0^\infty \sum_{k=1}^n \left( \beta_{1,k} e^{-\frac{a_k \tau}{\tau}} - \beta_{1,k} e^{-\frac{a_k t}{\tau}} \right) dt}_{S_{21}}
\]

\[
+ \frac{R}{\tau^3} \sum_{k=1}^n \int_0^\infty \left( \alpha_k \beta_{1,k} e^{-\frac{a_k \tau}{\tau}} - \alpha_k \tilde{\beta}_{1,k} e^{-\frac{a_k t}{\tau}} \right) dt
\]

\[
+ \frac{R}{\tau^4} \sum_{k=1}^n \int_0^\infty \alpha_k \beta_{1,k} e^{-\frac{a_k \tau}{\tau}} dt
\]

\[
+ \frac{R}{\tau^5} \sum_{k=1}^n \int_0^\infty \alpha_k \beta_{1,k} e^{-\frac{a_k t}{\tau}} dt
\]

(D.37)

Each of the four parts of (D.37) is evaluated next. It is still assumed that $n$ is large. First note that $S_{21} \sim O(1/n)$, cf the result of Lemma 8.10 and (D.32). Recall the definition of $f_{k,n}(t)$ in Lemma 8.1,

\[
f_{k,n}(t) = \alpha_k \beta_{1,k} e^{-\frac{a_k t}{\tau}} - \alpha_k \tilde{\beta}_{1,k} e^{-\frac{a_k t}{\tau}}
\]

(D.38)

This function was shown to change sign for certain time indices, $t_{k,n}$. The function $f_{k,n}(t)$ is plotted for some indices $k$ in Figure D.1. Observe that the axes are scaled differently. It can be noted that the function (D.38) consists of a positive part as well as a negative part for most indices $k$, cf Figure D.1. For small indices $k$, $f_{k,n}(t)$ is negative for small $t$ and positive for larger $t$. For increasing values of $k$, the curve moves to the left in the graph, and for larger $k$, $f_{k,n}(t)$ is positive for small $t$ and negative for larger $t$. It, however, always holds that the positive part of the curve exceeds the negative part in size. The graphs look coarse for large $k$ since only discrete points of $f_{k,n}(t)$ are plotted.

Using Lemma 8.1, $S_{22}$ can be expressed in a different form by dividing the
Figure D.1: The function $f_{k,n}$ as given in (D.38). It is shown to change sign depending on the value of $k$. Note that for some indices $k$, e.g., $k = 8$ shown here, there is no change of signs since $t_{k,n} \leq 0$ for this particular $k$. In this example, the model order was chosen to equal 100.

The integral into two parts. Then the following expression is obtained,

\[
S_{22} = \frac{R}{\tau^3} \int_{0}^{\infty} \left| \sum_{k=1}^{n} f_{k,n}(t) \right| dt
\]

\[
\leq \frac{R}{\tau^3} \int_{0}^{\infty} \sum_{k=1}^{n} |f_{k,n}(t)| dt
\]

\[
= \frac{R}{\tau^3} \sum_{k=1}^{n} \int_{0}^{\infty} |f_{k,n}(t)| dt
\]

\[
= \frac{R}{\tau^3} \sum_{k=1}^{n} \left( \int_{0}^{t_{k,n}} f_{k,n}(t) dt \right. + \left. \int_{t_{k,n}}^{\infty} |f_{k,n}(t)| dt \right)
\]

\[
= \frac{R}{\tau^3} \sum_{k=1}^{n} \left( \int_{0}^{t_{k,n}} f_{k,n}(t) dt - \int_{t_{k,n}}^{\infty} f_{k,n}(t) dt \right) \quad (D.39)
\]

where the integrands of (D.39) have constant signs within their respective interval. It is also worth repeating that the two integrals have opposite signs!
Further, introduce the notation

\[ x_{k,n} \triangleq \int_0^{t_{k,n}} f_{k,n}(t) dt \quad (D.40) \]

\[ z_{k,n} \triangleq \int_{t_{k,n}}^\infty f_{k,n}(t) dt \quad (D.41) \]

The integrals in (D.40) and (D.41) are computed next.

\[
x_{k,n} = \int_0^{t_{k,n}} f_{k,n}(t) dt = \int_0^{t_{k,n}} \left( \alpha_k \beta_{1,k} te^{-\frac{\alpha_k t}{\tau}} - \alpha_k \bar{\beta}_{1,k} te^{-\frac{\alpha_k t}{\tau}} \right) dt
\]

\[
= \frac{\tau^2}{\alpha_k} \beta_{1,k} \left( e^{-\frac{\alpha_k t_{k,n}}{\tau}} \left( -\frac{\alpha_k t_{k,n}}{\tau} - 1 \right) + 1 \right)
\]

\[
- \frac{\tau^2}{\alpha_k} \bar{\beta}_{1,k} \left( e^{-\frac{\alpha_k t_{k,n}}{\tau}} \left( -\frac{\alpha_k t_{k,n}}{\tau} - 1 \right) + 1 \right) \quad (D.42)
\]

\[
z_{k,n} = \int_{t_{k,n}}^\infty f_{k,n}(t) dt = \int_{t_{k,n}}^\infty \left( \alpha_k \beta_{1,k} te^{-\frac{\alpha_k t}{\tau}} - \alpha_k \bar{\beta}_{1,k} te^{-\frac{\alpha_k t}{\tau}} \right) dt
\]

\[
= \frac{\tau^2}{\alpha_k} \beta_{1,k} \left( e^{-\frac{\alpha_k t_{k,n}}{\tau}} \left( -\frac{\alpha_k t_{k,n}}{\tau} - 1 \right) + 1 \right)
\]

\[
- \frac{\tau^2}{\alpha_k} \bar{\beta}_{1,k} \left( e^{-\frac{\alpha_k t_{k,n}}{\tau}} \left( -\frac{\alpha_k t_{k,n}}{\tau} - 1 \right) + 1 \right) \quad (D.43)
\]

Inserting (D.42) and (D.43) into (D.39) give

\[
S_{22} \leq \frac{2R}{\tau} \sum_{k=1}^n \beta_{1,k} e^{-\frac{\alpha_k t_{k,n}}{\tau}} - \frac{\bar{\beta}_{1,k} e^{-\frac{\alpha_k t_{k,n}}{\tau}}}{\alpha_k} + \frac{2R}{\tau} \sum_{k=1}^n \beta_{1,k} e^{-\frac{\alpha_k t_{k,n}}{\tau}} - \frac{\bar{\beta}_{1,k} e^{-\frac{\alpha_k t_{k,n}}{\tau}}}{\alpha_k} + \frac{R}{\tau} \sum_{k=1}^n \frac{\beta_{1,k} - \bar{\beta}_{1,k}}{\alpha_k} \quad (D.44)
\]
The three parts of (D.44) are studied next. For convenience, the third part, \( S_{22c} \), is considered first. Using Lemma 8.13 and 8.14, it is easily given that

\[
S_{22c} \sim O\left( \frac{1}{n} \right)
\]  

(D.45)

This result will be used in the following to investigate the behaviour of \( S_{22a} \) and \( S_{22b} \).

When comparing \( S_{22b} \) and \( S_{22c} \), it is noted that the difference is an exponential term and its approximated counterpart. To see what impact these terms have on the sum, they are approximated for small respectively large indices \( k \). The expression for \( t_{k,n} \) is found in Lemma 8.1, (8.16). Lemma 8.2 gives the approximate expressions of the exponential terms for small respective large indices \( k \). The values of \( \alpha_k t_{k,n} / \tau \) and \( \tilde{\alpha}_k t_{k,n} / \tau \) vary, of course, with \( k \) even for medium-sized indices. The important aspect, however, is to realize that for small \( k \), the quantities (8.17) and (8.18) are approximately equal. For larger \( k \), both are rather large. Hence, the contribution to the sum \( S_{22b} \) decreases for increasing \( k \), since \( e^{-\alpha_k t_{k,n} / \tau} \) decreases rapidly when \( \alpha_k t_{k,n} / \tau \) increases. Let \( k' \) be an integer close to \( \sqrt{n} \). For indices \( k \in [1, k' - 1] \), \( k \) is considered small, and when \( k \in [k', n] \), \( k \) is considered large. The sum in \( S_{22b} \) can be separated into two parts,

\[
S_{22b} = \sum_{k=1}^{n} \frac{\beta_1 k e^{-\alpha_k t_{k,n} / \tau}}{\alpha_k} - \frac{\tilde{\beta}_1 k e^{-\tilde{\alpha}_k t_{k,n} / \tau}}{\tilde{\alpha}_k} + \sum_{k=k'}^{n} \frac{\beta_1 k e^{-\alpha_k t_{k,n} / \tau}}{\alpha_k} - \frac{\tilde{\beta}_1 k e^{-\tilde{\alpha}_k t_{k,n} / \tau}}{\tilde{\alpha}_k}
\]  

