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Hybrid observers for systems with intrinsic pulse-modulated feedback

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Abstract

This licentiate thesis deals with a special class of hybrid systems, where the continuous linear part is controlled by an intrinsic impulsive feedback that contributes discrete dynamics. The impacting pulsatile feedback signal is not available for measurement and, therefore, has to be reconstructed. To estimate all the elements of the hybrid state vector, an observation problem is considered.

The motivation for the research performed in this thesis comes from mathematical modelling of pulsatile endocrine regulation, where one of the hormones (a releasing hormone) is secreted in pulses from neurons in the hypothalamus of the brain. Thus a direct measurement of the concentration of this hormone in the human is not possible for ethical reasons and it has to be estimated.

Several hybrid observer structures are proposed and evaluated. The observer design is reduced to a problem of synchronizing the impulsive sequence produced by the observer with that of the plant. It utilizes a local approach of assigning, through the output error feedback in both the discrete and continuous parts of the plant model, a guaranteed convergence rate to the local dynamics of a synchronous mode. Performance of the proposed observer schemes is analyzed by means of pointwise discrete (Poincaré) maps.

The first two papers of the thesis address the effects of observer design degrees of freedom on the convergence of the hybrid state estimation error. A generalization of the proposed observation scheme to hybrid impulsive systems with a time delay in continuous part of the plant is investigated in Paper III and Paper IV.

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List of Papers

This thesis is based on the following papers

- I** D. Yamalova, A. Churilov, and A. Medvedev, Design Degrees of Freedom in a Hybrid Observer for a Continuous Plant under an Intrinsic Pulse-modulated Feedback, *IFAC–PapersOnLine*. Modelling, Identification and Control of Nonlinear Systems, 48(11):1080–1085, 2015
- II** D. Yamalova and A. Medvedev, Design of a Hybrid Observer for an Oscillator with an Intrinsic Pulse-modulated Feedback, In *Proc. of the 2017 American Control Conference*. Seattle, WA, USA, May 24 – 26 2017. (accepted)
- III** D. Yamalova, A. Churilov, and A. Medvedev, Finite-Dimensional Hybrid Observer for Delayed Impulsive Model of Testosterone Regulation, *Mathematical Problems in Engineering*, vol. 2015, Article ID 190463, 12 pages, 2015
- IV** D. Yamalova, A. Churilov, and A. Medvedev. Hybrid state observer for time-delay systems under intrinsic impulsive feedback. In *Proc. 21st Intern. Symp. Math. Theory Networks Syst.* (MTNS), pages 977–984, Groningen, The Netherlands, July 7 – 11 2014

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List of Notation

\mathbb{N}	natural numbers $\{1, 2, 3, \dots\}$
\mathbb{N}_0	natural numbers $\mathbb{N} \cup \{0\}$
\mathbb{R}	real numbers
\mathbb{R}^n	linear space of n -dimensional real vectors
\mathbb{C}	complex numbers
$\operatorname{Re} \lambda$	real part of $\lambda \in \mathbb{C}$
$\operatorname{Im} \lambda$	imaginary part of $\lambda \in \mathbb{C}$
i	imaginary unit, $\sqrt{-1}$, unless otherwise specified
\mathbb{C}^n	linear space of n -dimensional complex vectors
\top	vector or matrix transpose
$*$	complex conjugate, or for vectors and matrices, the conjugate transpose
$\ x\ $	Euclidian norm $\ x\ = \sqrt{x^*x}$ of a vector $x \in \mathbb{R}^n$ or $x \in \mathbb{C}^n$, unless otherwise specified
0	denotes zero number, zero vector, or zero matrix, depending on the context
I_n	identity matrix (in some cases the index n is omitted)
A^{-1}	matrix inverse to A
$\det A$	determinant of a matrix A
$\operatorname{tr} A$	trace of a matrix A (the sum of its diagonal elements)
$\operatorname{adj} A$	adjunct of a matrix A
$\operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$	diagonal matrix whose diagonal elements are $\lambda_1, \dots, \lambda_n$
$\lim_{\tau \nearrow t} f(\tau)$	left-hand-side limit of a function $f(\tau)$ of a real variable τ as τ approaches a point t from below, i.e. $\lim_{\tau \nearrow t} f(\tau) = \lim_{\tau \rightarrow t-0} f(\tau)$
$\lim_{\tau \searrow t} f(\tau)$	right-hand-side limit of a function $f(\tau)$ of a real variable τ as τ approaches a point t from above, i.e. $\lim_{\tau \searrow t} f(\tau) = \lim_{\tau \rightarrow t+0} f(\tau)$
$C[-\tau, 0]$	space of \mathbb{R}^n -valued piecewise continuous functions on $[-\tau, 0]$
$\ f\ _\tau$	uniform norm, $\ f\ _\tau = \sup_{\theta \in [-\tau, 0]} \ f(\theta)\ $
$\operatorname{dist}(x, \mathcal{A})$	distance of a vector x to a set \mathcal{A} , i.e. $\operatorname{dist}(x, \mathcal{A}) = \min_{y \in \mathcal{A}} \ x - y\ $

Chapter 1

Introduction

Hybrid mathematical models combining continuous dynamics with an amplitude and frequency pulse-modulated feedback appear in various problems of engineering and science, including those related to biology and medicine [35, 27], population dynamics [7, 92], pharmacokinetics [78, 111, 110], mathematical economy [133], theoretical physics [71], chemistry [8], telecommunications [40, 132], radio engineering [16], communication security [67, 68], and mechanics [16].

One of the main biomedical applications that hybrid impulsive systems currently have is mathematical modeling in the field of neuroendocrinology that studies the numerous interactions between the nervous system and the endocrine system. The continuous metabolic processes in the organism are controlled in a feedback manner by impulses of neurotransmitter concentrations that are generated by ensembles of neurons in the brain. Typically, the interactions between the endocrine glands (or their cells) are usually described as temporal excursions in the blood (plasma) concentrations of the involved hormones.

It is difficult or impossible to measure the concentrations of all the hormones involved in the regulation chain without causing harm to human life or health. This poses the problem of estimating unmeasured hormone concentrations in an endocrine loop, based on the measured ones. Mathematical modeling of neuroendocrine regulation by hybrid dynamical systems can thus provide a non-invasive tool to solve this estimation problem. Further, based on actual clinical data, deeper insights into endocrine regulation can be acquired, thus facilitating timely diagnostics of medical conditions and planning of individualized treatments.

1.1 Contribution

This licentiate thesis concerns the hybrid state estimation problem in a continuous linear time-invariant system under an intrinsic pulse-modulated feedback, where the firing times in the feedback mechanism are treated as an unknown discrete state that, along with the continuous states, has to be reconstructed from the available continuous outputs. The problem in hand is exemplified by an endocrine system where episodically firing neurons control the production of hormones in endocrine glands. The time variation of some hormone concentrations can be obtained by taking and analyzing blood samples while the concentrations of others are unaccessible for direct measurement in the blood stream. Thus, the immeasurable concentrations and the episodes of the pulse-modulated feedback interactions with the continuous part have to be estimated in some manner from the available data, for instance by applying an observer. Design and analysis of hybrid observers solving this estimation problem is the focal point of the thesis.

The impulsive observers proposed below are based on synchronization of impulse sequences generated by the pulse-modulated feedback, an approach originating from [32]. For observer design, the hybrid state estimation problem is recast as a synchronization problem between the impulsive sequence in the plant and that in the observer. The main drawback of the observer scheme introduced in [32] is that only the continuous but not discrete state estimates are updated by means of the output estimation error feedback, which structure may result in a slow and oscillative convergence of the estimates. It is remedied in the present work by introduction of an additional feedback of the (continuous) output estimation error in order to as well correct the estimates of the discrete states.

As known from [32, 131], the critical case of a 1-cycle results in slow observer convergence and stands in need of detailed analysis. It also constitutes the most frequent behavior in pulse-modulated models of endocrine systems [85, 86]. Correct balancing of the contributions from the continuous and discrete parts of the observer is a prerequisite for achieving beneficial overall (hybrid) convergence and is secured by the proposed design tools. Further, useful connections between the cascade structure of the continuous part of the plant and the continuous observer gain matrix structure are revealed. The designed observer demonstrates in simulation clearly superior convergence compared with that obtained in [32].

The presence of time delays in closed loop is inevitable in endocrine systems where the hormones are transported in the blood by the circulatory system to target distant organs. Delays also arise due the time necessary for an endocrine gland to produce a certain hormone quantity. With the time delay taken into account, the pulse-modulated model of endocrine regulation

acquires an infinite-dimensional continuous part. The closed-loop dynamics become therefore both hybrid and infinite-dimensional. Departing from the concept of a finite-dimension reducibility (FD-reducibility) introduced in [30, 28], hybrid observer structures that handle the delay phenomena in the continuous part of the system are developed.

1.2 Thesis outline

The thesis is composed in two parts: First, the necessary theoretical background and motivation are summarized. Second, reprints of the papers resulting from the research performed for this licentiate are provided.

Paper I

In Paper I, an improved version of the observer suggested in [32] is considered where feedback action is also used for the calculation of the impulse firing times in the observer. Therefore, a faster convergence of the hybrid state estimation error has resulted from this structural improvement. The observer design degrees of freedom influencing the observer performance are investigated by means of a pointwise discrete (Poincaré) map and illustrated by extensive numerical simulations and calculations.

Paper II

In Paper II, the same observer structure as in Paper I is considered and the analysis is focused on the case of a 1-cycle that is known from [32] to be a worst-case scenario for observer convergence. Local stability of the synchronous mode is secured through assigning the spectral radius of the Jacobian of a Poincaré mapping. A method for achieving a certain guaranteed convergence rate to the local dynamics of a synchronous mode is derived.

Paper III

In Paper III, a hybrid observer for a finite-dimensional delay-free model is proposed, that can be used for a hybrid time-delay plant. This observer allows to estimate the parameters of impulses thus reconstructing both the continuous and discrete states in the hybrid time-delay plant. However, in this case, the class of observation plants is more restrictive.

Paper IV

Paper IV also deals with a hybrid state observer for FD-reducible impulsive time-delay system. Unlike the case treated in Paper III, the observer proposed here explicitly involves a time delay in the description of its continuous part.

Additional material pertaining to the topic of this thesis is presented in the following publications that are not part of this manuscript:

- Diana Yamalova, Alexander Churilov, and Alexander Medvedev. Hybrid State Observer with Modulated Correction for Periodic Systems under Intrinsic Impulsive Feedback. *IFAC-PapersOnLine*. Periodic Control Systems. Volume 46, Issue 12, 2013, Pages 119–124.
- Diana Yamalova, Alexander Churilov, and Alexander Medvedev. State estimation in a delayed impulsive model of testosterone regulation by a finite-dimensional hybrid observer. *The 14th European Control Conference*. 15 - 17 July 2015, Linz, Austria.
- Diana Yamalova, Alexander Churilov, and Alexander Medvedev. Hybrid Observer for an Intrinsic Impulsive Feedback System. *The 20th World Congress of the International Federation of Automatic Control*. 9-14 July 2017, Toulouse, France (submitted).

1.3 Hybrid systems

Hybrid systems involve both continuous and discrete dynamics. Such a combination is quite common for various dynamical phenomena in, e.g., networked control systems [134, 136, 4, 56, 128, 99, 89, 79], coordination of multi-agent networks [70, 129, 119, 81], continuous switching systems [75, 17], continuous systems under intrinsic pulsatile feedback [51, 130, 107, 77].

The nonlinear dynamics of hybrid dynamical systems are considerably richer and more complicated than those of smooth dynamical systems. Non-smooth dynamics even generate special classes of bifurcations [91]. Thus, the study of hybrid systems is generally more challenging than that of purely discrete or purely continuous systems, because of the interaction between dynamics of different nature. Due to the great diversity of such interactions, hybrid dynamical systems have not yet been formulated by a common mathematical description. From the viewpoint of deterministic dynamics, one of general formulations is described as follows [18, 13, 80, 1]

$$\begin{aligned}
 \dot{x}(t) &= F_{i(t)}(x(t), u(t), \mu), \\
 i(t) &= G(i(t^-), x(t^-), u(t), \mu), \\
 x(t) &= R(i(t^-), x(t^-), u(t), \mu), \\
 y(t) &= O(i(t), x(t), u(t), \mu),
 \end{aligned} \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$ is the continuous state at time $t \in \mathbb{R}$; $i(t) \in \{1, \dots, N\}$ is the discrete state at time t , $F_{i(t)}$ is the vector-valued smooth function specified by $i(t)$, $u(t) \in \mathbb{R}^m$ is the external input, $\mu \in \mathbb{R}^l$ is the system (bifurcation) parameters, G is a map of discrete-state transitions from $i(t^-)$ to $i(t)$ with $i(t^-) \equiv \lim_{\tau \nearrow t} i(\tau)$, R is a reset map of continuous states implementing a discrete-state transition, $y(t) \in \mathbb{R}^k$ is the output, and O is the output function.