(D.46)

Lemma 8.2 may be used in order to make some simplifications of the expressions. In the first sum, where \( k \) is regarded small, the exponentials are approximately equal. In the second sum, \( \tilde{\alpha}_k t_{k,n} / \tau \) can be rewritten as a function of \( \alpha_k t_{k,n} / \tau \), cf (8.18).

\[
S_{22b} \approx \sum_{k=1}^{k' - 1} \frac{\beta_1 k e^{-\alpha_k t_{k,n} / \tau}}{\alpha_k} - \frac{\tilde{\beta}_1 k e^{-\tilde{\alpha}_k t_{k,n} / \tau}}{\tilde{\alpha}_k} + \sum_{k=k'}^{n} \frac{\beta_1 k e^{-\alpha_k t_{k,n} / \tau}}{\alpha_k} - \frac{\tilde{\beta}_1 k e^{-\tilde{\alpha}_k t_{k,n} / \tau}}{\tilde{\alpha}_k}
\]  

(D.47)

In the first sum of (D.47), the exponentials are approximately constant, cf (8.17). Further, the second sum is small compared to the first sum since the exponentials
are small for large \( k \) (cf Lemma 8.2), and can thus be neglected. This discussion results in, using (D.45), that

\[
S_{22a} \sim O \left( \frac{1}{n} \right) \quad (D.48)
\]

Next continue with the term \( S_{22a} \) in (D.44). From Lemma 8.2 it is obtained that the following holds

\[
t_{k,n} \approx \frac{3\tau}{2\alpha_k}, \quad k \text{ small} \quad (D.49)
\]

\[
t_{k,n} \approx \frac{2\pi^2 \tau}{(\pi^2 - 4)\alpha_k} \ln \left( \frac{n}{n - k + 1} \right), \quad k \text{ large} \quad (k \approx n) \quad (D.50)
\]

First \( S_{22a} \) is separated into two parts, \( k \in [1, k' - 1] \) and \( k \in [k', n] \) where \( k' \) still is an integer close to \( \sqrt{n} \). In a second step the approximate expressions for \( t_{k,n} \), (D.49), (D.50), are inserted in \( S_{22a} \), where it is noted that \( t_{k,n} \sim 1/\alpha_k \). This gives the following result,

\[
S_{22a} = \sum_{k=1}^{n} \left| \beta_{1,k} t_{k,n} e^{\frac{a_{k,n}}{2\alpha_k}} - \beta_{1,k} t_{k,n} e^{\frac{a_{k,n}}{2\alpha_k}} \right|
\]

\[
= \sum_{k=1}^{k'-1} \left| \beta_{1,k} t_{k,n} e^{\frac{a_{k,n}}{2\alpha_k}} - \beta_{1,k} t_{k,n} e^{\frac{a_{k,n}}{2\alpha_k}} \right|
\]

\[
+ \sum_{k=k'}^{n} \left| \beta_{1,k} t_{k,n} e^{\frac{a_{k,n}}{2\alpha_k}} - \beta_{1,k} t_{k,n} e^{\frac{a_{k,n}}{2\alpha_k}} \right|
\]

\[
\approx \frac{3\tau}{2} \sum_{k=1}^{k'-1} \left| \frac{\beta_{1,k} e^{\frac{a_{k,n}}{2\alpha_k}}}{\alpha_k} - \frac{\beta_{1,k} e^{\frac{a_{k,n}}{2\alpha_k}}}{\alpha_k} \right|
\]

\[
+ \frac{2\pi^2 \tau}{\pi^2 - 4} \ln \left( \frac{n}{n - k + 1} \right)
\]

\[
\times \sum_{k=k'}^{n} \left| \frac{\beta_{1,k} e^{\frac{a_{k,n}}{2\alpha_k}}}{\alpha_k} - \frac{\beta_{1,k} e^{\frac{a_{k,n}}{2\alpha_k}}}{\alpha_k} \right| \quad (D.51)
\]

The second sum is negligible compared to the first sum since \( \alpha_k t_{k,n}/\tau \sim \frac{2\pi^2}{(\pi^2 - 4)} \ln(n/(n - k + 1)) \) is large for large \( k \) and \( \beta_{1,k}/\alpha_k \sim O(1/k^2) \). The first sum is \( O(1/n) \) since the exponentials are approximately constant for small indices \( k \), cf Lemma 8.2. Hence,

\[
\frac{3\tau}{2} \sum_{k=1}^{k'-1} \left| \frac{\beta_{1,k} e^{\frac{a_{k,n}}{2\alpha_k}}}{\alpha_k} - \frac{\beta_{1,k} e^{\frac{a_{k,n}}{2\alpha_k}}}{\alpha_k} \right| \approx \frac{3\tau}{2} e^{-\frac{\pi^2}{2}} \sum_{k=1}^{k'-1} \left| \frac{\beta_{1,k}}{\alpha_k} - \frac{\beta_{1,k}}{\alpha_k} \right|
\]

\[
\approx O \left( \frac{1}{n} \right) \quad (D.52)
\]
Summarizing, (D.44)–(D.52) give that

\[ S_{22} = \frac{R}{\tau^2} \int_0^\infty \left[ \sum_{k=1}^n \left( \alpha_k \beta_{1,k} e^{-\frac{\alpha_k t}{\tau}} - \alpha_k \beta_{1,1} e^{-\frac{\alpha_1 t}{\tau}} \right) \right] dt = O\left( \frac{1}{n} \right) \quad (D.53) \]

So far, the first two terms in (D.37) have been treated. The term \( S_{23} \) in (D.37) is next shown to be \( O(1/n) \).

\[
S_{23} = \frac{R}{\tau^2} \sum_{k=n+1}^{\infty} \int_0^\infty \beta_{1,k} e^{-\frac{\beta_{1,k} t}{\tau}} dt
= \frac{R}{\tau} \sum_{k=n+1}^{\infty} \frac{\beta_{1,k}}{\alpha_k}
= \frac{8R}{\pi^2 \tau} \sum_{k=n+1}^{\infty} \frac{1}{(2k-1)^2}
\leq \frac{8R}{\pi^2 \tau} \int_{k=n}^{\infty} \frac{1}{(2x-1)^2} dx
= \frac{4R}{\pi^2 (2n-1)}
\sim O\left( \frac{1}{n} \right)
\quad (D.54)
\]

where integral approximation was used in order to determine the sum. Finally, the last term in (D.37) is evaluated. The result (D.54) is reused in the following, since the expression in (D.55) below becomes similar to (D.54).

\[
S_{24} = \frac{R}{\tau^3} \sum_{k=n+1}^{\infty} \int_0^\infty \alpha_k \beta_{1,k} e^{-\frac{\alpha_k t}{\tau}} dt
= \frac{R}{\tau} \sum_{k=n+1}^{\infty} \frac{\beta_{1,k}}{\alpha_k}
\leq \frac{4R}{\pi^2 (2n-1)}
\sim O\left( \frac{1}{n} \right)
\quad (D.55)
\]

The approximations with upper bounds made in (D.54) and (D.55) contribute
only with small errors when \( n \) is large. At last (D.37) is determined to be

\[
\int_{0}^{\infty} \left| \frac{\partial \tilde{g}_2(t)}{\partial \tau} \right| dt \leq S_{21} + S_{22} + S_{23} + S_{24} \sim O \left( \frac{1}{n} \right) \quad (D.56)
\]

So far two of four terms in (D.24) have proven to approach zero as \( O(1/n) \). Next continue with the third term which is shown to be simpler to determine. Equation (D.27) gives that the result of \( \int_{0}^{\infty} \frac{\partial \tilde{g}_2 \tau \psi(t)}{\partial \tau} dt \) is zero since the integrand is zero.

\[
\int_{0}^{\infty} \left| \frac{\partial \tilde{g}_2(t)}{\partial R} \right| dt = \int_{0}^{\infty} \left| \frac{\partial g_2(t)}{\partial R} - \frac{\partial \tilde{g}_2(t)}{\partial R} \right| dt = 0 \quad (D.57)
\]

The last term

\[
\int_{0}^{\infty} \left| \frac{\partial \tilde{g}_2(t)}{\partial \tau} \right| dt
\]

is treated in a similar way as (D.35) was. The main difference here is that the series \( \tilde{g}_2(t) \) is alternating for increasing indices \( k \).

\[
\int_{0}^{\infty} \left| \frac{\partial \tilde{g}_2(t)}{\partial \tau} \right| dt = \int_{0}^{\infty} \left| \frac{\partial g_2(t)}{\partial \tau} dt - \frac{\partial \tilde{g}_2(t)}{\partial \tau} dt \right|
\]

\[
= \int_{0}^{\infty} \left[ \frac{1}{2} \sum_{k=1}^{\infty} \left( \frac{-\beta_{2,k}e^{-\frac{a_{2,k}}{\tau}} + \frac{\alpha_{2,k}\beta_{2,k}}{\tau}e^{-\frac{a_{2,k}}{\tau}}}{\tau} \right) \right]
\]

\[
- \frac{1}{2} \sum_{k=1}^{\infty} \left( \frac{-\beta_{2,k}e^{-\frac{a_{2,k}}{\tau}} + \frac{\alpha_{2,k}\beta_{2,k}}{\tau}e^{-\frac{a_{2,k}}{\tau}}}{\tau} \right) dt \quad (D.58)
\]
An upper bound is determined next.

\[
\int_0^\infty \left| \frac{\partial \hat{p}_t^\ast (t)}{\partial \tau} \right| \, dt \leq \int_0^\infty \left[ \frac{1}{\tau^2} \sum_{k=1}^n \left( -\beta_{2,k} e^{-\frac{\alpha_k t}{\tau}} + \hat{\beta}_{2,k} e^{-\frac{\alpha_k t}{\tau}} \right) \right] \, dt \\
+ \int_0^\infty \left[ \frac{1}{\tau^3} \sum_{k=1}^n \left( \alpha_k \beta_{2,k} e^{-\frac{\alpha_k t}{\tau}} - \alpha_k \hat{\beta}_{2,k} e^{-\frac{\alpha_k t}{\tau}} \right) \right] \, dt \\
+ \int_0^\infty \left[ \frac{1}{\tau^2} \sum_{k=n+1}^\infty \left( -\beta_{2,k} e^{-\frac{\alpha_k t}{\tau}} \right) \right] \, dt \\
+ \int_0^\infty \left[ \frac{1}{\tau^3} \sum_{k=n+1}^\infty \left( \alpha_k \beta_{2,k} e^{-\frac{\alpha_k t}{\tau}} \right) \right] \, dt 
\] (D.59)

The approach to determine how the model order influences \( S_{41} \) in (D.59) is as follows. An expression on the indices \( k \) for which the term \( |\beta_{2,k} e^{-\alpha_k t/\tau}| \) is dominating over \( |\hat{\beta}_{2,k} e^{-\alpha_k t/\tau}| \) is sought,

\[
|\beta_{2,k} e^{-\alpha_k t/\tau}| \gg |\hat{\beta}_{2,k} e^{-\alpha_k t/\tau}| \quad \text{(D.60)}
\]

Let the index corresponding to the smallest \( k \) fulfilling this requirement be denoted by \( k_0 \). The two intervals \( k \in [1, k_0 - 1] \) and \( k \in [k_0, n] \) are then treated separately.

The time interval (0 to infinity) is also divided into two intervals, one regarding small times and one regarding large. This separation is done since different methods to show the behaviour is needed for large respectively small \( t \). Introduce \( t_c \) such that

\[
\tau - \varepsilon < t_c < \tau 
\] (D.61)

where \( \varepsilon \) is a small value. Hence, the time index \( t_c \) is close to \( \tau \). The following
subintegrals and sums are then to be studied,

\[
S_{41} = \int_0^\infty \frac{1}{t^2} \sum_{k=1}^n \left( -\beta_{2,k}e^{-\frac{\omega_k^2}{t}} + \bar{\beta}_{2,k}e^{-\frac{\omega_k^2}{t}} \right) dt
\]

\[
= \frac{1}{t^2} \int_0^t \left[ \sum_{k=1}^n \left( -\beta_{2,k}e^{-\frac{\omega_k^2}{t}} + \bar{\beta}_{2,k}e^{-\frac{\omega_k^2}{t}} \right) \right] dt
\]

\[
+ \frac{1}{t^2} \int_t^\infty \left[ \sum_{k=1}^{k_{n-1}} \left( -\beta_{2,k}e^{-\frac{\omega_k^2}{t}} + \bar{\beta}_{2,k}e^{-\frac{\omega_k^2}{t}} \right) \right] dt
\]

\[
+ \sum_{k=k_{n-1}}^n \left( -\beta_{2,k}e^{-\frac{\omega_k^2}{t}} + \bar{\beta}_{2,k}e^{-\frac{\omega_k^2}{t}} \right) dt
\]

\[
< \frac{1}{t^2} \int_0^t \left[ \sum_{k=1}^n \left( \beta_{2,k}e^{-\frac{\omega_k^2}{t}} - \bar{\beta}_{2,k}e^{-\frac{\omega_k^2}{t}} \right) \right] dt
\]

\[
+ \frac{1}{t^2} \int_t^\infty \left[ \sum_{k=1}^{k_{n-1}} \left( \beta_{2,k}e^{-\frac{\omega_k^2}{t}} - \bar{\beta}_{2,k}e^{-\frac{\omega_k^2}{t}} \right) \right] dt
\]

\[
+ \frac{1}{t^2} \int_t^\infty \sum_{k=k_{n-1}}^n \left( \beta_{2,k}e^{-\frac{\omega_k^2}{t}} - \bar{\beta}_{2,k}e^{-\frac{\omega_k^2}{t}} \right) dt
\]  \hspace{1cm} (D.62)

Now, an expression for the indices $k$ is sought, for which (D.60) holds. It is assumed that $k_n/n \ll 1$. The definition for $\omega_k$ is found in (8.11).

\[
\frac{\beta_{2,k}e^{-\frac{\omega_k^2}{t}}}{\bar{\beta}_{2,k}e^{-\frac{\omega_k^2}{t}}} = \frac{\pi(2n+1)(2k-1)e^{-\frac{\pi^2 \omega_k^2}{4}}}{4(n+1)^2 \sin(\omega_k/2) \cos(\omega_k/2) e^{-\frac{\pi^2 \omega_k^2}{4}}}
\]

\[
\approx \frac{\pi(2n+1)(2k-1)e^{-\frac{\pi^2 \omega_k^2}{4}}}{8(n+1)^2 \sin(\omega_k/2) \cos^2(\omega_k/2) e^{-\frac{\pi^2 \omega_k^2}{4}}} \]  \hspace{1cm} (D.63)
Further approximations give

\[
\frac{\beta_{2,k} e^{-\alpha_{k}/\tau}}{\beta_{2,k} e^{-\alpha_{k}/\tau}} \approx \frac{\pi(2n+1)(2k-1)e^{4\pi^2(2k-1)^2/(4(2n+1)^2)}}{8(n+1)^2 \pi(2k-1)^2 \left(1 - \frac{\pi^2(2k-1)^2}{4(2n+1)^2}\right)^2}\]

\[
\approx \frac{(2n+1)^2}{4(n+1)^2} \left(1 - \frac{\pi^2(2k-1)^2}{4(2n+1)^2}\right)^2 e^{\frac{\pi^2(2k-1)^2}{4(2n+1)^2}}
\]  

(D.64)

where the first approximation originates from the use of the parameter expressions in Lemma 4.2. The second approximation is due to the substitution of the sine function by its arguments, \(\sin(x) \approx x\). The error that is introduced is small since \(k_n/n \ll 1\). Also note the use of the well-known trigonometric rule \(\sin^2(x) + \cos^2(x) = 1\).