If only the output $y(t)$ can be measured and the whole state $x(t)$ is unknown then one is faced with a state observation problem. Roughly speaking, system (1.1) is said to be observable if the state $x(t)$ can be determined from the knowledge of $y(t)$ and $u(t)$. More formally, consider the observation problem for the case when system (1.1) is a switched linear system with state jumps described as in [117]

$$\begin{aligned} \dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \quad t \neq \{t_q\}, \\ x(t_q) &= E_{\sigma(t_q^-)}x(t_q^-) + F_{\sigma(t_q^-)}v_q, \quad q \geq 1, \\ y(t) &= C_{\sigma(t)}x(t) + D_{\sigma(t)}u(t), \quad t \geq t_0, \end{aligned} \quad (1.2)$$

where $x(t) \in \mathbb{R}^n$ is the state, $y(t) \in \mathbb{R}^k$ is the output, $u(t) \in \mathbb{R}^m$ and $v(t) \in \mathbb{R}^l$ are the inputs. For some index set Σ , the switching signal $\sigma : \mathbb{R} \rightarrow \Sigma$ is a piecewise constant and right-continuous function that changes its value at switching times t_q , $q \in \mathbb{N}$. A_σ , B_σ , C_σ , D_σ , E_σ , F_σ are the matrices of appropriate dimensions. Introduce the formal definition of observability.

Definition 1 ([117]). *Let $(\sigma^i, u^i, v^i, y^i, x^i)$, for $i = 1, 2$, be the signals that satisfy (1.2) over an interval $[t_0, T^+)$, where $T^+ = T + \varepsilon$, and $\varepsilon > 0$ is arbitrarily small. The system (1.2) is said to be $[t_0, T^+)$ -observable if the equality $(\sigma^1, u^1, v^1, y^1, x^1) = (\sigma^2, u^2, v^2, y^2, x^2)$ implies that $x^1(T) = x^2(T)$.*

In switched systems, if the switching signal is known, then, even though the individual subsystems are not observable, it could be still possible to recover the initial state by appropriately processing the measured signals over a time interval that involves multiple switching instants. In e.g. [117], a characterization of observability and an observer design for a switching system, where the subsystems are not required to be observable, are proposed. On the other hand, if each subsystem is observable, but the switching signal is treated as an unknown discrete state, there is a nontrivial problem of simultaneous recovery of the discrete and continuous state.

Introduce a general stability concept for hybrid systems. Hybrid systems may be formally described as a differential inclusion by the following model [42, 43]:

$$\begin{aligned} \dot{x} &\in F(x), \quad x \in C, \\ x^+ &\in G(x), \quad x \in D, \end{aligned} \quad (1.3)$$

where $x \in \mathbb{R}^n$ is the state vector of a hybrid system, x^+ is the state of a hybrid system after a jump, F is a set-valued mapping (the flow map), $C \subset \mathbb{R}^n$ (the flow set), G is a set-valued mapping (the jump map), $D \subset \mathbb{R}^n$ (the jump set).

Denote hybrid system (1.3) by $\mathcal{H} = (F, G, C, D)$. The solution to the hybrid system \mathcal{H} [44] is a piecewise absolutely continuous function satisfying $\dot{x}(t) \in F(x(t))$ almost everywhere and $x(t) \in C$ on each interval of continuity and whose right limits $x(t^+) = \lim_{\tau \searrow t} x(\tau)$ at jump times t , which are determined by $x(t) \in D$, are related to $x(t)$ through $x(t^+) \in G(x)$. However, this choice of the time domain does not allow for more than one jump at a given time.

An alternate approach [83, 6, 82, 44, 42] is to consider the state of a system not only as a function of time but also of the number of jumps that occurred. A subset E of $[0, \infty) \times \mathbb{N}_0$ is a *hybrid time domain* if it is the union of infinitely many intervals of the form $[t_j, t_{j+1}] \times \{j\}$, where $0 \leq t_0 \leq t_1 \leq t_2 \leq \dots$, or of finitely many such intervals, with the last one possibly of the form $[t_j, t_{j+1}] \times \{j\}$, $[t_j, t_{j+1}) \times \{j\}$, or $[t_j, \infty) \times \{j\}$. This concept gives a more explicit role to the "discrete" variable j : the state of the system is parameterized by $(t; j)$. Therefore, one can define a solution to a hybrid system as a function, defined on a hybrid time domain, that satisfies the dynamics and constraints given by the data of the hybrid system.

A *hybrid arc* is a function $x : \text{dom } x \rightarrow \mathbb{R}^n$, where $\text{dom } x$ is a hybrid time domain and, for each fixed j , $t \rightarrow x(t, j)$ is a locally absolutely continuous function on the interval $I_j = \{t : (t, j) \in \text{dom } x\}$.

The hybrid arc x is a **solution to the hybrid system** \mathcal{H} [42] if $x(0, 0) \in C \cup D$ and the following conditions are satisfied:

- *Flow condition.* For each $j \in \mathbb{N}_0$ such that I_j has nonempty interior, $\dot{x}(t, j) \in F(x(t, j))$ for almost all $t \in I_j$, and $x(t, j) \in C$ for all $t \in [\min I_j, \sup I_j]$.
- *Jump condition.* For each $(t, j) \in \text{dom } x$ such that $(t, j + 1) \in \text{dom } x$, $x(t, j + 1) \in G(x(t, j))$, $x(t, j) \in D$.

The solution x to \mathcal{H} is *nontrivial* if $\text{dom } x$ contains at least one point different from $(0, 0)$; *maximal* if it cannot be extended, that is, the hybrid system has no solution x' whose domain $\text{dom } x'$ contains $\text{dom } x$ as a proper subset and such that x' agrees with x on $\text{dom } x$; and *complete* if $\text{dom } x$ is unbounded. Every complete solution is maximal.

Since the solutions of a hybrid dynamical system may contain logic variables, timers, counters, they do not necessarily converge to an equilibrium point. Therefore, in theory of hybrid systems, it is common to study asymptotic stability of compact sets rather than of simple equilibrium point.

Definition 2 ([42]). *Let \mathcal{A} be a compact set in \mathbb{R}^n .*

- *The set \mathcal{A} is **stable** for \mathcal{H} if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\text{dist}(x(0, 0), \mathcal{A}) \leq \delta$ implies $\text{dist}(x(t, j), \mathcal{A}) \leq \varepsilon$ for all solutions x to \mathcal{H} and all $(t, j) \in \text{dom } x$. The notation $\text{dist}(x, \mathcal{A}) = \min_{y \in \mathcal{A}} \|x - y\|$ indicates the distance of the vector x to the set \mathcal{A} . If \mathcal{A} is the origin then $\text{dist}(x, \mathcal{A}) = \|x\|$.*
- *The set \mathcal{A} is **pre-attractive** if there exists a neighborhood of \mathcal{A} from which each solution is bounded and the complete solutions converge to \mathcal{A} , that is, $\text{dist}(x(t, j), \mathcal{A}) \rightarrow 0$ as $t + j \rightarrow \infty$, where $(t, j) \in \text{dom } x$.*
- *The set \mathcal{A} is **pre-asymptotically stable** if it is stable and pre-attractive.*

The term "pre-" indicates the possibility of a maximal solution that is not complete, even though it may be bounded. Therefore, a compact set \mathcal{A} is *asymptotically stable* if solutions that start close to \mathcal{A} stay close to \mathcal{A} , and complete solutions that start close to \mathcal{A} converge to \mathcal{A} .

1.4 Pulse-modulated systems

Impulsive systems or systems with pulse-modulated feedback can be viewed as a special class of hybrid systems, where the continuous dynamics (the flow map and the flow set) are presented by the motion of the dynamical system in between impulsive or resetting events. The discrete dynamics (the jump map and the jump set) are defined as follows:

- *the jump map* is given by a difference equation that governs the way in which the hybrid system state is instantaneously changed when a resetting event occurs;
- *the jump set* is defined by a criterion that determines when the state of the hybrid system has to be reset.

The principal element of an impulsive system is a pulse modulator that converts a continuous-time input signal into a sequence of pulses. It can be mathematically described by a nonlinear operator [40] that maps a continuous input function $\sigma(t)$ to a piecewise continuous output function $f(t)$:

$$M : \sigma(t) \mapsto f(t),$$

where the real-valued functions $\sigma(t)$, $f(t)$ are defined for $t \geq 0$. Any sequence of pulses is associated with an increasing sequence $t_0 = 0 < t_1 < t_2 < \dots$, where $\lim_{k \rightarrow \infty} t_k = \infty$, whose elements are called *sampling moments* or *firing*

times. The time interval $[t_n, t_n + 1)$ is called *the n-th sampling interval* and $T_n = t_{n+1} - t_n$ is *the length of the n-th sampling interval*. If $T_n = T = \text{const}$, $n = 0, 1, 2, \dots$, then the value T is called *a sampling period*. Thus, the function $f(t)$ can be represented as follows

$$f(t) = f_n(t), \quad t_n \leq t < t_{n+1}, \quad n = 0, 1, \dots,$$

where $f_n(t)$ describes the shape of the n -th pulse. One-sided limits $f_{n-1}(t_n - 0)$ and $f_n(t_n + 0)$ exist and are finite but not necessarily equal to each other.

The simplest and most common pulse shape is a rectangular one (see Fig. 1.1), when

$$f_n(t) = \begin{cases} 0, & t_n \leq t < t'_n, \\ \lambda_n, & t'_n \leq t < t''_n, \\ 0, & t''_n \leq t < t_{n+1}. \end{cases} \quad (1.4)$$

Here t'_n, t''_n, λ_n are real numbers. The parameter λ_n is called *pulse amplitude*. The numbers t'_n and t''_n define the position of the leading and trailing edge of a pulse, respectively. The parameter $\vartheta_n = t'_n - t_n$ is called *pulse phase* (the displacement of the leading edge relative to the beginning of the n -th sampling interval). The parameter $\tau_n = t''_n - t'_n$ is called *pulse width* (or *pulse duration*, *pulse length*). Note that when $\tau_n = 0$ then the modulator is said to produce a train of impulses, i.e. instant pulses (pulses of zero duration), while the term "pulse" is used for pulses of finite duration.

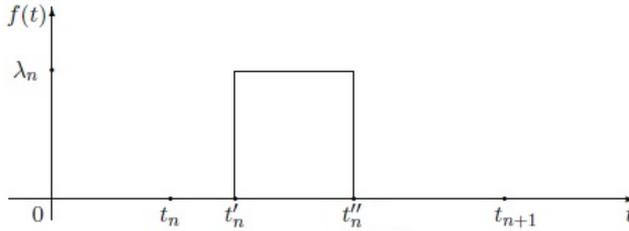


Figure 1.1: A square pulse.

Thus, (1.4) can be equivalently written as

$$f_n(t) \equiv f_n(t, p_n), \quad p_n = \{\lambda_n, \vartheta_n, \tau_n, T_n\}. \quad (1.5)$$

Some parameters of $f(t)$ are considered fixed and known, while the others are treated as functions or functionals of $\sigma(t)$ and called *modulated parameters*. Thereby, there are the following types of modulation (see Fig. 1.2):

- *Pulse-frequency modulation (PFM)*. The length of impulsive interval T_n depends on $\sigma(t)$ and the other parameters are fixed.

- *Pulse-amplitude modulation (PAM)*. The value of λ_n depends on $\sigma(t)$ and the other parameters are fixed.
- *Pulse-width modulation (PWM)*. The varying parameter is τ_n .
- *Pulse-position or pulse-phase modulation (PPM)*. The value of t'_n is modulated while the other parameters are fixed.
- *Combined pulse modulation*, when several parameters depend on $\sigma(t)$.

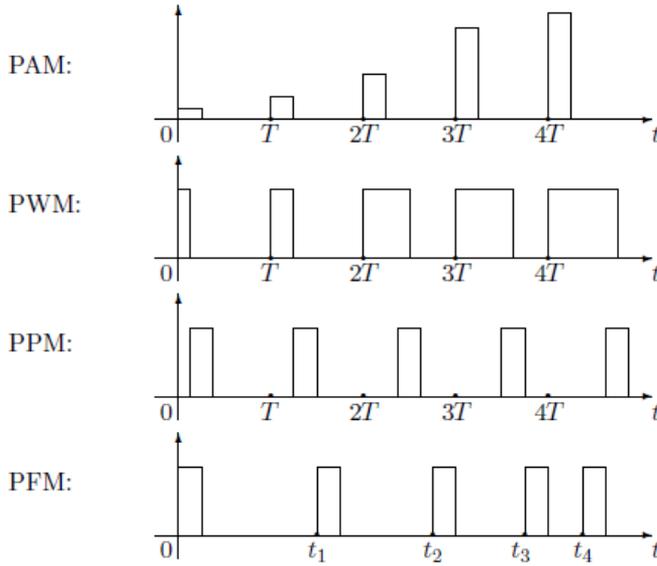


Figure 1.2: Types of pulse modulation.

1.5 Time-delay systems

Consider the general form of a retarded type time-delay system

$$\dot{x}(t) = f(t, x(t), x(t - \tau)) \quad (1.6)$$

that belongs to a class of functional differential equations. Here $x \in \mathbb{R}^n$, $\tau > 0$ is the time delay, and the vector function f is continuous in its variables. As can be seen from (1.6), the value of time derivative of the function $x(t)$ is defined not only by the behavior of the function x at the current time t , but also by the behavior of the function x at the previous time instant $t - \tau$.

To define a solution of system (1.6), one needs to formulate the initial value problem: given a piecewise continuous function $\varphi : [-\tau, 0] \rightarrow \mathbb{R}^n$ and an initial time instant t_0 , find $x(t)$ such that the condition $x(t_0 + \theta) = \varphi(\theta)$, $\theta \in [-\tau, 0]$ is satisfied. A solution to system (1.6) can be found with the approach known as step-by-step method. The idea is as follows. Define the solution on the segment $[t_0, t_0 + \tau]$ by substituting the function $\varphi(t - t_0 - \tau)$ into the right hand side of (1.6) instead of $x(t - \tau)$, and solving the resulting ordinary differential equation for $x(t_0) = \varphi(0)$. Next, one can find the solution on the segment $[t_0 + \tau, t_0 + 2\tau]$ in the same way, defining $\varphi(t) = x(t)$ for $t \in [t_0, t_0 + \tau]$. Therefore, the calculation of the solution to time-delay system (1.6) can be reduced to a step-by-step computation of solutions to ordinary differential equations [69, 50].

Introduce the state concept for time-delay system (1.6). In general, the state of a system at a given time instant t_* should include all the information that allows to determine the system dynamics for $t \geq t_*$. And, as can be seen from the initial value problem, unlike a system described by an ordinary differential equation, where the state is just $x(t_*)$, for time-delay system (1.6), one needs to know the whole function $x(t_* + \theta)$ for $\theta \in [-\tau, 0]$ to extend a solution to $t \geq t_*$. Therefore, a proper notation for the state of a time-delay system is the following function

$$x_t(\theta) = x(t + \theta), \quad \theta \in [-\tau, 0]. \quad (1.7)$$

Note that, since the state of a time-delay system belongs to an infinite-dimensional space of functions, time-delay system (1.6) is *infinite-dimensional*.

The following theorem gives the conditions for existence and uniqueness of solutions to time-delay system (1.6).

Theorem 1 ([39]). *Suppose that function $f : \mathbb{R}^{1 \times n \times n} \rightarrow \mathbb{R}^n$ in (1.6) is piecewise continuous in the first and the third arguments and is Lipschitz-continuous in the second argument, i.e.*

$$\|f(t, x_1, y) - f(t, x_2, y)\| \leq L \|x_1 - x_2\|,$$

where $L > 0$, and the function φ is continuous on $[t_0 - \tau, t_0]$. Then the solution of time-delay system (1.6) exists and is unique on the segment $[0, T]$ for any $T > 0$.

Next, a definition of uniform stability of the trivial solution of a time-delay system is given.

Definition 3. *The trivial solution to system (1.6) is said to be uniformly asymptotically stable if,*

1. for any $\varepsilon > 0$ and $t_0 \geq 0$, there exists $\delta(\varepsilon)$ such that if $\|x_{t_0}\|_\tau < \delta(\varepsilon)$ then $\|x(t)\| < \varepsilon$ for $t \geq t_0$;
2. for any $\eta > 0$ and $t_0 \geq 0$, there exist δ_a and $T(\delta_a, \eta)$ such that if $\|x_{t_0}\|_\tau < \delta_a$ then $\|x(t)\| < \eta$ for $t \geq t_0 + T(\delta_a, \eta)$,
where the norm $\|x_{t_0}\|_\tau = \sup_{\theta \in [-\tau, 0]} \|x_{t_0}(\theta)\|$.

1.5.1 Linear time-delay systems

Consider the following linear time-delay system of retarded type:

$$\dot{x}(t) = A_0x(t) + A_1x(t - \tau), \quad (1.8)$$

where A_0, A_1 are given real $n \times n$ matrices. Let the initial conditions be given by the piecewise continuous function $\varphi : [-\tau, 0] \rightarrow \mathbb{R}^n$ and $x(\theta) = \varphi(\theta)$, $\theta \in [-\tau, 0]$ with $t_0 = 0$.

The characteristic equation of system (1.8)

$$\rho(s) = \det [sI - A_0 - A_1e^{-s\tau}] = 0. \quad (1.9)$$

is a transcendental and has an infinite number of roots. They have the following property: there is a finite number of the roots of (1.9) on the right hand side from any vertical line $\operatorname{Re} s = \text{const}$.

Introduce a definition of exponential stability of system (1.8) that is equivalent to its asymptotic stability [69].

Definition 4. *System (1.8) is said to be exponentially stable if there exist $\gamma \geq 1$ and $\sigma > 0$ such that any solution $x(t)$ of the system satisfies the inequality*

$$\|x(t)\| \leq \gamma e^{-\sigma t} \|\varphi\|_\tau, \quad t \geq 0.$$

Definition 5. *A complex number s_0 is said to be an eigenvalue of system (1.8) if it is a root of the system characteristic equation $\rho(s)$. The set $\Lambda = \{s \mid \rho(s) = 0\}$ is known as the spectrum of system (1.8).*

Theorem 2 ([12]). *System (1.8) is exponentially stable if and only if all its eigenvalues lie in the open left half-plane of the complex plane*

$$\operatorname{Re} s_0 < 0, \quad \forall s_0 \in \Lambda.$$

1.5.2 Finite-dimensional reducibility

Definition 6. *System (1.8) is said to be finite-dimensional (FD) reducible if there exists a matrix D such that any solution of (1.8) satisfies a delay-free system*

$$\dot{x}(t) = Dx(t)$$

for all $t \geq \tau$.

The property of FD-reducibility can be characterized as follows.

Lemma 1 ([29]). *System (1.8) is FD-reducible if and only if*

1) *the equalities $A_1 A_0^k A_1 = 0$ hold for all $k = 0, 1, \dots, n - 1$;*

or

2) *there exists a nonsingular matrix $S \in \mathbb{R}^n$ such that*

$$S^{-1}A_0S = \begin{bmatrix} U & 0 \\ W & V \end{bmatrix}, \quad S^{-1}A_1S = \begin{bmatrix} 0 & 0 \\ \bar{W} & 0 \end{bmatrix}, \quad (1.10)$$

where the blocks U, V are square, the blocks W, \bar{W} are of the same dimension. Further, if system (1.8) is FD-reducible, then $D = A_0 + A_1 e^{-A_0 \tau}$.

For an FD-reducible system, the eigenvalue spectrum of the matrix A_0 coincides with that of D and, thus, the spectrum of D is independent of τ , i.e.

$$\det [sI - A_0 - A_1 e^{-A_0 \tau}] = \det [sI - A_0]$$

for all complex s and any τ . Additionally, FD-reducible time-delay linear system (1.8) obviously possesses a finite spectrum, since

$$\det(sI_n - A_0 - A_1 e^{-\tau s}) = \det(sI_n - A_0),$$

for all complex s .

1.6 Hybrid models in life science

The research field of hybrid systems that covers interactions of discrete and continuous dynamics is gaining more attention in mathematical modeling and analysis within life science and medicine. Basically, every form of life exhibits hybrid dynamics. Hybrid behaviours commonly occur within living organisms – such as microorganisms, plants, animals, and human beings – due to their complex dynamical nature, as their evolvement is subject to discontinuities. Typical examples of such discontinuities appear in threshold-triggered firing in neurons [62], on–off switching of gene expression by a transcription factor [61, 95, 109], division in cells [24, 11, 93], in disease progression e.g. prostate cancer under intermittent hormonal therapy, where continuous tumor dynamics are switched by interruption and reinstatement of medication [52, 60, 106, 115, 116, 57, 114], in endocrine regulation, where neural processes interact with the hormone kinetics thus giving rise to hybrid models with relatively slow continuous dynamics that are controlled through impulsive action of firing neurons [65, 66, 125]. These processes admixing discrete events with continuous system evolution cannot be adequately described by linear models that are based on proportionality

between two variables and/or relationships described by linear differential equations. Nonlinear modeling alone still is not able to explain all of the dynamical diversity present in a living organism.

Periodic, quasi-periodic, and chaotic modes are inherent phenomena in any living organism governed by biological rhythms and self-regulatory mechanisms. Oscillations are unavoidable in the cardiac rhythm observed in electrocardiograms [97, 41], breathing [96, 10, 98], neural system dynamics [36, 48], hormone concentrations [55, 113, 112], in enzymatic control processes [47, 46, 122], gene regulatory processes [49]. Consequently, the question for the existence of periodic and chaotic solutions is a central problem of the qualitative investigation of the corresponding dynamical systems.

Looking for oscillations in cardiac rhythm, brain, or population dynamics has also the merit of replacing the study of such systems within the framework of hybrid dynamics. These systems have been traditionally studied by means of systems of differential equations or by purely discrete models. In general, such a representation efficiently comports with clinical data; it has in some cases a good predictive value, but no explanatory value. Since it does not point at the actual underlying mechanisms, nor a regulatory value, it is not able to evidence control parameters of the observed behaviour and to capture all possible interactions between the continuous and discrete dynamics of such systems. A hybrid dynamics viewpoint provides useful tools to investigate the nature of the complex dynamics in biological system and to understand the origin of the presented dynamic behaviour. Further, mathematical modeling by hybrid dynamical systems is particularly important for understanding the nonlinear dynamics of the human organism in health and in disease in efforts to predict medical conditions, help appropriate diagnoses, and optimize treatments.

Hybrid systems can be conventionally divided into three major classes: hybrid dynamical systems, hybrid control systems, and hybrid automata. It is worth noting that, in control engineering, the concept of hybrid systems is applied mainly in design, optimisation, and control synthesis. In life sciences, hybrid systems are used to assist in interpreting, explaining, and predicting dynamical phenomena. However, existing rich theoretical framework in technical applications of hybrid systems often cannot be straightforwardly spread to problems arising in biological and medical systems. This is due to the different nuances and ethical issues arising in biomedical control and observation problems. In particular, the fact that discrete events triggered in deep parts of the brain are typically not accessible in living organisms poses a specific and seldom addressed in control theory problem of estimating discrete states of a hybrid system from only continuous measurements. Furthermore, rhythmic behaviour is one of the main properties of the living organism, meaning that corresponding dynamical systems lacks equilibria,

which is also not typical for most of engineering problems. Therefore, hybrid systems in biomedical applications require separate consideration.

1.7 The impulsive Goodwin's oscillator

Investigation of the dynamics of complex systems has shown that the interplay between components in chemical, mechanical [63, 5, 34, 58, 126] and/or genetic networks [121, 54] readily leads to oscillatory behaviors as a consequence of self-organization and the periodic changes of the system in time [135, 74, 104, 14, 59]. Oscillations are an essential property of living systems, from primitive bacteria to the most sophisticated life forms. Oscillations can take place in a biological system in a multitude of ways. Positive feedback loops, on their own or in combination with negative feedback, are a common feature of oscillating biological systems [20, 22].

The equations of Goodwin's oscillator [47, 46] may describe a basic mechanism for oscillation in a cascaded biochemical system of three or more variables and is based on negative feedback. Goodwin's oscillator was originally proposed to model oscillatory processes in enzymatic control, with improvements and generalizations presented later in a large number of publications [49, 123, 122, 127, 3, 94]. The prototypical Goodwin's model is a third-order linear continuous system with a static nonlinear feedback parameterized by a Hill function. The feedback is capable of causing self-sustained closed-loop oscillations when the Hill function is at least of eighth order.

The variants of Goodwin's equations are also commonly used to model circadian and other genetic oscillators in biology [127, 102, 103, 76, 45, 23, 120, 49]. In 1980s, the Goodwin's model was adopted by R. Smith for describing biological phenomena associated with periodic behaviours in the endocrine system of testosterone regulation (referred to as the Goodwin-Smith model [113, 112]).

As was demonstrated in [49], the original Goodwin Oscillator needs biologically infeasible slopes of the feedback nonlinearity (i.e. Hill function order greater than eight) to possess periodic solutions. Consider an impulsive model of non-basal endocrine regulation based on Goodwin's equations that exhibits oscillation under much more reasonable conditions. The presence of nonlinear amplitude and frequency modulation functions (PAM and PFM, see section 1.4) is actually sufficient to cause oscillation. System (1.11),(1.12) was considered in [31] but equivalently described with Dirac delta functions.

The model is comprised of a continuous linear part

$$\dot{x}(t) = Ax(t), \quad z(t) = Cx(t), \quad y(t) = Lx(t), \quad (1.11)$$

and a discrete part

$$\begin{aligned} x(t_n^+) &= x(t_n^-) + \lambda_n B, & t_{n+1} &= t_n + T_n, \\ T_n &= \Phi(z(t_n)), & \lambda_n &= F(z(t_n)). \end{aligned} \quad (1.12)$$

Here $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x}$, $C \in \mathbb{R}^{1 \times n_x}$, $L \in \mathbb{R}^{n_y \times n_x}$ are constant matrices, z is the scalar controlled output, y is the vector of measurable output, and x is the state vector of (1.11).

The recursion in (1.12) gives rise to discrete dynamics in the closed-loop system and adds a discrete state variable to the hybrid system expressed by (1.11)–(1.12).

The matrix A is Hurwitz, i.e. all its eigenvalues have strictly negative real parts, the matrix pair (A, L) is observable, and the relationships

$$CB = 0, \quad LB = 0 \quad (1.13)$$

apply.

The elements of the state vector $x(t)$ experience jumps at time instants $t = t_n$ with corresponding weights λ_n , $n = 0, 1, 2, \dots$, $x(t_n^-)$, $x(t_n^+)$ are left-sided and right-sided limits of $x(t)$ at t_n , respectively. However, the outputs $y(t)$, $z(t)$ are continuous due to assumption (1.13).

The amplitude and frequency modulation functions $\Phi(\cdot)$ and $F(\cdot)$ are continuous, strictly monotonic and bounded,

$$0 < \Phi_1 \leq \Phi(\cdot) \leq \Phi_2, \quad 0 < F_1 \leq F(\cdot) \leq F_2, \quad (1.14)$$

where Φ_1 , Φ_2 , F_1 , F_2 are strictly positive constant numbers. Unlike modulators used in technical applications, $\Phi(\cdot)$ is non-decreasing and $F(\cdot)$ is non-increasing. Thus (1.12) denotes a combined (pulse-frequency and pulse-amplitude) modulation [40].