For large model orders and small indices \(k\), the exponential term is a good approximation of (D.64). This approximation is done for simplicity. The term \(\beta_{2,k} e^{-\alpha_{k}/\tau}\) is assumed to be substantially larger than its approximated counterpart if it is at least twice as large. Then the following expression can be used to find \(k_n\), cf (D.64),

\[
e^{-\frac{\pi^2(2k-1)^2(4n+3)}{4(2n+1)^2\tau}} > 2
\]  

(D.65)

which gives

\[
\frac{\pi^2(2k-1)^2(4n+3)t}{4(2n+1)^2\tau} > \ln(2)
\]  

(D.66)

This leads to an expression for the indices \(k\) that were sought,

\[
k > \frac{\sqrt{n+1}}{\pi} + \frac{1}{16n+12} \sqrt{\frac{\tau}{t} + \frac{1}{2}}
\]

\[
\approx \frac{\sqrt{n+1}}{\pi} \sqrt{\frac{\tau}{t} + \frac{1}{2}}
\]  

(D.67)

To study \(S_{11}\), start with large times, \(t \geq t_c\). Then the auxiliary variable mentioned on page 170 can be introduced as

\[
k_n = \frac{\sqrt{\ln(2)} \sqrt{n+1}}{\sqrt{\tau} + \frac{1}{2}}
\]  

(D.68)

This leads to that \(k > k_n\) gives (D.66). The index \(k_n\) should be the smallest integer value of \(k\) fulfilling (D.66). But, since the exact value of (D.68) is unknown for general \(n\) and \(t_c\), the closest integer value is not known unless \(n\) and
$t_c$ are specified. Therefore, (D.68) will be used in the calculations to come, even though $k_n$ should be an integer. It can be noted that $k_n$ is small as compared to $n$ since $t_c$ is close to $\tau$, fulfilling the assumption $k_n/n \ll 1$. For $k > k_n$,

$$\left| \tilde{\beta}_{2,k}e^{-\frac{\alpha k}{T}} \right| > 2 \left| \frac{\beta_{2,k}e^{-\alpha k / T}}{\tau} \right| $$

(D.69)

holds. The sums $S_{41b}$ and $S_{41c}$ respectively, are studied, cf (D.62).

$$S_{41c} = \int_{t_c}^{\infty} \left| \sum_{k=k_n}^{n} \left( \beta_{2,k}e^{-\frac{\alpha k}{T}} - \tilde{\beta}_{2,k}e^{-\frac{\alpha k}{T}} \right) \right| dt$$

$$\leq \int_{t_c}^{\infty} \sum_{k=k_n}^{n} \left| \beta_{2,k}e^{-\frac{\alpha k}{T}} - \tilde{\beta}_{2,k}e^{-\frac{\alpha k}{T}} \right| dt$$

$$\leq 2 \int_{t_c}^{\infty} \sum_{k=k_n}^{n} \left| \beta_{2,k}e^{-\frac{\alpha k}{T}} \right| dt$$

$$\leq 2n \int_{t_c}^{\infty} \left| \beta_{2,k}e^{-\frac{\alpha k}{T}} \right| dt$$

$$\leq \frac{2n|\beta_{2,k}|\tau e^{-\frac{\alpha k \epsilon}{T}}}{\alpha k_n}$$

$$= \frac{8nT}{\pi (2k_n - 1)} e^{-\frac{\pi^2 (2k_n - 1)^2}{4T^2}}$$

(D.70)

where (3.23) and (3.25) were used. Using the expressions (D.68) and (D.70) give

$$S_{41c} \leq \frac{4 \sqrt{\pi T T_c}}{\sqrt{\ln(2)}} \ln(2n)$$

(D.71)

Since $t_c < \tau$ and $n$ is large, the contribution of (D.71) is negligible.

Next, $S_{41b}$ is studied. Hence,

$$S_{41b} = \int_{t_c}^{\infty} \left| \sum_{k=1}^{k-1} \left( \beta_{2,k}e^{-\frac{\alpha k}{T}} - \tilde{\beta}_{2,k}e^{-\frac{\alpha k}{T}} \right) \right| dt$$

(D.72)

Let the terms in the sum be denoted by $r_k$.

$$r_k \triangleq \beta_{2,k}e^{-\frac{\alpha k}{T}} - \tilde{\beta}_{2,k}e^{-\frac{\alpha k}{T}}$$

(D.73)

It was shown that $k_n/n \ll 1$ since $t_c$ is close to $\tau$, see (D.68), which gives that the trigonometric functions in $\beta_{2,k}$ in $r_k$ can be approximated using $\sin(x) \approx x$, ...
introducing small errors only. In the expression below, the argument of the trigonometric functions is $\omega_k/2$, whose definition is found in (8.11).

\[
\begin{align*}
\rho_k &= \pi(2k - 1)(-1)^{k+1} e^{-\frac{\pi^2 |\omega_k - 1|^2}{4\rho_t}} \\
& \quad - \frac{8(n + 1)^2(-1)^{k+1}}{2n + 1} \sin(\omega_k/2) \cos^2(\omega_k/2) e^{-\frac{4(n + 1)^2 |\omega_k - 1|^2}{4(2n + 1)^2}} \\
& \approx \pi(2k - 1)(-1)^{k+1} e^{-\frac{\pi^2 |\omega_k - 1|^2}{4\rho_t}} \\
& \quad - \frac{4(n + 1)^2}{(2n + 1)^2} \left(1 - \frac{\pi^2 (2k - 1)^2}{4(2n + 1)^2} e^{-\frac{4(n + 1)^2 |\omega_k - 1|^2}{4(2n + 1)^2}}\right)
\end{align*}
\] (D.74)

The knowledge that $k \ll n$ can be used. Then further approximations give

\[
\begin{align*}
\rho_k &\approx \pi(2k - 1)(-1)^{k+1} e^{-\frac{\pi^2 |\omega_k - 1|^2}{4\rho_t}} \\
& \quad - \frac{4(n + 1)^2}{(2n + 1)^2} \left(1 - \frac{\pi^2 (2k - 1)^2}{4(2n + 1)^2} e^{-\frac{4(n + 1)^2 |\omega_k - 1|^2}{4(2n + 1)^2}}\right)
\end{align*}
\] (D.75)

Then an upper bound of the absolute value of $\rho_k^{\text{app}}$ is

\[
|\rho_k^{\text{app}}| \leq \pi(2k - 1)e^{-\frac{\pi^2 |\omega_k - 1|^2}{4\rho_t}} \left(1 - \frac{\pi^2 (2k - 1)^2}{4(2n + 1)^2} e^{-\frac{4(n + 1)^2 |\omega_k - 1|^2}{4(2n + 1)^2}}\right)
\] (D.76)
Expressions (D.72), (D.73) (D.75) and (D.76) give

\[
S_{44b} = \int_{t_i}^{t_f} \left| \sum_{k=1}^{k_i} r_k \right| dt \\
\approx \int_{t_i}^{t_f} \left| \sum_{k=1}^{k_i} r_k^{app} \right| dt \\
\leq \int_{t_i}^{t_f} \left| \sum_{k=1}^{k_i} r_k^{app} \right| dt \\
\leq \sum_{k=1}^{k_i} \int_{t_i}^{t_f} \pi(2k-1) \left( e^{-\frac{\pi^2 (2k+1)^2}{4r} t} - e^{-\frac{\pi^2 (2k+1)^2}{4r} \frac{t}{n+1}} \right) dt \\
= \sum_{k=1}^{k_i} \pi(2k-1) \left( \frac{4r}{\pi^2 (2k-1)^2} e^{-\frac{\pi^2 (2k+1)^2}{4r}} \right) \\
\quad - \frac{4r}{\pi^2 (2k-1)^2} \left( 1 + \frac{4n+3}{(2n+1)^2} \right) e^{-\frac{\pi^2 (2k+1)^2}{4r} \frac{t}{n+1}} \\
\triangleq S_{44b}^{I} \tag{D.77}
\]

Using the simple rewriting

\[
\frac{1}{1 + \frac{4n+3}{(2n+1)^2}} = 1 - \frac{n + 3/4}{(n + 1)^2}
\]

the expression in (D.77) can be rewritten as follows.

\[
S_{44b}^{I} = \frac{4r}{\pi} \sum_{k=1}^{k_i} \frac{1}{2k-1} \left( e^{-\frac{\pi^2 (2k+1)^2}{4r}} - \left( 1 - \frac{n + 3/4}{(n + 1)^2} \right) e^{-\frac{\pi^2 (2k+1)^2}{4r} \frac{t}{n+1}} \right) \\
= \frac{4r}{\pi} \sum_{k=1}^{k_i} \frac{1}{2k-1} e^{-\frac{\pi^2 (2k+1)^2}{4r}} \\
\times \left( 1 - e^{-\frac{\pi^2 (2k+1)^2}{4r} \frac{t}{n+1}} + \frac{n + 3/4}{(n + 1)^2} e^{-\frac{\pi^2 (2k+1)^2}{4r} \frac{t}{n+1}} \right) \tag{D.78}
\]
Further,

\[
S_{41b}' \approx \frac{4\tau}{\pi} \sum_{k=1}^{k_{n}-1} \frac{1}{2k-1} e^{-\frac{x^2(2k-1)^2}{4\tau}} (\frac{\pi^2(2k-1)^2(4n+3)\tau}{4\tau(4n+3)^2}) + \frac{(n+3/4)}{\tau(n+1)^2} \left( 1 - \frac{x^2(2k-1)^2(4n+3)\tau}{4\tau(4n+3)^2} \right)
\]

\[
\triangleq S_{41b}''
\]

where series expansion of exponentials were used,

\[
e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots \approx 1 - x
\]

The error introduced by the truncation of the series is small since \(k_n/n\) is small, and \(\tau \ll \tau\). To continue, investigate

\[
\sum_{k=1}^{k_{n}-1} (2k-1)e^{-\frac{x^2(2k-1)^2}{4\tau}}
\]

and determine how it depends on the model order.