From the stability of A and the boundness of $\Phi(\cdot)$ and $F(\cdot)$, it readily follows that all the solutions of (1.11)–(1.12) are bounded from below and above. In addition, system (1.11)–(1.12) does not have equilibria because all the modulation characteristics are positive. It was established that system (1.11)–(1.12) may have (orbitally) stable and unstable periodic solutions [31] and, for certain values of parameters, the system may have chaotic behaviour [137].

The hybrid plant is subject to initial conditions $(x(0), t_0)$ and the first firing instant of the pulsatile feedback occurs after the initial time instant, $t_0 \geq 0$. To study the hybrid system dynamics of (1.11)–(1.12), the initial conditions can be specified as $(x(t_0^-), t_0)$.

1.7.1 Mathematical model of testosterone regulation

The case of a third order system (1.11)–(1.12) with the matrices

$$A = \begin{bmatrix} -b_1 & 0 & 0 \\ g_1 & -b_2 & 0 \\ 0 & g_2 & -b_3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$C = [0 \quad 0 \quad 1],$$

can be used to model testosterone regulation in the human male [31, 29, 86]. Where b_1, b_2, b_3, g_1, g_2 are positive parameters, reflecting the kinetics of the involved hormones. From the biology of the system, one has $b_i \neq b_j$ for $i \neq j$.

In the endocrine system of testosterone (Te) regulation in the male, an essential role is played by three different hormones, namely gonadotropin-releasing hormone (GnRH), luteinizing hormone (LH), and testosterone (Te). GnRH is released from the hypothalamus of the brain in pulsatile fashion with short latency. Reaching the pituitary gland, GnRH initiates the production of LH, which in turn stimulates production of Te in the testes. Finally, both the GnRH outflow and the LH secretion are subject to feedback inhibition by Te [124]. The latter is however not modeled in this thesis. The action of the hypothalamic GnRH neurons can be described by a pulse element (pulse modulator) performing amplitude and frequency modulation, i.e. the Te concentration acts as a modulation signal, and the GnRH concentration as a pulse-modulated signal. Thus, the elements of x correspond to the concentrations of GnRH (x_1), LH (x_2), and of Te (x_3).

Such an impulsive model has been demonstrated to be capable of explaining actual endocrine data [66] and to be feasible for analysis of closed-loop behaviors, [85]. The capability of model is illustrated by Fig. 1.3 and Fig. 1.4.

1.7.2 Solutions

Define $x_n = x(t_n^-)$. Then any solution $x(t)$ of (1.11)–(1.12) satisfies the discrete-time equation

$$x_{n+1} = P(x_n), \tag{1.15}$$

where

$$P(x) = e^{A\Phi(Cx)}(x + F(Cx)B). \tag{1.16}$$

Together with the equation $t_{n+1} = t_n + \Phi(x_n)$, (1.15) completely defines the dynamics of (1.11)–(1.12), [31].

Consider periodic solutions of (1.15), and, consequently, of (1.11)–(1.12). A set of points $S(x_0) = \{x_0, x_1, \dots\}$ with $x_{n+1} = P(x_n)$ is usually termed as an orbit of system (1.15) through the point x_0 .

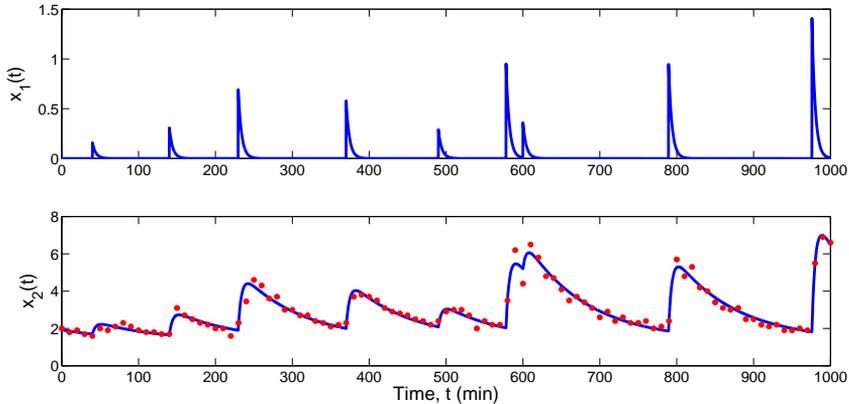


Figure 1.3: LH data measured with 10 min sampling in a healthy 27 years old man (red). Estimated GnRH and simulated LH (blue). The figure provided by Per Mattsson, per.mattsson@it.uu.se

A solution x_n , $n = 0, 1, \dots$ is called m -periodic (for some $m \geq 1$), if m is the smallest value for which the relationships

$$\begin{aligned} x_1 &= P(x_0), \\ x_2 &= P(x_1), \\ &\vdots \\ x_m &= P(x_{m-1}), \\ x_m &= x_0 \end{aligned}$$

hold.

Then the orbit of such a solution is $S(x_0) = S_m(x_0) = \{x_0, x_1, \dots, x_{m-1}\}$, which sequence defines an m -periodic orbit of discrete system (1.15). The initial value x_0 for an m -periodic solution satisfies $x_0 = P^{(m)}(x_0)$, where

$$P^{(m)}(x_0) = \underbrace{P(P(\dots))}_{m}(x_0).$$

Let x_n be an m -periodic solution of (1.15). Pick a solution $x(t)$ of (1.11)–(1.12) for $t \geq t_0$ with some initial values $t_0 > 0$ and $x(t_0^-) = x_0$. Then $x(t)$ is periodic with the period $T = \Phi(x_0) + \dots + \Phi(x_{m-1})$ and has exactly m impulses on the periodicity interval $[0, T)$. Moreover, $t_m = t_0 + T$. Such a solution $x(t)$ is called an m -cycle.

For system (1.11)–(1.12), the stability notion is understood as orbital asymptotic stability with respect to small perturbation in the initial condi-

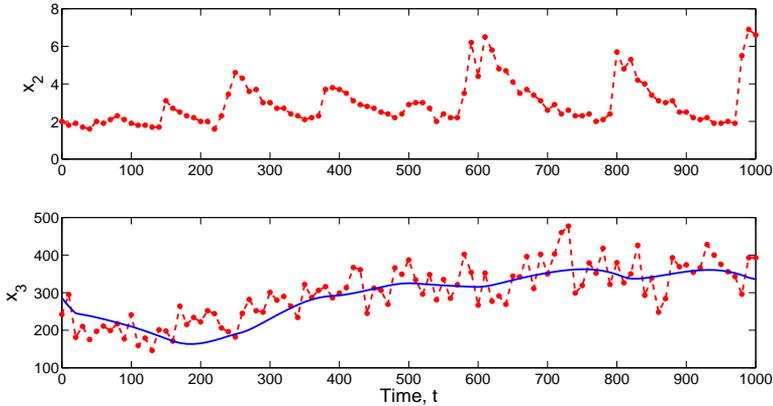


Figure 1.4: LH (upper plot) and GnRH (lower plot) data measured with 10 min sampling in a healthy 27 years old man (red). Estimated GnRH and simulated LH (blue). The figure provided by Per Mattsson, per.mattsson@it.uu.se.

tions. This does not generally imply stability in Lyapunov sense, see [53, 108].

An orbit $S(x_0)$ of discrete equation (1.15) is called *asymptotically stable* [72] if

- (i) for any neighborhood $V \supset S$, there exists a neighborhood $U \supset S$ such that $x_n \in V$ for all $x_0 \in U$ and $n \geq 0$;
- (ii) there exists a neighborhood $U_0 \supset S$ such that the distance $\text{dist}(x_n, S) \rightarrow 0$ for all $x_0 \in U_0$, as $n \rightarrow \infty$.

For (1.15), an m -periodic orbit $\{x_0, x_1, \dots, x_{m-1}\}$ is locally asymptotically stable if all the eigenvalues of matrix product $J_0 J_1 \dots J_{m-1}$ lie inside the unit circle, where J_k is Jacobian of $P(x)$ at a point x_k . Thus, for (1.11)–(1.12), local stability of an m -cycle can be checked by linearizing the mapping $P(x)$ in a neighborhood of each fixed point $x_i = 0, \dots, m-1$.

Since the hybrid system under consideration is of dimension $(n+1)$ (one continuous state and one discrete), consider a $(n+1)$ -dimensional orbit (a set of ordered pairs) $\tilde{S}(x_0, t_0) = \{(x_n, t_n), n = 0, 1, \dots\}$. Clearly, it is not periodic even in the case when $S(x_0)$ is an m -periodic orbit of (1.15). Since $t_{n+1} = t_n + \Phi(x_n)$, for an asymptotically stable S , the orbit \tilde{S} will not be asymptotically stable. Indeed, small perturbations in t_0 remain small as n increases, but do not vanish.

For biologically motivated values of parameters, autonomous system (1.11)–(1.12) normally exhibits either a stable 1-cycle, or a stable 2-cycle (i.e. periodic solutions with either one, or two impulses fired in the least period) [31], but chaotic solutions are also possible. Being extended with a time delay [137] or a continuous exogenous signal in the continuous part [88], the model may exhibit more complex nonlinear dynamics such as cycles of higher periodicity, chaos, bistability, and quasiperiodical solutions.

With respect to pulsatile endocrine regulation, the state vector $x(t)$ of (1.11)–(1.12) is composed of the concentrations of the involved hormones, $y(t)$ represents the concentrations of the measured (in the bloodstream) hormones, and $z(t)$ stands for the hormone concentration that modulates the pulsatile feedback. The release hormone pulses are secreted at the time instances t_n , have the weight of λ_n , and are immeasurable.

The purpose of the observation in hybrid system (1.11)–(1.12) is to produce estimates $(\hat{t}_n, \hat{\lambda}_n)$ of the impulse parameters (t_n, λ_n) under unknown initial conditions $(x(0), t_0)$. Given the sequence $(t_n, \lambda_n), n = 0, \dots, \infty$, estimates of the state vector x of the continuous part can be obtained by conventional state estimation techniques. The main challenge of hybrid observation is to ensure asymptotic convergence of the sequence $\{\hat{t}_n\}$ to $\{t_n\}$, i.e. to synchronize impulses in the observer with those of the plant.

1.8 Hybrid observers

A considerable number of papers is devoted to the investigation of hybrid observers. Yet a significant part of the literature covers hybrid observer design to achieve a performance objective in non-hybrid plants [118, 33, 100, 101, 25, 26]. Most of the currently existing observer design approaches for hybrid systems target engineered systems and typically assume that the discrete states of the plant are known, i.e both the jump magnitude and timing [2, 19, 21, 117, 38].

In many biomedical applications, where the jumps (or switching) corresponds an unknown discrete state, there is a nontrivial problem of simultaneous recovery of the discrete and continuous states only from measurable continuous output. This problem is typical to biomedicine as measurements of discrete events are often not accessible in living organisms, where the continuous and discrete dynamics may not only co-exist but can also interact, and changes occur in response to discrete events and/or continuous inputs, both internally or exogenously generated.

In, e.g. [9], the knowledge of the discrete state is not required, but the observer design assumes the knowledge of the hybrid plant inputs and outputs (either discrete or continuous) while the equations of the impulsive

Goodwin's oscillator in (1.11)–(1.12) constitute an autonomous system. To recapitulate, the problem of estimating discrete states of a hybrid system from only continuous measurements is not so well studied in the literature.

In endocrine systems with pulsatile secretion, the highest degree of uncertainty is associated with the discrete (impulsive) part whose states have to be reconstructed from hormone concentration measurements. Two model-based estimation approaches are currently known. The first one is based on batch deconvolution techniques (blind system identification) [64, 125, 37, 84, 87], while the relatively recent second one employs a state observer, whose estimates are corrected by output estimation error feedback [32]. While being completely functional and able to handle both periodic and chaotic solutions in the plant, this observer structure suffered from slow convergence, motivating further investigation carried out in this thesis.

Another complication in state estimation of linear continuous time-invariant systems under inaccessible for measurement pulse-modulated feedback is in stability analysis of the resulting observer structures. Indeed, the states of the considered system undergo jumps at certain time instants (firing times) modulated by other states and, in general, the firing times produced by an observer do not coincide with those of the plant. Consequently, on the time interval defined by the mismatch in the firing times, the error between the solution to the plant and that to the observer is large and does not monotonically converge to zero as time increases, so-called "peaking phenomenon" [15, 73, 90, 105].

To illustrate the stability issue arising in impulsive systems, consider a nonlinear system described by an ordinary differential equation $\dot{x}(t) = f(x(t))$, assuming without loss of generality that $f(0) = 0$. Recall that the trivial solution is said to be *Lyapunov stable*, if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that, if $\|x(0)\| < \delta$, then $\|x(t)\| < \varepsilon$ for every $t \geq 0$. Now consider two solutions of the impulsive Goodwin's oscillator. Let $(x^1(t), t_n^1)$ be a stable 1-cycle, and $(x^2(t), t_n^2)$ be the same solution with the perturbed initial conditions, so that $(x_0^1, t_0^1) \approx (x_0^2, t_0^2)$. However, the relations $x_0^1 \rightarrow x_0^2$ and $t_0^1 \rightarrow t_0^2$ do **not** imply $\|x^1(t) - x^2(t)\| \rightarrow 0$ as $t \rightarrow \infty$, see. Fig. 1.5 and Fig. 1.6). In other words, the inequality $\|x^1(t) - x^2(t)\| < \varepsilon$ is not fulfilled for any ε even for sufficiently small δ . Therefore, the convergence for pulse-modulated systems cannot be considered in Lyapunov sense within the standard analytical framework.