\[
\sum_{k=1}^{k_{n}-1} (2k-1)e^{-\frac{x^2(2k-1)^2}{4\tau}} \leq \sum_{k=1}^{\infty} (2k-1)e^{-\frac{x^2(2k-1)^2}{4\tau}}
\]

\[
= \sum_{k=1}^{\infty} (2k-1)e^{-\frac{x^2k^2}{4\tau}} e^{-\frac{x^2}{4\tau}}
\]

\[
= 2e^{-\frac{x^2}{4\tau}} \sum_{k=1}^{\infty} ke^{\frac{x^2k^2}{4\tau}} - e^{-\frac{x^2}{4\tau}} \sum_{k=1}^{\infty} e^{\frac{x^2k^2}{4\tau}}
\]

\[
= \frac{2e^{-\frac{x^2}{4\tau}} e^{-\frac{x^2}{4\tau}}}{(1 - e^{-\frac{x^2}{4\tau}})^2} - \frac{e^{-\frac{x^2}{4\tau}}}{(1 - e^{-\frac{x^2}{4\tau}})^2} + e^{\frac{x^2}{4\tau}}
\]

\[
= \frac{2e^{-\frac{x^2}{4\tau}}}{(1 - e^{-\frac{x^2}{4\tau}})^2} - \frac{e^{-\frac{x^2}{4\tau}}}{(1 - e^{-\frac{x^2}{4\tau}})^2} + e^{\frac{x^2}{4\tau}}
\]

\[
= e^{\frac{x^2}{4\tau}} \left( 1 + 3e^{-\frac{x^2}{4\tau}} \right)
\]

\[
\approx O(1)
\]

(D.79)
where the expressions of the sums can be found in e. g. [15]. The solution (D.81) will be small since $t_c \approx \tau$. The result of (D.81) is then used in (D.79), giving

$$
S_{116}^H = \frac{\pi t_c(4n + 3)}{(2n + 1)^2} \sum_{k=1}^{b_n-1} \frac{1}{(2k-1)} e^{-\frac{\pi (2k-1)^2 \tau}{2}}
+ \frac{4\pi (n + 3/4)}{(n + 1)^2} \sum_{k=1}^{b_n-1} \frac{1}{(2k-1)} e^{-\frac{\pi (2k-1)^2 \tau}{4}}
- \pi t_c(4n + 3) \sum_{k=1}^{b_n-1} \frac{1}{(2k-1)} e^{-\frac{\pi (2k-1)^2 \tau}{2n}}
\leq \pi t_c(4n + 3) \sum_{k=1}^{b_n-1} \frac{1}{(2k-1)} e^{-\frac{\pi (2k-1)^2 \tau}{2n}}
+ \frac{\pi t_c(4n + 3)}{(2n + 1)^2} \sum_{k=1}^{b_n-1} \frac{1}{(2k-1)} e^{-\frac{\pi (2k-1)^2 \tau}{n}}
\leq \frac{\pi t_c(4n + 3)}{(2n + 1)^2} + \frac{\pi t_c(4n + 3)}{(2n + 1)^2} \sum_{k=1}^{b_n-1} \frac{1}{(2k-1)} e^{-\frac{\pi (2k-1)^2 \tau}{2n}}
\sim O \left( \frac{1}{n} \right)
$$

(D.82)

The expressions (D.72)–(D.82) hence give that

$$
S_{116} = \int_{t_c}^{\infty} \left| \sum_{k=1}^{b_n-1} (\beta_{2,k} e^{-\frac{\pi (2k-1)^2 \tau}{2n}} - \beta_{2,k} e^{-\frac{\pi (2k-1)^2 \tau}{2n}}) \right| dt \sim O \left( \frac{1}{n} \right)
$$

(D.83)

So far $S_{416}$ and $S_{416}$ in (D.62) have been investigated. Next $S_{416}$, corresponding to the time interval $t \in [0, t_c)$ where $\tau - \varepsilon < t < \tau$, is studied. It was shown in Lemma 8.11 that

$$
\sum_{k=1}^{n} (\beta_{2,k} e^{-\frac{\pi (2k-1)^2 \tau}{2n}} - \beta_{2,k} e^{-\frac{\pi (2k-1)^2 \tau}{2n}})
$$

(D.84)

changes sign at most two times. This means that (D.84) has a constant sign for $t \approx 0$. (Negative sign if $n$ is even, and a positive sign if $n$ is odd). Let the integral $S_{416}$ in (D.62) be divided into three parts, all of which have constant sign. It does not necessary need to be exactly three parts, but can also be one or two, depending on how many times (D.84) changes sign. The number of changes
does not need to be known in the analysis. Hence,

\[
S_{11e} = \int_0^{t_a} \left| \sum_{k=1}^{n} \left( \beta_{2,k} e^{-\frac{\alpha_k}{\tau}} - \beta_{2,k} e^{-\frac{\alpha_k}{\tau}} \right) \right| dt \\
= \int_0^{t_a} \left| \sum_{k=1}^{n} \left( \beta_{2,k} e^{-\frac{\alpha_k}{\tau}} - \beta_{2,k} e^{-\frac{\alpha_k}{\tau}} \right) \right| dt \\
+ \int_{t_a}^{t_e} \left| \sum_{k=1}^{n} \left( \beta_{2,k} e^{-\frac{\alpha_k}{\tau}} - \beta_{2,k} e^{-\frac{\alpha_k}{\tau}} \right) \right| dt \\
+ \int_{t_e}^{t_i} \left| \sum_{k=1}^{n} \left( \beta_{2,k} e^{-\frac{\alpha_k}{\tau}} - \beta_{2,k} e^{-\frac{\alpha_k}{\tau}} \right) \right| dt \tag{D.85}
\]

Of course, \(0 \leq t_a \leq t_b \leq t_e < \tau\) holds. Study each part in (D.85) separately. Start with \(I_1\). If \(t_a\) is sufficiently small, which can be assumed, then the sum has constant sign for all \(t \in [0, t_a)\).

\[
I_1 \triangleq \int_0^{t_a} \left| \sum_{k=1}^{n} \left( \beta_{2,k} e^{-\frac{\alpha_k}{\tau}} - \beta_{2,k} e^{-\frac{\alpha_k}{\tau}} \right) \right| dt \\
= \int_0^{t_a} \left| \sum_{k=1}^{n} \left( \beta_{2,k} e^{-\frac{\alpha_k}{\tau}} - \beta_{2,k} e^{-\frac{\alpha_k}{\tau}} \right) \right| dt \\
= \tau \left| \sum_{k=1}^{n} \left( \beta_{2,k} e^{-\frac{\alpha_k}{\tau}} - \beta_{2,k} e^{-\frac{\alpha_k}{\tau}} \right) \right| \\
\leq \tau \left( \sum_{k=1}^{n} \left( \beta_{2,k} - \beta_{2,k} \right) \right) \\
+ \tau \left( \sum_{k=1}^{n} \left( \frac{\beta_{2,k}}{\alpha_k} - \frac{\beta_{2,k}}{\alpha_k} \right) \right) \tag{D.86}
\]

Let \(I_1^1\) denote the first sum in (D.86) and let \(I_1^2\) denote the second sum. Then \(I_1^1\) and \(I_1^2\) in (D.86) can be examined separately. The first sum is already treated.
in Lemma 8.15, provided \( n \) is large,

\[
I_1 = \sum_{k=1}^{n} \left( \frac{\beta_{2,k}}{\alpha_k} - \frac{\beta_{2,k}}{\alpha_k} \right) \sim O \left( \frac{1}{n} \right) \tag{D.87}
\]

The sum \( I_1^2 \) is studied next. \( I_1^2 \) can be divided into two separate sums, one for small \( k \) and one for large. Let \( k' \) be an integer close to \( \sqrt{n} \). Then

\[
I_1^2 = \sum_{k=1}^{k'-1} \left( \frac{\beta_{2,k}}{\alpha_k} e^{-\frac{a_t a}{\alpha_k}} - \frac{\beta_{2,k}}{\alpha_k} e^{-\frac{a_t a}{\alpha_k}} \right) + \sum_{k=k'}^{n} \left( \frac{\beta_{2,k}}{\alpha_k} e^{-\frac{a_t a}{\alpha_k}} - \frac{\beta_{2,k}}{\alpha_k} e^{-\frac{a_t a}{\alpha_k}} \right) \tag{D.88}
\]

For small indices \( k \), the exponential terms are approximately equal, since \( \alpha_k \approx \alpha_k \). Then Lemma 8.15 gives that this sum is \( O(1/n) \). Next consider the second sum in (D.88). For larger \( k \), e.g. \( k \in [k', \sqrt{n}] \), the parameters \( \alpha_k \) and \( \alpha_k \) differ in size too much to be considered (approximately) equal. The size of the exponentials also differs depending on \( a_t \). Recall \( 0 < a_t < \tau \). (Note that \( a_t \) is close to one if there only exists one integral in (D.85). If (D.84) changes sign two times, \( a_t \) will be closer to zero). If \( a_t \approx \tau \), the exponentials will be large, since \( k \) is large in the considered case. This implies that the second sum in (D.88) is much smaller than the first sum, and can be neglected. On the other hand, when \( a_t \) is close to zero, the second sum will dominate over the first sum. It will, however, still be \( O(1/n) \), since small \( a_t \) gives small exponents. Then the sum can be compared to Lemma 8.15. This discussion leads to the conclusion that the sum \( I_1^2 \) in (D.86) is \( O(1/n) \). Equations (D.86)–(D.88) and the discussion above then give that

\[
I_1 \sim O \left( \frac{1}{n} \right) \tag{D.89}
\]

Continue with \( I_2 \) in (D.85). Recollect that \( I_1, I_2 \) and \( I_3 \) in (D.85) all have constant signs, and that one or two of them might be zero, if (D.84) changes sign.
less than two times.

\[
I_2 \triangleq \int_{t_x}^{t_y} \left| \sum_{k=1}^{n} \left( \beta_{2,k} e^{-\frac{\alpha_k t}{x}} - \tilde{\beta}_{2,k} e^{-\frac{\alpha_k t}{x}} \right) \right| \, dt
\]

\[
= \int_{t_x}^{t_y} \left| \sum_{k=1}^{n} \left( \beta_{2,k} e^{-\frac{\alpha_k t}{x}} - \tilde{\beta}_{2,k} e^{-\frac{\alpha_k t}{x}} \right) \right| \, dt
\]

\[
= \sum_{k=1}^{n} \left( -\frac{\tau \beta_{2,k}}{\alpha_k} e^{-\frac{\alpha_k t}{x}} + \frac{\tau \tilde{\beta}_{2,k}}{\alpha_k} e^{-\frac{\alpha_k t}{x}} + \frac{\tau \tilde{\beta}_{2,k}}{\alpha_k} e^{-\frac{\alpha_k t}{x}} - \frac{\tau \beta_{2,k}}{\alpha_k} e^{-\frac{\alpha_k t}{x}} \right)
\]

\[
\leq \tau \sum_{k=1}^{n} \left( \frac{\beta_{2,k}}{\alpha_k} e^{-\frac{\alpha_k t}{x}} - \frac{\tilde{\beta}_{2,k}}{\alpha_k} e^{-\frac{\alpha_k t}{x}} \right) + \tau \sum_{k=1}^{n} \left( \frac{\beta_{2,k}}{\alpha_k} e^{-\frac{\alpha_k t}{x}} - \frac{\tilde{\beta}_{2,k}}{\alpha_k} e^{-\frac{\alpha_k t}{x}} \right)
\]

\[
\sim O \left( \frac{1}{n} \right) \quad (D.90)
\]

It can now be noted that the second sum in (D.90) is exactly \( I_2^2 \) in (D.86). Also, the first sum is similar except from the time index. Since (D.86) was shown to be \( O(1/n) \), so is \( I_2 \) in (D.90).

\[
I_2 \sim O \left( \frac{1}{n} \right) \quad (D.91)
\]

The result of the third interval \( I_3 \) is provided in a similar way.

\[
I_3 \triangleq \int_{t_x}^{t_y} \left| \sum_{k=1}^{n} \left( \beta_{2,k} e^{-\frac{\alpha_k t}{x}} - \tilde{\beta}_{2,k} e^{-\frac{\alpha_k t}{x}} \right) \right| \, dt
\]

\[
\leq \tau \sum_{k=1}^{n} \left( \frac{\beta_{2,k}}{\alpha_k} e^{-\frac{\alpha_k t}{x}} - \frac{\tilde{\beta}_{2,k}}{\alpha_k} e^{-\frac{\alpha_k t}{x}} \right) + \tau \sum_{k=1}^{n} \left( \frac{\beta_{2,k}}{\alpha_k} e^{-\frac{\alpha_k t}{x}} - \frac{\tilde{\beta}_{2,k}}{\alpha_k} e^{-\frac{\alpha_k t}{x}} \right)
\]

\[
\sim O \left( \frac{1}{n} \right) \quad (D.92)
\]

Summarizing, from (D.85), (D.89), (D.91) and (D.92),

\[
S_{41a} \sim O \left( \frac{1}{n} \right) \quad (D.93)
\]
is obtained. Using (D.71), (D.83), and (D.93) in (D.62), it is obtained that

$$S_{41} = \frac{1}{\tau} \int_0^\infty \sum_{k=1}^n \left( \beta_{2,k} e^{-\frac{\nu k}{\tau}} - \bar{\beta}_{2,k} e^{-\frac{\nu k}{\tau}} \right) dt \sim O \left( \frac{1}{n} \right)$$  \hspace{1cm} (D.94)

Next, $S_{42}$ in (D.59) is studied. Recall the definition of $f_{k,n}(t)$ in Lemma 8.3, (8.19),

$$f_{k,n}(t) = \alpha_k \beta_{2,k} e^{-\frac{\alpha_k}{\tau}} - \bar{\alpha}_k \bar{\beta}_{2,k} e^{-\frac{\alpha_k}{\tau}}$$  \hspace{1cm} (D.95)

For each specific index $k$, this function changes sign for a certain time index, $I_{k,n}$. Introduce

$$F_n(t) \triangleq \sum_{k=1}^n f_{k,n}(t)$$  \hspace{1cm} (D.96)

To analytically derive $S_{42}$, the expression is rewritten in order to remove the absolute value sign from within the integral. Assume that $F_n(t) = 0$ only in a finite number of points. This assumption holds true, otherwise $S_{42}$ would equal zero. Also, assume that

$$\int_0^\infty F_n(t) dt \neq 0$$

when $n < \infty$. It is, however, allowed that $n$ is large. If the assumptions hold, then the following upper approximation is allowed,

$$\int_0^\infty |F_n(t)| dt \leq C_2 \int_0^\infty F_n(t) dt$$  \hspace{1cm} (D.97)

where $C_2$ is a bounded constant. Next

$$\left| \int_0^\infty F_n(t) dt \right|$$

is derived. It will be shown that (D.98) is larger than zero, thus fulfilling the
assumption.

\[
\left| \int_0^\infty F_n(t) \, dt \right| = \left| \int_0^\infty \sum_{k=1}^n f_{k,n}(t) \, dt \right|
\]

\[
= \left| \int_0^\infty \sum_{k=1}^n \left( \alpha_k \beta_{2,k} e^{-\frac{a_k t}{\tau}} - \frac{\tilde{\alpha}_k \tilde{\beta}_{2,k} t e^{-\frac{a_k t}{\tau}}}{\tau} \right) \, dt \right|
\]

\[
= \left| \sum_{k=1}^n \int_0^\infty \left( \alpha_k \beta_{2,k} e^{-\frac{a_k t}{\tau}} - \frac{\tilde{\alpha}_k \tilde{\beta}_{2,k} t e^{-\frac{a_k t}{\tau}}}{\tau} \right) \, dt \right|
\]

\[
= \tau^2 \left| \sum_{k=1}^n \left( \frac{\beta_{2,k}}{\alpha_k} - \frac{\tilde{\beta}_{2,k}}{\alpha_k} \right) \right| \tag{D.99}
\]

The expression (D.99) was investigated in Lemma 8.15, giving

\[
\left| \int_0^\infty F_n(t) \, dt \right| \sim O \left( \frac{1}{n} \right) \tag{D.100}
\]

As long as \( n \) is finite, (D.100) will be larger than zero. Now an approximation of \( S_{i2} \) in (D.59) can be presented using (D.95), (D.97), (D.99) and (D.100).

\[
S_{i2} = \frac{1}{\tau^2} \int_0^\infty \left| \sum_{k=1}^n \left( \frac{\beta_{2,k}}{\alpha_k} - \frac{\tilde{\beta}_{2,k}}{\alpha_k} \right) \right| \, dt
\]

\[
= \frac{1}{\tau^2} \int_0^\infty \left| \sum_{k=1}^n f_{k,n}(t) \right| \, dt
\]

\[
\leq C_2 \frac{1}{\tau^2} \int_0^\infty \left| \sum_{k=1}^n f_{k,n}(t) \right| \, dt
\]

\[
= C_2 \frac{1}{\tau^2} \left| \sum_{k=1}^n \left( \frac{\beta_{2,k}}{\alpha_k} - \frac{\tilde{\beta}_{2,k}}{\alpha_k} \right) \right|
\]

\[
\sim O \left( \frac{1}{n} \right) \tag{D.101}
\]