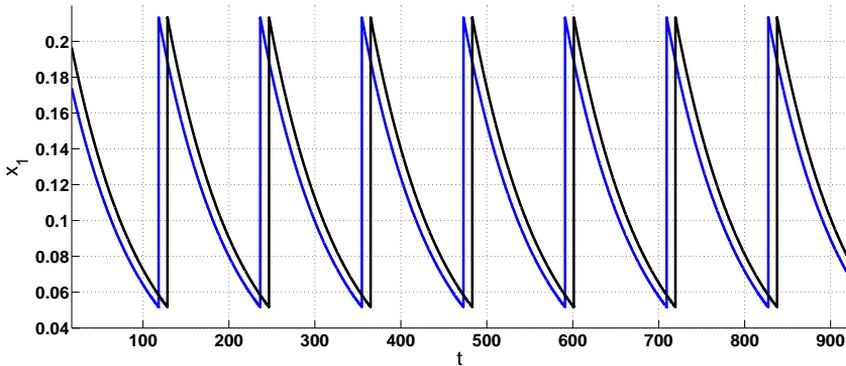


Figure 1.5: The impulsive Goodwin's oscillator. Transients in the continuous states of two 1-cycles with the initial conditions close to each other. The trajectory corresponds to the state that jumps.

Summary

Hybrid systems is a relatively young research field where the development has been predominately driven by theoretical interests. During the last decade, the tools and concepts of hybrid systems have found applications across many areas of engineering and science, including life sciences.

Dynamical processes resulting from interaction of continuous and discrete dynamics are often encountered in living organisms. Time evolutions of such processes constitute continuous variables that are subject to instant changes at discrete points of time. Usually, these discrete events cannot be observed directly and have to be reconstructed from the accessible for measurement continuous variables. Thus, the problem of hybrid state estimation from measurements of continuous outputs is important to and naturally arises in life sciences but, so far, scarcely covered in the existing literature.

The contribution of the present thesis is twofold. First, it proposes and studies in detail several observer structures for hybrid systems that are able to reconstruct discrete states from only continuous measurements. Second, the performance of the proposed observers applied to a model of non-basal testosterone regulation in the human male is carefully assessed.

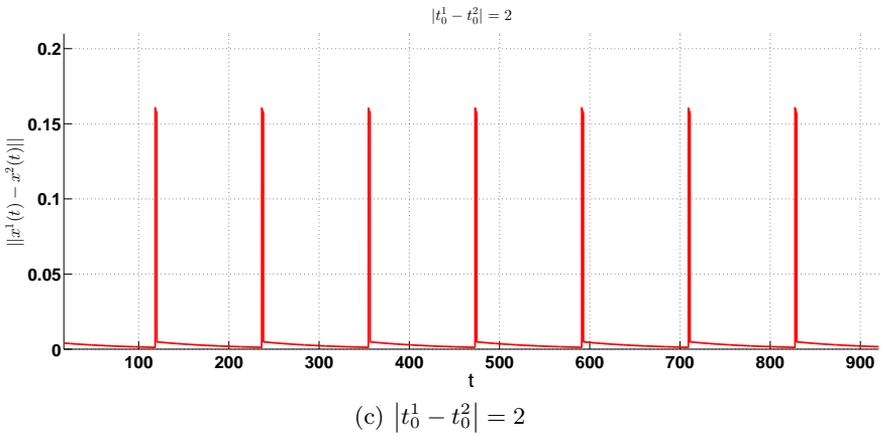
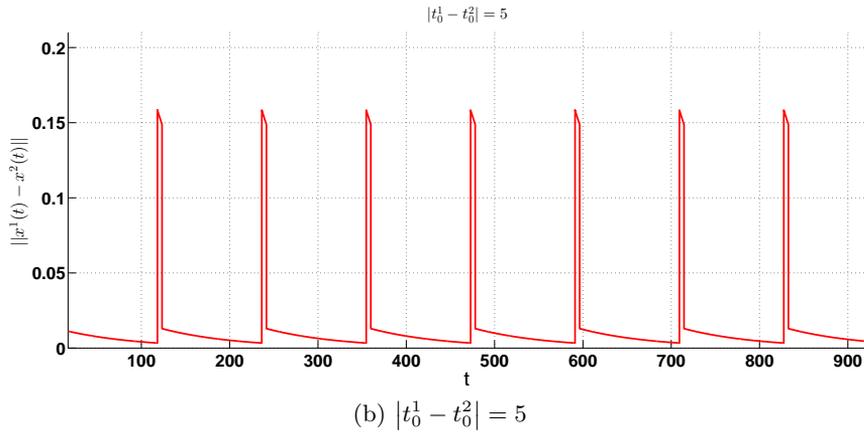
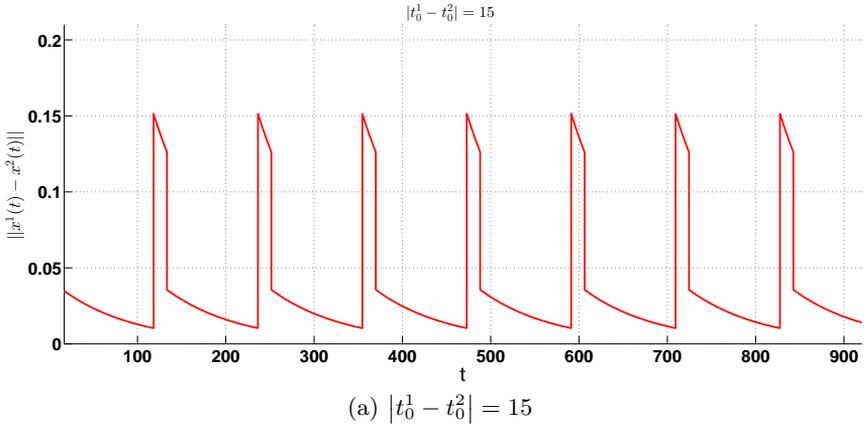


Figure 1.6: "Peaking phenomenon". $x_0^1 = x_0^2$. Relation $t_0^1 \rightarrow t_0^2$ does not imply $\|x^1(t) - x^2(t)\| \rightarrow 0$ as $t \rightarrow \infty$

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Paper I

Design Degrees of Freedom in a Hybrid Observer for a Continuous Plant under an Intrinsic Pulse-modulated Feedback

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Abstract

A hybrid observer for a linear time-invariant continuous plant under an intrinsic pulse-modulated feedback is considered. The firing times of the feedback representing the discrete state of the hybrid system to be observed are thus unknown. The observer possesses two feedback gains making use of the continuous output estimation error in order to correct the estimates of the continuous and discrete states, respectively. By driving the hybrid state estimation error to zero, the observer solves a synchronization problem between the firing times of the pulse-modulated feedback of the plant and that of the observer. Equivalence between a synchronous mode in the observer and zero output estimation error is proved. The influence of the observer design degrees of freedom on the observer performance is investigated by extensive numerical experiments. The introduction of the discrete observer gain clearly improves the observer performance for low-multiplicity periodic solutions in the plant but not for those with high multiplicity. In the latter case, sufficiently fast observer convergence is achieved even for a zero value of the discrete gain.

Introduction

Systems where continuous dynamics interact with discrete events in closed-loop manner appear often in biology and medicine, see [9]. A prominent example of such hybrid interaction comes from the field of neuroendocrinology that studies numerous interconnections of the endocrine and nervous systems. The hypothalamus controls secretion of pituitary gland hormones and exerts episodic feedback action on the relatively slow dynamics of the hormone kinetics, [7].

An impulsive model of testosterone regulation was proposed in [2] and analyzed in detail in [1]. Following the biological evidence, the model was based on the principles of pulse modulation, see e.g. [4]. It demonstrated a good agreement with clinical data [5, 11, 10] and provided an explanation to the experimentally observed complex dynamical phenomena in endocrine systems, including deterministic chaos [14]. In fact, the model lacks equilibria and appears to exhibit only periodic, quasiperiodic, and chaotic behaviors.

Unlike most of engineered systems where the generated control signal is typically known, the impacting pulsatile neural signal is not available for measurement and has to be estimated. Therefore, a special class of hybrid systems where discrete states are related to the modulation law and observed through continuous measurements has to be considered. The firing times of the pulse-modulated feedback can be portrayed as a discrete state variable in a hybrid system and estimated by a hybrid observer.

Theory of impulsive systems was extensively treated in the literature. However, the topic of observers for hybrid systems that are able to reconstruct discrete states from only continuous measurements are, to the best knowledge of the authors, currently lacking, with the exception of the one proposed in [3] and subsequently generalized by including a time delay in [12]. For a hybrid system with modulated jumps, a state observer is supposed to not only drive the continuous state estimation error to zero, but also synchronize the jump instants estimated by the observer with those of the plant. The main drawback of the observer in [3] is that only the continuous but not discrete state estimates are updated by means of an output estimation error feedback which results in slow convergence.

For desynchronized modulation in the plant and the observer, the hybrid state estimation error becomes significant. This causes an additional and unnecessary transient in the continuous part. The idea advocated in [13] is to use the feedback action for the calculation of the impulse firing times in both continuous and discrete parts of the observer. Significant enhancement of the observer transient performance relative to [3], both in convergence time and accuracy, has resulted from this structural improvement.

The theoretical contribution of the present paper with respect to the results of [13] is twofold. First, it is demonstrated that the problem of synchronizing the firings of the plant pulse-modulated feedback with those of the observer is equivalent to a zero output estimation error of the observer. Second, the smoothness and stability properties of the discrete mapping governing the observer dynamics from one firing instant to the next one are worked out in detail.

The paper is composed as follows. First, the equations of the mathematical model (i.e. the plant) in hand are recapitulated. Then, the observer introduced in [3] along with the underlying mathematical theory necessary for observer performance analysis are briefly reviewed. Further, an improved observer structure proposed in [13] and exploiting a discrete gain in addition to the continuous one of the original observer from [3] is analyzed. Extensive numerical examples illustrating the effect of the discrete gain on the observer performance are presented.

System equations

Consider a plant governed by the hybrid system comprised of a continuous linear part

$$\dot{x}(t) = Ax(t), \quad z(t) = Cx(t), \quad y(t) = Lx(t), \quad (1)$$

and a discrete part

$$\begin{aligned} x(t_n^+) &= x(t_n^-) + \lambda_n B, & t_{n+1} &= t_n + T_n, \\ T_n &= \Phi(z(t_n)), & \lambda_n &= F(z(t_n)). \end{aligned} \quad (2)$$

Here $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x}$, $C \in \mathbb{R}^{1 \times n_x}$, $L \in \mathbb{R}^{n_y \times n_x}$, z is the scalar controlled output, y is the vector measurable output, and x is the state vector of (1). The minus and plus in the superscript denote a left-sided and a right-sided limits, respectively. The matrix A is Hurwitz stable, the matrix pair (A, L) is observable, and the relationships

$$CB = 0, \quad LB = 0 \quad (3)$$

apply. The functions $\Phi(\cdot)$ and $F(\cdot)$ are smooth, strictly monotonic and bounded

$$\Phi_1 \leq \Phi(\cdot) \leq \Phi_2, \quad F_1 \leq F(\cdot) \leq F_2, \quad (4)$$

where Φ_1, Φ_2, F_1, F_2 are strictly positive numbers.

The elements of the vector $x(t)$ in (1) undergo jumps at the times $t = t_n$. However, due to (3), the outputs $y(t), z(t)$ are still continuous. It is known from [1] that all the solutions of system (1), (2) are bounded and the system does not have equilibria, i.e. all the existing attractors are hidden (see [8]).

The hybrid plant is subject to unknown initial conditions $(x(0), t_0)$ and the first firing instant of the pulsatile feedback occurs after the initial time instant, $t_0 \geq 0$. To study the hybrid system dynamics of (1), (2), the initial conditions are specified as $(x(t_0^-), t_0)$.

State Observer with Hybrid Correction

The purpose of observation in hybrid closed-loop system (1), (2) is to produce estimates $(\hat{t}_n, \hat{\lambda}_n)$ of the jump parameters (t_n, λ_n) . Given the sequence $(t_n, \lambda_n), n = 0, 1, \dots$, estimates of the state vector x of the continuous part can be obtained by continuous state estimation techniques as the discrete state variables are readily known. The main challenge of hybrid observation is to ensure asymptotic convergence of the sequence $\{\hat{t}_n\}$ to $\{t_n\}$, i.e. to synchronize jumps in the observer with those of the plant.

In order to estimate the state vector of (1), (2), a hybrid observer is introduced in [3] as follows:

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + K(y(t) - \hat{y}(t)), & \hat{y}(t) &= L\hat{x}(t), \\ \hat{z}(t) &= C\hat{x}(t), & \hat{x}(t_n^+) &= \hat{x}(t_n^-) + \hat{\lambda}_n B, \\ \hat{t}_{n+1} &= \hat{t}_n + \hat{T}_n, & \hat{\lambda}_n &= F(\hat{z}(\hat{t}_n)), \end{aligned} \quad (5)$$

and

$$\hat{T}_n = \Phi(\hat{z}(\hat{t}_n)). \quad (6)$$

Here K is a continuous feedback gain chosen to render $D = A - KL$ Hurwitz. Without loss of generality, it is assumed that $\hat{t}_0 \geq t_0$.

Since the state vector $x(t)$ undergoes jumps at certain times, the closeness of $x(t)$ and $\hat{x}(t)$ cannot be ensured for all t . Indeed, suppose that \hat{t}_n and t_n are close, but do not coincide exactly, and the vector $\hat{x}(t)$ has jumps at \hat{t}_n . Let $t_n < \hat{t}_n$ for definiteness. Then if $t_n < t < \hat{t}_n$, the vector $x(t)$ already has jumps, while the vector $\hat{x}(t)$ does not. Thus $x(t)$ and $\hat{x}(t)$ can differ significantly in such time intervals. However, the closeness of $x(t)$ and its estimate $\hat{x}(t)$ can be ensured in the sense that there exists a constant integer $a > 0$ depending on initial conditions and such that $\hat{t}_n - t_{n+a} \rightarrow 0$ and $\|\hat{x}(\hat{t}_n^-) - x(t_{n+a}^-)\| \rightarrow 0$ as $n \rightarrow +\infty$.

The main issue with observer (5), (6) is slow convergence because only the continuous but not discrete state estimates are updated by the output estimation error feedback. It is therefore suggested in [13] to substitute (6) with

$$\hat{T}_n = \Phi(\hat{z}(\hat{t}_n) + K_f(\hat{y}(\hat{t}_n) - y(\hat{t}_n))). \quad (7)$$

In contrast with (5), (6) the observer above has two feedback gain matrices: $K_f \in \mathbb{R}^{1 \times n_y}$ is a constant gain matrix in the discrete part of the observer while K is still chosen to render $D = A - KL$ Hurwitz.

The novelty of observer (5), (7) is thus in the additional gain K_f explicitly making the firing times \hat{T}_n influenced by the output estimation error. With $K_f = 0$ observer (5), (7) is identical to (5), (6).

Stability analysis of synchronous mode

Let $(x(t), t_n)$ be a solution of plant equations (1), (2) with the parameters λ_k, T_k , and $x_k = x(t_k^-)$. Suppose that the hybrid state $(x(t), t_n)$ is reconstructed by means of observer (5), (7). Without loss of generality, assume that $t_0 \leq \hat{t}_0 < t_1$.

Obviously, the solution $(\hat{x}(t), \hat{t}_n)$ of observer equations (5), (7) subject to the initial conditions $\hat{t}_0 = t_0, \hat{x}(\hat{t}_0^-) = x(t_0^-)$, yields $\hat{x}(t) = x(t)$ for all $t \geq t_0$. Such a solution $(\hat{x}(t), \hat{t}_n)$ is called a *synchronous mode* of observer (5), (7) with respect to $(x(t), t_n)$, see [3].

A synchronous mode with respect to $(x(t), t_n)$ will be called *locally asymptotically stable* (see [3]) if for any solution $(\hat{x}(t), \hat{t}_n)$ of (5), (7) such that the initial estimation errors $|\hat{t}_0 - t_0|$ and $\|\hat{x}(\hat{t}_0^-) - x(t_0^-)\|$ are sufficiently small, it follows that $\hat{t}_n - t_n \rightarrow 0$ and $\|\hat{x}(\hat{t}_n^-) - x(t_n^-)\| \rightarrow 0$ as $n \rightarrow \infty$. The latter implies $\hat{\lambda}_n - \lambda_n \rightarrow 0$ as $n \rightarrow \infty$.

Proposition 1. *The following statements are equivalent*

1. $(\hat{x}(t), \hat{t}_n)$ is a synchronous mode with respect to $(x(t), t_n)$;
2. $\hat{y}(t) \equiv y(t)$ for all $t \geq t_0$.

Proof. Obviously (1) implies (2). Prove now the converse statement. Since the matrix pair (A, L) is observable, the matrix pair (D, L) is also observable (see [6], ex.3.3-5). Denote $\Delta(t) = x(t) - \hat{x}(t)$. The elements of $\Delta(t)$ can experience jumps at the times $t = t_n$ and $t = \hat{t}_n, n = 0, 1, \dots$, while at the other points $\Delta(t)$ satisfies the linear differential equation $\dot{\Delta} = D\Delta(t)$. From $y(t) \equiv \hat{y}(t)$ it follows that $L\Delta(t) \equiv 0$ for all $t \geq t_0$. Thus $LD^k\Delta(t) \equiv 0$ for any $k \geq 1$ at any point t , where $\Delta(t)$ is continuous. Then observability of the matrix pair (D, L) implies that $\Delta(t)$ has no jumps and $\Delta(t) \equiv 0$ for all $t \geq t_0$. \square

Of special interest is observing an m -periodic sequence of the sampled plant states (an m -cycle) with $x_{n+m} = x_n, \lambda_{n+m} = \lambda_n, T_{n+m} = T_n$, where $m \geq 1$ is some integer. Then the sampled states of the corresponding synchronous mode are also m -periodic.

By local stability analysis of (5), (6) it was shown in [3] that with a proper observer gain, one can obtain an asymptotically converging estimate of the plant states and a synchronization between the impulse sequences of the plant and that of the observer.

To carry out a similar analysis of observer (5), (7), derive a discrete mapping for the evolution of new observer state.

$$(\hat{x}(\hat{t}_n^-), \hat{t}_n) \mapsto (\hat{x}(\hat{t}_{n+1}^-), \hat{t}_{n+1}). \quad (8)$$

Denote for brevity $\hat{x}_n = \hat{x}(\hat{t}_n^-)$.

For any $(\zeta, \tau) \in \mathbb{R}^{n_x} \times \mathbb{R}$, select integers r and s , $r \leq s$, such that $t_r \leq \tau < t_{r+1}$, $t_s \leq \tau + \Phi(\alpha(\zeta, \tau)) < t_{s+1}$. Define

$$\alpha(\zeta, \tau) = R\zeta - K_f y(\tau), \quad R = C + K_f L,$$

and $P(\zeta, \tau) = P_{r,s}(\zeta, \tau)$ with

$$\begin{aligned} P_{r,s}(\zeta, \tau) = & e^{A(\tau + \Phi(\alpha(\zeta, \tau)) - t_s)} x(t_s^+) \\ & + e^{D\Phi(\alpha(\zeta, \tau))} \left[\zeta + F(C\zeta)B - e^{A(\tau - t_r)} x(t_r^+) \right] \\ & - \sum_{k=r+1}^s \lambda_k e^{D(\tau + \Phi(\alpha(\zeta, \tau)) - t_k)} B. \end{aligned}$$

Theorem 1. *Pointwise mapping (8) for observer (5), (7) is given by*

$$\hat{x}_{n+1} = P(\hat{x}_n, \hat{t}_n), \quad \hat{t}_{n+1} = \hat{t}_n + \Phi(\alpha(\hat{x}_n, \hat{t}_n)). \quad (9)$$

Proof. Consider the difference $\Delta(t) = x(t) - \hat{x}(t)$ in the interval $(\hat{t}_n, \hat{t}_{n+1})$. Obviously, $\Delta(t)$ satisfies $\dot{\Delta} = D\Delta(t)$ at all the points t , where Δ has no jumps. The function Δ has jumps $\Delta(t^+) - \Delta(t^-) = \lambda_k B$ at the points $t = t_k$, $r + 1 \leq k \leq s$. Thus one concludes that

$$\Delta(\hat{t}_{n+1}^-) = e^{D\hat{T}_n} \Delta(\hat{t}_n^+) + \Delta_{r,s}, \quad (10)$$

where

$$\Delta_{r,s} = \sum_{k=r+1}^s \lambda_k e^{D(\hat{t}_{n+1} - t_k)} B.$$

Then (10) can be rewritten as

$$\hat{x}_{n+1} = x(\hat{t}_{n+1}^-) + e^{D\hat{T}_n} (\hat{x}_n + \hat{\lambda}_n B - x(\hat{t}_n^+)) - \Delta_{r,s}. \quad (11)$$

Since

$$x(\hat{t}_{n+1}^-) = e^{A(\hat{t}_{n+1} - t_s)} x(t_s^+), \quad x(\hat{t}_n^+) = e^{A(\hat{t}_n - t_r)} x(t_r^+),$$

equality (11) implies (9). □

Since the pair (A, L) is observable, the matrix K can always be chosen in such a way that the matrix $D = A - KL$ has arbitrary pre-defined eigenvalues. It becomes apparent from (9) and the definition of $\alpha(\cdot, \cdot)$ that the mapping P boasts an additional, compared to the static feedback observer in [3], design degree of freedom K_f that directly influences the discrete dynamics of (5), (7). In the original observer, the convergence of the hybrid state estimation error is safeguarded by the convergence of the continuous part, resulting in a slow synchronization between the jumps. Despite the modification in the discrete part of the observer, the pointwise mapping characterizing its dynamics is still continuous.

Theorem 2. *The mapping $P(\zeta, \tau)$ is smooth.*

Proof. Since

$$x(t_k^-) = e^{A(t_k - t_{k-1})} x(t_{k-1}^-), \quad k \geq 1,$$

it is straightforward to see that

$$P_{r,s} - P_{r-1,s} = \lambda_r e^{D\Phi} \left[e^{D(\tau - t_r)} - e^{A(\tau - t_r)} \right] B, \quad (12)$$

$$P_{r,s} - P_{r,s-1} = \lambda_s \left[e^{A(\tau + \Phi - t_s)} - e^{D(\tau + \Phi - t_s)} \right] B, \quad (13)$$

for $r \geq 1$, $s \geq 1$. Here the values of the functions $P_{i,j}$ are taken at (ζ, τ) and the function Φ is taken at $\alpha(\zeta, \tau)$. Obviously, the function $P(\zeta, \tau)$ can have gaps only on the surfaces in the space (ζ, τ) , where either $\tau = t_r$ or $\tau + \Phi(\zeta, \tau) = t_s$ for some r, s . Yet, from (12), (13), it follows that

$$P_{r,s} - P_{r-1,s}|_{\tau=t_r} = 0, \quad P_{r,s} - P_{r,s-1}|_{\tau+\Phi=t_s} = 0,$$

and the function P is continuous everywhere. Moreover, (3) implies $(D - A)B = 0$. The latter fact together with (12), (13) leads to the partial derivatives of P in ζ and in τ being also continuous. \square

The sampled at the pulse modulation firing times hybrid states that correspond to the synchronous mode are $\hat{t}_n = t_n$, $\hat{x}_n = \hat{x}(\hat{t}_n^-) = x(t_n^-)$. Being the observer states, they satisfy (9). Thus local stability of such a solution can be derived from that of discrete time system (9).

By Theorem 2, the discrete mapping (9) is smooth, so local stability of an m -periodic solution of (9) can be studied considering the eigenvalues of a monodromy matrix, i.e. of the matrix product $J_{m-1} \dots J_0$, where J_k is the Jacobi matrix of mapping (9) calculated at the point (x_k, t_k) , $k = 0, \dots, m-1$. Technical details can be found in [3].

The following theorem gives a sufficient local stabilizability condition for the state estimation error in the observer by a constant gain matrix K , for a given constant matrix K_f .

Theorem 3. *Let $(x(t), t_n)$ be an m -cycle. Suppose*

$$-1 < \prod_{k=0}^{m-1} (\Phi'_k R A x_{k+2} + (1 - \Phi'_k K_f L A x_k)) < 1. \quad (14)$$

Then there exists a matrix K such that $D = A - KL$ is Hurwitz and the matrix product $J_0 \dots J_{m-1}$ is Schur stable.

Proof. It can be easily checked that for all $k \geq 0$

$$J_k = \begin{bmatrix} (J_k)_{11} & (J_k)_{12} \\ (J_k)_{21} & (J_k)_{22} \end{bmatrix}$$

comprised of the following blocks

$$\begin{aligned} (J_k)_{11} &= \Phi'_k A x_{k+1} R + e^{D\Phi(Cx_k)} (I_{n_x} + F'_k B C), \\ (J_k)_{12} &= A x_{k+1} (1 - \Phi'_k K_f L A x_k) \\ &\quad - e^{D\Phi(Cx_k)} (A(x_k + \lambda_k B)), \\ (J_k)_{21} &= \Phi'_k R, \\ (J_k)_{22} &= 1 - \Phi'_k K_f L A x_k. \end{aligned}$$

The matrix J_k can be decomposed as

$$J_k = \tilde{J}_k + W_k(D),$$

where

$$\begin{aligned} \tilde{J}_k &= \begin{bmatrix} \Phi'_k A x_{k+1} R & A x_{k+1} (1 - \Phi'_k K_f L A x_k) \\ \Phi'_k R & 1 - \Phi'_k K_f L A x_k \end{bmatrix} \\ &= \begin{bmatrix} A x_{k+1} \\ 1 \end{bmatrix} \begin{bmatrix} \Phi'_k R & 1 - \Phi'_k K_f L A x_k \end{bmatrix} \end{aligned}$$

and

$$W_k(D) = \begin{bmatrix} e^{DT_k} \\ 0 \end{bmatrix} \begin{bmatrix} I_{n_x} + F'_k B C & -A(x_k + \lambda_k B) \end{bmatrix}.$$

The elements of e^{DT_k} , $k = 0, 1, \dots, m-1$, and hence those of $W_k(D)$, can be made arbitrarily small by choosing K . At the same time,

$$\begin{aligned} \tilde{J}_0 \dots \tilde{J}_{m-1} &= \begin{bmatrix} A x_1 \\ 1 \end{bmatrix} \begin{bmatrix} \Phi'_{m-1} R & 1 - \Phi'_{m-1} K_f L A x_{m-1} \end{bmatrix} \\ &\quad \times \prod_{k=0}^{m-2} (\Phi'_k R A x_{k+2} + (1 - \Phi'_k K_f L A x_k)), \end{aligned}$$

where $x_0 = x_m$ and $x_1 = x_{m+1}$. Thus $\text{rank}(\tilde{J}_0 \dots \tilde{J}_{m-1}) \leq 1$ and the matrix product has only one eigenvalue that may be non-zero, that is the one equal to

$$\text{tr}(\tilde{J}_0 \dots \tilde{J}_{m-1}) = \prod_{k=0}^{m-1} (\Phi'_k R A x_{k+2} + (1 - \Phi'_k K_f L A x_k)).$$

If (14) is satisfied, the matrix $\tilde{J}_0 \dots \tilde{J}_{m-1}$ is Schur stable. Thus, for sufficiently small elements of e^{DT_k} , the matrix $J_0 \dots J_{m-1}$ is also Schur stable. \square

Notice that in (14) $x_m = x_0$ and $x_{m+1} = x_1$. In the case of a 1-cycle, (14) turns into

$$-2 < \Phi'_0 C A x_0 < 0, \quad (15)$$

and, in the case of 2-cycle, (14) becomes

$$-1 < (\Phi'_0 C A x_0 + 1)(\Phi'_1 C A x_1 + 1) < 1 \quad (16)$$

while in the case of 4-cycle, (14) is as follows

$$\begin{aligned} -1 < & (\Phi'_0 R A x_2 + (1 - \Phi'_0 K_f L A x_0)) \times \\ & (\Phi'_1 R A x_3 + (1 - \Phi'_1 K_f L A x_1)) \times \\ & (\Phi'_2 R A x_0 + (1 - \Phi'_2 K_f L A x_2)) \times \\ & (\Phi'_3 R A x_1 + (1 - \Phi'_3 K_f L A x_3)) < 1. \quad (17) \end{aligned}$$

Thus, the influence of K_f on the validity of stabilizability condition (14) is more prominent for the periodic solutions of higher multiplicity. Yet, the conditions are not shown to be necessary.