Finally, the two last parts of (D.59) are shown to approach zero as \( O(1/n) \).

\[
S_{i3} = \frac{1}{\tau^2} \left| \sum_{k=1}^\infty \left( -\beta_{2,k} e^{-\frac{a_k t}{\tau}} \right) \right| \, dt \tag{D.102}
\]
The fact that the terms in \( \{\beta_{2,k}\}^n \) are alternating will be used, cf (3.25). The terms within the summation are added together in pairs. It is noted that
\[
\beta_{2,k+1} = \pi(2k+1)(-1)^{k+1} \\
= -\pi(2k-1)(-1)^{k+1} - 2\pi(-1)^{k+1} \\
= -\beta_{2,k} - 2\pi(-1)^{k+1}
\]
and
\[
\alpha_{k+1} = \frac{\pi^2(2k+1)^2}{4} \\
= \alpha_k + \pi^2(2k-1) + \pi^2 \\
= \alpha_k + 2\pi^2k
\]
Then
\[
\left|\beta_{2,k}e^{-\frac{\alpha_k}{\tau}} + \beta_{2,k+1}e^{-\frac{\alpha_{k+1}}{\tau}}\right| = \\
\left|\beta_{2,k}e^{-\frac{\alpha_k}{\tau}} - (\beta_{2,k} + 2\pi(-1)^{k+1})e^{-\frac{\alpha_k}{\tau}}\right| \\
\left|\beta_{2,k}e^{-\frac{\alpha_k}{\tau}} - \beta_{2,k}e^{-\frac{\alpha_k}{\tau}}e^{-\frac{2\pi^2k}{\tau}} - 2\pi(-1)^{k+1}e^{-\frac{\alpha_k}{\tau}}e^{-\frac{2\pi^2k}{\tau}}\right| \\
\left|\beta_{2,k}e^{-\frac{\alpha_k}{\tau}}(1 - e^{-\frac{2\pi^2k}{\tau}}) - 2\pi(-1)^{k+1}e^{-\frac{\alpha_k}{\tau}}e^{-\frac{2\pi^2k}{\tau}}\right| \\
\leq \left|\beta_{2,k}\right|e^{-\frac{\alpha_k}{\tau}}(1 - e^{-\frac{2\pi^2k}{\tau}}) + 2\pi e^{-\frac{\alpha_k}{\tau}}e^{-\frac{2\pi^2k}{\tau}}
\]
This approximate expression is inserted into (D.102).
\[
S_{13} \leq \frac{1}{\tau} \int \sum_{k=1}^{\infty} \int \sum_{k=odd}^{\infty} \left[\left|\beta_{2,k}\right|e^{-\frac{\alpha_k}{\tau}}(1 - e^{-\frac{2\pi^2k}{\tau}}) + 2\pi e^{-\frac{\alpha_k}{\tau}}e^{-\frac{2\pi^2k}{\tau}}\right] dt \\
\leq \frac{1}{\tau} \sum_{k=odd}^{\infty} \left[\frac{\tau\left|\beta_{2,k}\right|}{\alpha_k} - \frac{\tau\left|\beta_{2,k}\right|}{\alpha_k + 2\pi^2k} + \frac{2\pi\tau}{\alpha_k + 2\pi^2k}\right] \\
= \frac{2\pi}{\tau} \sum_{k=odd}^{\infty} \left[\frac{1}{\alpha_k + 2\pi^2k} + \frac{\pi k\left|\beta_{2,k}\right|}{\alpha_k(\alpha_k + 2\pi^2k)}\right] \\
\sim O\left(\frac{1}{n}\right)
\]
since \( \alpha_k \sim O(k^2) \) and \( \beta_{2,k} \sim O(k) \). The indices in the sums in (D.106) are said to be odd. It is hence assumed that the model number \( n \) is even. If \( n \) is odd, the indices \( k \) will obviously be even.

Finally consider the remaining part of (D.59),

\[
S_{44} = \int_0^\infty \frac{1}{r^3} \sum_{k=n+1}^{\infty} \alpha_k \beta_{2,k} t e^{-\frac{a_{2,k}}{r}} \, dt \tag{D.107}
\]

Similarly as was performed for \( S_{33} \), the terms in (D.107) are added together in pairs,

\[
\begin{align*}
&\left| \alpha_k \beta_{2,k} e^{-\frac{a_{2,k}}{r}} + \alpha_{k+1} \beta_{2,k+1} e^{-\frac{a_{2,k+1}}{r}} \right| = \\
= &\left| \alpha_k \beta_{2,k} e^{-\frac{a_{2,k}}{r}} + (\alpha_k + 2\pi k) (-\beta_{2,k} - 2\pi(-1)^{k+1}) e^{-\frac{a_{2,k} + 2\pi^2 k}{r}} \right| \\
= &\left| \alpha_k \beta_{2,k} e^{-\frac{a_{2,k}}{r}} \left(1 - e^{-\frac{2\pi^2 k}{r}}\right) \\
& - (2\pi \alpha_k (-1)^{k+1} + 4\pi^3 k (-1)^{k+1} + 2\pi^2 k \beta_{2,k}) e^{-\frac{a_{2,k} + 2\pi^2 k}{r}} \right| \\
\leq &\alpha_k |\beta_{2,k}| e^{-\frac{a_{2,k}}{r}} \left(1 - e^{-\frac{2\pi^2 k}{r}}\right) \\
& + 2\pi (\alpha_k + 2\pi^2 k + \pi k |\beta_{2,k}|) e^{-\frac{a_{2,k} + 2\pi^2 k}{r}} \tag{D.108}
\end{align*}
\]

Inserted into (D.107) the following is obtained,

\[
S_{44} \leq \frac{1}{r^3} \sum_{k=\text{odd}}^{\infty} \int_0^\infty \left[ \alpha_k |\beta_{2,k}| t e^{-\frac{a_{2,k}}{r}} - \alpha_k |\beta_{2,k}| t e^{-\frac{a_{2,k} + 2\pi^2 k}{r}} \right] dt \\
+ 2\pi (\alpha_k + 2\pi^2 k + \pi k |\beta_{2,k}|) t e^{-\frac{a_{2,k} + 2\pi^2 k}{r}} \, dt \tag{D.109}
\]
Solving the integral gives

\[
S_{44} \leq \frac{1}{\tau} \sum_{k=\text{o}dd}^{\infty} \left[ \frac{\beta_{2,k}}{\alpha_k} - \frac{\alpha_k \beta_{2,k}}{(\alpha_k + 2\pi^2 k)^2} + \frac{2\pi (\alpha_k + 2\pi^2 k + \pi k \beta_{2,k})}{(\alpha_k + 2\pi^2 k)^2} \right]
\]

\[
= \frac{1}{\tau} \sum_{k=\text{o}dd}^{\infty} \left[ \frac{4\pi^2 \beta_{2,k}}{\alpha_k (\alpha_k + 2\pi^2 k)^2} \right]
\]

\[
= \frac{1}{\tau} \sum_{k=\text{o}dd}^{\infty} \left[ \frac{4\pi^2 \beta_{2,k}}{\alpha_k (\alpha_k + 2\pi^2 k)^2} \right] + \frac{2\pi}{\alpha_k + 2\pi^2 k} + \frac{2\pi k \beta_{2,k}}{(\alpha_k + 2\pi^2 k)^2}
\]

\[
\leq \sum_{k=\text{o}dd}^{\infty} O \left( \frac{1}{k^2} \right)
\]

\[
\sim O \left( \frac{1}{n} \right)
\]

Expressions of \( \alpha_k \) and \( \beta_{2,k} \) are found in (3.23) and (3.25), giving

\[
S_{44} \leq \frac{1}{\tau} \sum_{k=\text{o}dd}^{\infty} \left[ \frac{4\pi^3 (2k - 1)}{\alpha_k (\alpha_k + 2\pi^2 k)^2} + \frac{2\pi^2 (2k - 1)^2}{\alpha_k (\alpha_k + 2\pi^2 k)^2} \right]
\]

\[
\leq \sum_{k=\text{o}dd}^{\infty} O \left( \frac{1}{k^2} \right)
\]

\[
\sim O \left( \frac{1}{n} \right)
\]

Summarizing, (D.59), (D.94), (D.101), (D.106) and, finally, (D.111) give that

\[
\int_0^\infty \left| \frac{\partial g^2(t)}{\partial t} \right| dt \sim O \left( \frac{1}{n} \right)
\]