Numerical example

To demonstrate the impact of an additional modulated feedback of the output estimation error on the observer performance, consider an illustrative numerical example:

Assume the following values in model (1), (2)

$$A = \begin{bmatrix} -b & 0 & 0 \\ 1.5 & -0.15 & 0 \\ 0 & 2.8 & -0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (18)$$

$$C = [0 \quad 0 \quad 1],$$

$$\Phi(z) = 40 + 80 \frac{(z/2.7)^2}{1 + (z/2.7)^2}, \quad F(z) = 0.05 + \frac{5}{1 + (z/2.7)^2}. \quad (19)$$

Here b can take the values $b_1 = 0.005$, $b_2 = 0.06$, $b_3 = 0.045$, that, respectively, result in a stable 1-cycle, 2-cycle and 4-cycle of the hybrid plant. These cycles are obtained via frequency doubling bifurcations of b .

Zero discrete gain K_f : For

$$K = \begin{bmatrix} 0 & 1.2 & 0 \\ 0 & 0 & 1.2 \end{bmatrix}^T$$

and $b = b_i$, $i = 1, 2, 3$, defined as above, the corresponding matrices $D = A - KL$ are Hurwitz stable. Let $K_f = [0 \ 0]$. Then the spectral radii of the corresponding monodromy $J_0 \cdot \dots \cdot J_{m-1}$ matrices for each of the three choices of $b = b_i$, $i = 1, 2, 3$ amount to:

$$\rho_1 = 0.994, \quad \rho_2 = 0.803, \quad \rho_3 = 0.060$$

rendering all $J_0 \cdot \dots \cdot J_{m-1}$ to be Schur stable (i.e., all its eigenvalues are strictly inside the unit circle). Therefore, stable synchronous modes is imposed on both observers by the choice of the observer continuous gain.

Hybrid observer performance can be measured in numerous ways. The convergence to a synchronous mode is characterized here by the first time instant when \hat{t}_n comes into ε_f -neighborhood of t_n and never leaves it:

$$\mathcal{P}(\varepsilon_f) = \hat{t}_{n^*},$$

$$\text{where } n^* = \min\{k : |\hat{t}_N - t_N| < \varepsilon_f \ \forall N > k\}. \quad (20)$$

Optimized discrete gain K_{fi} : A search for the values of K_{fi} that minimize the spectral radius of the monodromy matrix $J_0 \cdot \dots \cdot J_{m-1}$ yields $K_{f1} = [-547 \ -892]$, $K_{f2} = [-190 \ -215]$, $K_{f3} = [-0.6 \ -0.1]$. The corresponding spectral radii of the monodromy matrices are:

$$\rho_1 = 0.770, \quad \rho_2 = 0.391, \quad \rho_3 = 0.0008.$$

For optimized K_f , the observer convergence is much faster than for $K_f = 0$. Naturally, the selected value of the threshold ε_f has impact on criterion (20). To illustrate this effect, the plots of the functions $\mathcal{P}(\varepsilon_f)$ for the two considered choices of K_f are provided in Fig. 1–3 clearly demonstrating the superiority of the improved observer: the discrete gain (red lines) is beneficial for both convergence and accuracy of the state estimate for all ε_f .

The role of the plant cycle multiplicity: As becomes evident from Theorem 3, the exact knowledge of the periodic solution in the hybrid plant is necessary in order to guarantee local stability of the observer. A logical

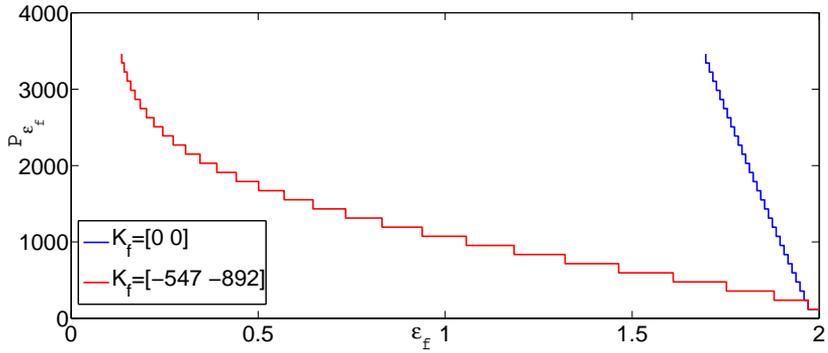


Figure 1: Settling time $\mathcal{P}(\varepsilon_f)$ as a function of the threshold ε_f for $K_f = [0 \ 0]$ and $K_f = [-547 \ -892]$ in case of a 1-cycle.

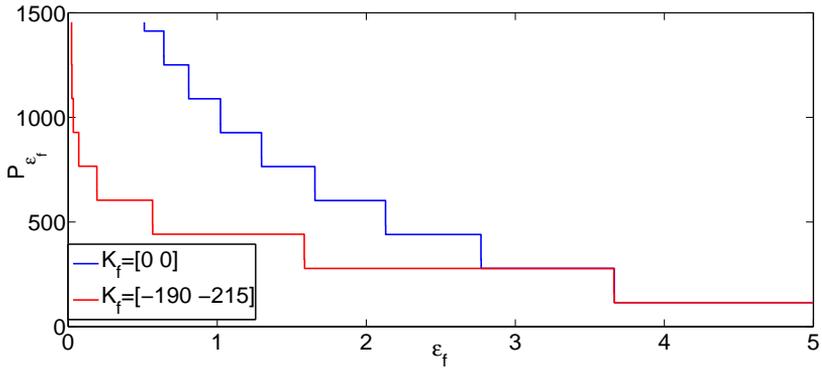


Figure 2: Settling time $\mathcal{P}(\varepsilon_f)$ as a function of the threshold ε_f for $K_f = [0 \ 0]$ and $K_f = [-190 \ -215]$ in case of a 2-cycle.

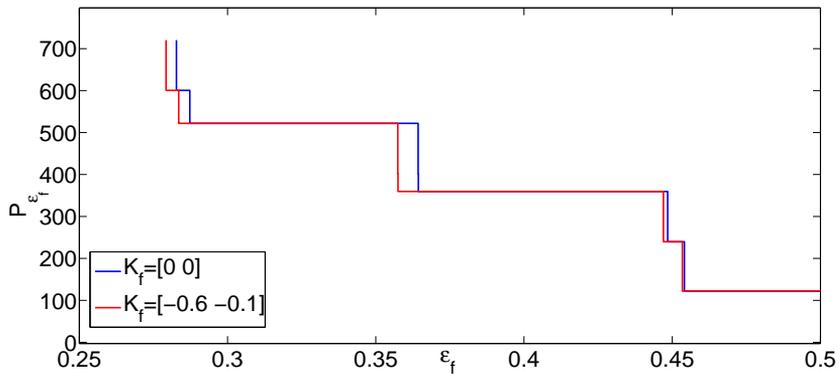


Figure 3: Settling time $\mathcal{P}(\varepsilon_f)$ as a function of the threshold ε_f for $K_f = [0 \ 0]$ and $K_f = [-0.6 \ -0.1]$ in case of a 4-cycle.

question is whether or not a gain K_f suitable for a number of periodic solutions can be found. By numerical analysis, it was concluded that the introduction of the discrete feedback gain has greater impact on the observer convergence in the case of lower multiplicity of periodical solutions. This conclusion is counterintuitive given the stabilizability conditions in (15), (16). Unlike cycles of a higher order, where a satisfactory convergence is typically achieved already with $K_f = 0$, the optimal choice of the gain in the discrete part allows to significantly reduce the transients in a low-order case (cf. Table 1).

	$K_f = [0 \ 0]$	Opt. K_f
1-cycle	0.9944	0.7702
2-cycle	0.8033	0.3910
4-cycle	0.0600	0.0008

Table 1: Spectral radii of the monodromy matrices for $b_1 = 0.005$ (1-cycle), $b_2 = 0.06$ (2-cycle), $b_3 = 0.045$ (4-cycle) and different values of K_f .

The mechanism behind better convergence of the observer for higher-order cycles in the plant is plausibly the same as for the exogenous excitation phenomena in system identification. The plant output signal under a 1-cycle or 2-cycle does not provide much information about the state vector of hybrid plant (18)–(19) during a few periods of the plant solution. Therefore, a significant number of periods is needed for the observer to converge. Notably, for a higher-order cycle, it is still possible to decrease the spectral radius of the monodromy matrix for say an order of magnitude. However, this does not translate into any notable improvement in the observer convergence.

Impact of K on the observer convergence. Since the improvement of the observer convergence by the introduction of the discrete gain is more noticeable in case of a low multiplicity periodical solution in the plant, the

impact of $K = \begin{bmatrix} 0 & k_1 & 0 \\ 0 & 0 & k_2 \end{bmatrix}^T$ on the observer convergence in case of

1-cycle is illustrated with fixed discrete gain on Fig. 4 ($K_f = [0 \ 0]$) and Fig. 5 ($K_f = [-500 \ -300]$).

From the proof of Theorem 3 and (15), it follows that the dependence of the spectral radius on K_f vanishes with simultaneous increase in k_1 and k_2 . This is confirmed by Fig. 4 and Fig. 5. At the same time, for a fixed nonzero

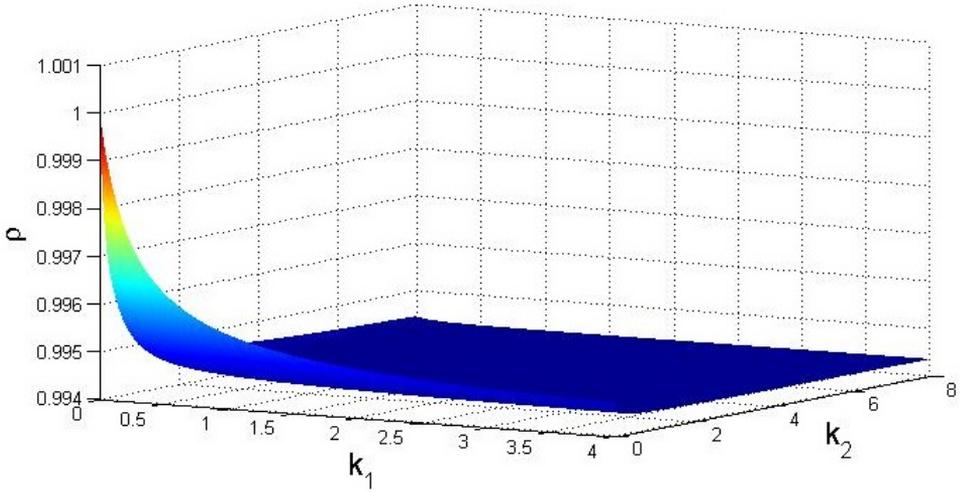


Figure 4: The dependence of the spectral radius of the monodromy matrix on the choice of the gain K for a fixed $K_f = [0 \ 0]$.

K_f , an optimal value of the spectral radius can be obtained for moderate values k_1 and k_2 (see Fig. 5 and Fig. 6).

The numerical experiments indicate that for better estimation performance the gains k_1 and k_2 should not differ significantly, i.e. it is beneficial to select k_1 and k_2 close to each other.

Conclusion

A recently suggested hybrid observer for continuous linear time-invariant systems under an intrinsic pulse-modulated feedback is considered. The observer makes use of correction terms in both the discrete and continuous parts of the plant model. It is proved that a zero output residual signal is equivalent to a synchronized mode of the observer. The impact of the discrete-time observer gain on the convergence of the hybrid state estimation error is investigated. While the introduction of the discrete gain is clearly beneficial to the observer convergence under a low multiplicity periodical solution in the plant, it does not significantly improve the observer performance for high multiplicity solutions. This presents a challenge in observer design when the type of the plant solution is not known beforehand.