As a concluding remark,

\[
\int_0^\infty \left| \frac{\partial g^{2i}(t)}{\partial \theta} \right| dt, \quad i = \{1, 2\}, \quad \theta = [R \quad \tau]^T
\]

was shown to go towards zero as \( O(1/n) \). This concludes the proof of Lemma 9.1.
D.3 Proof of Theorem 9.2

The integral of the approximation error $\tilde{g}_1(t)$ is investigated. The expressions for the true and approximated weighting functions are taken from (3.28) and (4.25).

$$
\int_0^\infty |\tilde{g}_1(t)| \, dt = \int_0^\infty |g_1(t) - \tilde{g}_1^n(t)| \, dt
$$

$$
= \int_0^\infty \left| \frac{R}{\tau} \sum_{k=1}^n \beta_{1,k} e^{-\frac{\alpha_{k,t}}{\tau}} - \frac{R}{\tau} \sum_{k=1}^n \beta_{1,k} e^{-\frac{\alpha_{k,t}}{\tau}} - R\tilde{D}_1 \delta(t) \right| \, dt
$$

$$
= \int_0^\infty \left( \frac{R}{\tau} \sum_{k=1}^n \left( \beta_{1,k} e^{-\frac{\alpha_{k,t}}{\tau}} - \tilde{\beta}_{1,k} e^{-\frac{\alpha_{k,t}}{\tau}} \right) \right) \, dt
$$

$$
+ \int_0^\infty \left( \frac{R}{\tau} \sum_{k=n+1}^\infty \beta_{1,k} e^{-\frac{\alpha_{k,t}}{\tau}} - R\tilde{D}_1 \delta(t) \right) \, dt
$$

$$
\leq \int_0^\infty \left( \left| \frac{R}{\tau} \sum_{k=1}^n \left( \beta_{1,k} e^{-\frac{\alpha_{k,t}}{\tau}} - \tilde{\beta}_{1,k} e^{-\frac{\alpha_{k,t}}{\tau}} \right) \right| \right) \, dt
$$

$$
+ \int_0^\infty \left. \int_0^\infty \left( \frac{R}{\tau} \sum_{k=1}^n \left( \beta_{1,k} e^{-\frac{\alpha_{k,t}}{\tau}} - \tilde{\beta}_{1,k} e^{-\frac{\alpha_{k,t}}{\tau}} \right) \right) \, dt \right| \right| R\tilde{D}_1 \delta(t) \right) \, dt
$$

$$
= \int_0^\infty \left( \frac{R}{\tau} \sum_{k=1}^n \left( \beta_{1,k} e^{-\frac{\alpha_{k,t}}{\tau}} - \tilde{\beta}_{1,k} e^{-\frac{\alpha_{k,t}}{\tau}} \right) \right) \, dt
$$

$$
+ \int_0^\infty \left. \int_0^\infty \left( \frac{R}{\tau} \sum_{k=1}^n \left( \beta_{1,k} e^{-\frac{\alpha_{k,t}}{\tau}} - \tilde{\beta}_{1,k} e^{-\frac{\alpha_{k,t}}{\tau}} \right) \right) \, dt \right| \right| R\tilde{D}_1 \delta(t) \right) \, dt
$$

(D.113)

Utilizing Lemma 8.10 gives

$$
\int_0^\infty |\tilde{g}_1(t)| \, dt \leq \frac{R}{\tau} \sum_{k=1}^n \int_0^\infty \left( \beta_{1,k} e^{-\frac{\alpha_{k,t}}{\tau}} - \tilde{\beta}_{1,k} e^{-\frac{\alpha_{k,t}}{\tau}} \right) \, dt
$$

$$
+ \frac{R}{\tau} \sum_{k=n+1}^\infty \int_0^\infty \left( \beta_{1,k} e^{-\frac{\alpha_{k,t}}{\tau}} - \tilde{\beta}_{1,k} e^{-\frac{\alpha_{k,t}}{\tau}} \right) \, dt
$$

(D.114)

+ R\tilde{D}_1 \delta(t) I_3

where $I_1 - I_3$ are introduced for convenience.
The last part of (D.114), \( I_3 \), is of course known exactly by (4.23). The remaining terms are studied next.

\[
I_1 = \frac{R}{\tau} \sum_{k=1}^{n} \int_0^{\infty} \left( \beta_{1,k} e^{-\frac{a_{1,k}}{\tau}} - \beta_{1,k} e^{-\frac{b_{1,k}}{\tau}} \right) dt
\]

\[
= R \sum_{k=1}^{n} \left( \frac{\beta_{1,k}}{\alpha_k} - \frac{\beta_{1,k}}{\alpha_k} \right)
\]

\[
\sim O \left( \frac{1}{n} \right) \quad \text{(D.115)}
\]

The final result of (D.115) was obtained from Lemma 8.14. Next proceed to the second part in (D.114).

\[
I_2 = \frac{R}{\tau} \sum_{k=n+1}^{\infty} \int_0^{\infty} \beta_{1,k} e^{-\frac{a_{1,k}}{\tau}} dt
\]

\[
= R \sum_{k=n+1}^{\infty} \frac{\beta_{1,k}}{\alpha_k}
\]

\[
= \frac{8R}{\pi^2} \sum_{k=n+1}^{\infty} \frac{1}{(2k-1)^2}
\]

\[
\leq \frac{8R}{\pi^2} \int_n^{\infty} \frac{1}{(2x-1)^2}dx
\]

\[
\sim O \left( \frac{1}{n} \right) \quad \text{(D.116)}
\]

The equations (D.114), (D.115) and (D.116) then give that

\[
\int_0^{\infty} |\tilde{g}_1(t)| dt \sim O \left( \frac{1}{n} \right) \quad \text{(D.117)}
\]

which proves the first part of Theorem 9.2.
The integral of the approximation error $\tilde{g}_2(t)$ is investigated next.

\[
\int_0^\infty |\tilde{g}_2(t)| \, dt = \int_0^\infty |g_2(t) - \tilde{g}_2^n(t)| \, dt \\
= \int_0^\infty \left[ \frac{1}{\tau} \sum_{k=1}^{\infty} \beta_{2,ke} \frac{n_k}{\tau} - \frac{1}{\tau} \sum_{k=1}^{n} \beta_{2,ke} \frac{n_k}{\tau} \right] \, dt \\
= \frac{1}{\tau} \int_0^\infty \left[ \sum_{k=1}^{\infty} \beta_{2,ke} \frac{n_k}{\tau} - \sum_{k=n+1}^{\infty} \beta_{2,ke} \frac{n_k}{\tau} \right] \, dt \\
\leq \frac{1}{\tau} \int_0^\infty \left[ \sum_{k=1}^{n} \beta_{2,ke} \frac{n_k}{\tau} - \sum_{k=n+1}^{\infty} \beta_{2,ke} \frac{n_k}{\tau} \right] \, dt \\
\underbrace{+ \frac{1}{\tau} \int_0^\infty \sum_{k=n+1}^{\infty} \beta_{2,ke} \frac{n_k}{\tau} \, dt}_{I_4} \\
\tag{D.118}
\]

First the expression $I_3$ is considered. It is not possible to make an upper bound by moving the absolute value sign inside the summation since this approximation is too coarse. However, it was shown in (D.94) and the discussion preceding it, that $I_3 = O(1/n)$.

Next, $I_4$ is treated.

\[
I_4 = \frac{1}{\tau} \int_0^\infty \sum_{k=n+1}^{\infty} \beta_{2,ke} \frac{n_k}{\tau} \, dt \\
= \frac{1}{\tau} \sum_{k=n+1}^{\infty} \int_0^\infty \beta_{2,ke} \frac{n_k}{\tau} \, dt \\
= \sum_{k=n+1}^{\infty} \frac{\beta_{2,k}}{\alpha k} \\
= \sum_{k=n+1}^{\infty} \frac{4}{\pi(2k - 1)} (-1)^{k+1} \\
\tag{D.119}
\]
Further,

\[ I_4 = \frac{4}{\pi} \sum_{k \text{ odd}}^{\infty} (-1)^{k+1} \left( \frac{1}{2k-1} - \frac{1}{2k+1} \right) \]
\[ = \frac{8}{\pi} \sum_{k \text{ odd}}^{\infty} \frac{1}{4k^2 - 1} (-1)^n \]
\[ \sim O \left( \frac{1}{n} \right) \quad \text{(D.120)} \]

In the sums in (D.120), the sum is taken over all odd indices \( k \). It is hence assumed that \( n \) is an even number. If instead \( n \) is odd, the sum is taken over all even indices \( k \). Equations (D.118) and (D.120) hence give

\[ \int_0^\infty |g_2(t)| dt \sim O \left( \frac{1}{n} \right) \quad \text{(D.121)} \]

This concludes the proof of Theorem 9.2.
References