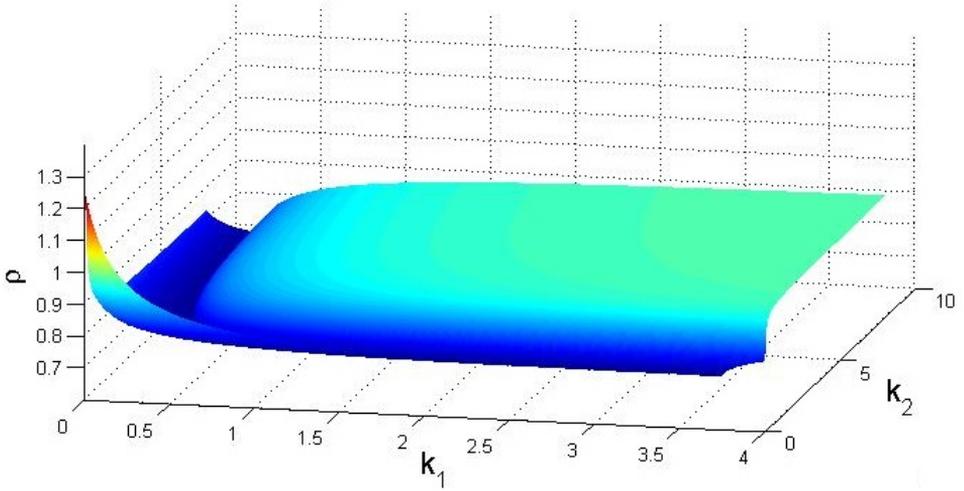


Figure 5: The dependence of the spectral radius of the monodromy matrix on the choice of the gain K for a fixed $K_f = [-500 \ -300]$.

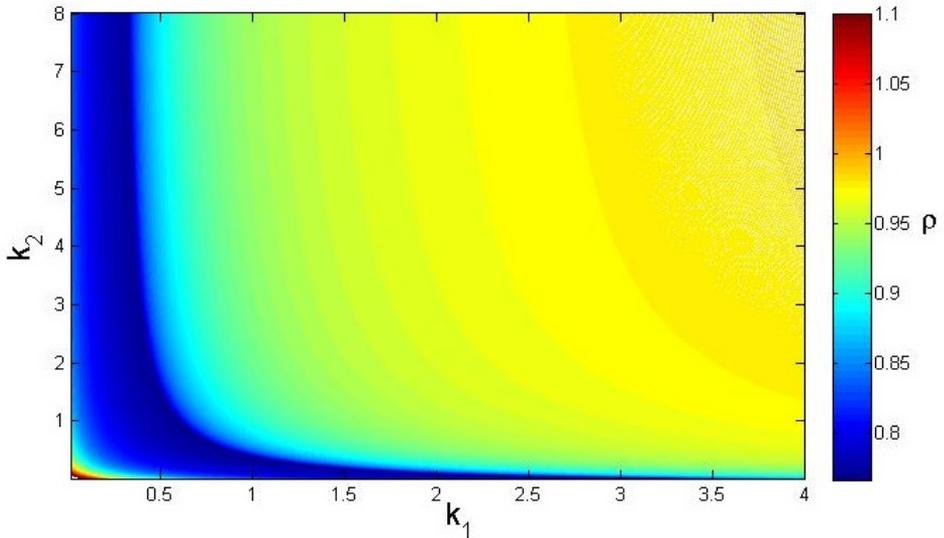


Figure 6: Contour map for the dependence of the spectral radius of the monodromy matrix on the choice of the gain K for a fixed $K_f = [-500 \ -300]$.

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Paper II

Design of a Hybrid Observer for an Oscillator with an Intrinsic Pulse-modulated Feedback

Diana Yamalova and Alexander Medvedev

Abstract

A hybrid observer estimating the states of an oscillating system composed of a linear chain structure and an intrinsic pulse-modulated feedback is considered. The observed plant corresponds to an intensively studied mathematical model of testosterone regulation in the male. The observer reconstructs the continuous states of the model describing the concentrations of the involved hormones as well as the firing times and weights of the feedback impulses. The pulse-modulated feedback is intrinsic and no measurements of the discrete part of the plant are available to the observer. The observer design is based on assigning, through the output error feedback gains, a guaranteed convergence rate to the local dynamics of a synchronous mode. A Poincaré mapping capturing the propagation of the continuous plant and observer states through the discrete cumulative sequence of the feedback firing instants is derived. Local stability properties of the synchronous mode are related to the spectral radius of the Jacobian of the mapping.

Introduction

Hybrid models, where continuous-time dynamics are admixed with discrete events, find a rapidly growing number of applications in life science and medicine [13],[1]. Complex nonlinear phenomena arising in such mathematical constructs are often too challenging to study in general terms and finding subclasses of hybrid systems that lend themselves to analytical treatment is an important task.

Periodicity (rhythmicity) is a prominent feature in living organisms and is captured mathematically in the concept of a dynamical oscillator. For instance, Kuramoto oscillator [12] and Goodwin's oscillator [8, 7] are popular modeling paradigms for smooth periodic, quasi-periodic, and chaotic behaviors in neuroscience and biochemistry [9, 6, 2, 5].

In endocrine regulation, neural processes interact with the hormone kinetics thus giving rise to hybrid models with relatively slow continuous dynamics that are controlled through impulsive action of firing neurons [10, 11, 18]. A relatively simple example of such a system is testosterone regulation in the male that has been intensively studied by means of a specialized version of Goodwin's oscillator called the Smith model [15, 16]. The mechanism of episodic secretion of the release hormones produced in the hypothalamus is typically described as pulse modulation, leading to a hybrid model called the impulsive Goodwin's oscillator [3].

The impulsive Goodwin's oscillator is characterized by two main features: a cascade structure of the continuous part of the model and a frequency and amplitude pulse-modulated feedback. While most of the continuous nonlinear oscillators produce self-sustained periodical solutions through Andronov-Hopf equilibrium-destabilizing bifurcations, the impulsive Goodwin's oscillator does not possess equilibria at all [3, 21]. Interestingly, the model has also been shown to describe actual endocrine data quite well, [14]. The pulse-modulated feedback of the impulsive Goodwin's oscillator is intrinsic, i.e. inaccessible for measurement, and poses a specific and seldom addressed in control theory problem of estimating discrete states of a hybrid system from only continuous measurements. This problem is more common in biomedical applications as measurements of discrete events are often not accessible in living organisms.

An observer making use of a single continuous output feedback for reconstructing the firing times and weights of the feedback pulses in the impulsive Goodwin's oscillator is proposed in [3]. For observer design, the hybrid state estimation problem is recast as a synchronization problem for the impulsive sequence of the plant and that of the observer. While being completely functional and able to handle both periodic and chaotic solutions in the plant, this observer structure suffered from slow convergence, especially in the case

of periodic solutions of low multiplicity. To enhance observer convergence, an additional feedback of the (continuous) output estimation error is proposed in [19]. Despite encouraging qualitative insights into the roles of the continuous and discrete feedback supported by simulation results, the latter publication has not produced a complete observer design approach.

The present paper extends further the synchronization approach to hybrid observation by examining in detail a special case of state estimation in the impulsive Goodwin's oscillator. This is in contrast with classical observer design, where the observer structure is derived from an observability notion. The same observer structure as in [19] is considered and local stability of the synchronous mode is secured through assigning the spectral radius of the Jacobian of a Poincaré mapping. A downside of this concept is that only local convergence results can be obtained. The analysis is focused on the critical case of a 1-cycle that is known from [4] to result in sluggish observer convergence.

The main contribution of this work is a tool for achieving fast hybrid observer convergence by coordinating its continuous and discrete parts. Further, useful connections between the cascade structure of the continuous part of the plant and the structure of the continuous observer gain matrix are revealed. Simulation of the designed observer demonstrates by far superior convergence compared to that in [4].

The rest of the paper is composed as follows. First, the mathematical model of testosterone regulation given by the impulsive Goodwin's oscillator is briefly summarized. Then, the observer structure in hand is revisited. Further, local stability of the synchronous observer mode is analyzed and shown to be guaranteed by a Jacobian matrix being Schur-stable. A method for achieving a certain pre-assigned spectral radius of the Jacobian by means of selecting the continuous and discrete gains of the observer is proposed. Finally, numerical simulations illustrating the theoretical contribution of the paper are provided.

Mathematical model of testosterone regulation

Consider an impulsive mathematical model of testosterone (Te) regulation in the male with non-basal secretion. Besides Te, it involves two hormones: gonadotropin-releasing hormone (GnRH) and luteinizing hormone (LH). GnRH is released from the hypothalamus of the brain in pulsatile fashion with short latency. Reaching the pituitary gland, GnRH initiates the production of LH, which in turn stimulates production of Te in the testes. Finally, both the GnRH outflow and the LH secretion are subject to feedback inhibition by Te [17].

On the continuity intervals $t_n < t < t_{n+1}$, $n = 0, 1, 2, \dots$, the model is given by the three ordinary differential equations

$$\begin{aligned} \dot{x}_1(t) &= -b_1x_1(t), \\ \dot{x}_2(t) &= g_1x_1(t) - b_2x_2(t), \\ \dot{x}_3(t) &= g_2x_2(t) - b_3x_3(t). \end{aligned} \tag{1}$$

The continuous state variables correspond to the (bloodstream) concentrations of GnRH – x_1 , LH – x_2 , and Te – x_3 . Here b_1, b_2, b_3, g_1 , and g_2 are positive parameters, reflecting the kinetics of the involved hormones.

The action of the hypothalamic GnRH neurons can be described by a pulse element (pulse modulator) performing amplitude and frequency modulation, with the Te concentration as a modulation signal. Thus the discrete part of the model is given by

$$\begin{aligned} x_1(t_n^+) &= x_1(t_n^-) + \lambda_n, & x_2(t_n^+) &= x_2(t_n^-), \\ x_3(t_n^+) &= x_3(t_n^-), & t_{n+1} &= t_n + T_n, \\ T_n &= \Phi(x_3(t_n)), & \lambda_n &= F(x_3(t_n)). \end{aligned} \tag{2}$$

The minus and plus in a superscript denote a left-sided and a right-sided limits, respectively, at the time instant t_n . Without loss of generality assume $t_0 = 0$. The functions $\Phi(\cdot)$ and $F(\cdot)$ stand for frequency and amplitude characteristics, respectively, and chosen to be the Hill functions

$$\Phi(x_3) = \Phi_1 + \Phi_2 \frac{(x_3/h)^p}{1 + (x_3/h)^p}, \quad F(x_3) = F_1 + \frac{F_2}{1 + (x_3/h)^p},$$

where the parameters $\Phi_1, \Phi_2, F_1, F_2, h$ are positive, and $p \geq 1$ is an integer. The choice of the modulation functions is in line with the physiological nature of the system, yielding smooth characteristics, bounded from above and below. It is straightforward to see that the following inequalities apply

$$0 < \Phi_1 \leq \Phi(\cdot) < \Phi_1 + \Phi_2, \quad 0 < F_1 < F(\cdot) \leq F_1 + F_2. \tag{3}$$

With respect to the state vector $x(t) = [x_1(t) \ x_2(t) \ x_3(t)]^T$, model (1)-(2) can be rewritten as

$$\begin{aligned} \dot{x}(t) &= Ax(t), & y(t) &= Lx(t), \\ z(t) &= Cx(t), & t_n &< t < t_{n+1}, \end{aligned} \tag{4}$$

where

$$A = \begin{bmatrix} -b_1 & 0 & 0 \\ g_1 & -b_2 & 0 \\ 0 & g_2 & -b_3 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = [0 \ 0 \ 1],$$

$y(t)$ represents the measured hormone concentrations of LH and Te, and $z(t)$ is the hormone concentration that modulates the pulsatile feedback (Te). Obviously, A is Hurwitz, i.e. all its eigenvalues are negative real, which property simply corresponds to the fact that all organic molecules eventually decay. It is easy to check that (A, L) is observable.

The vector $x(t)$ experiences jumps at time instants t_n , when GnRH pulses are fired with the corresponding weights λ_n :

$$\begin{aligned} x(t_n^+) &= x(t_n^-) + \lambda_n B, & t_{n+1} &= t_n + T_n, \\ T_n &= \Phi(z(t_n)), & \lambda_n &= F(z(t_n)), \end{aligned} \quad (5)$$

where $B = [1 \ 0 \ 0]^T$. Note that $CB = 0$, $LB = 0$, hence the outputs $y(t)$, $z(t)$ are continuous in t and have no jumps.

As shown in [3], the solutions to (4)–(5) are bounded and the system does not have equilibria. For biologically motivated parameter values, system (4)–(5) normally exhibits either a stable 1-cycle, or a stable 2-cycle (i.e. periodic solutions with either one or two GnRH impulses in the least period), but chaotic solutions are also possible. Being extended with a time delay, the model possesses more complex nonlinear dynamics such as cycles of higher periodicity, chaos, bistability, and quasiperiodical solutions [21].

The GnRH pulses are secreted at the time instants t_n with the weights λ_n and are immeasurable outside of the brain. Therefore, the task of observation is to produce estimates $(\hat{t}_n, \hat{\lambda}_n)$ of the impulse parameters (t_n, λ_n) . Note that, with a completely known impulse sequence, estimates of the continuous state vector x can be obtained by conventional state estimation techniques.

Observer

In [20], for system (4) under intrinsic impulsive feedback (5), the following hybrid observer with modulated correction is introduced

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + K(y(t) - \hat{y}(t)), & \hat{y}(t) &= L\hat{x}(t), \\ \hat{z}(t) &= C\hat{x}(t), & \hat{t}_n < t < \hat{t}_{n+1}, \end{aligned} \quad (6)$$

$$\begin{aligned} \hat{x}(t_n^+) &= \hat{x}(t_n^-) + \hat{\lambda}_n B, & \hat{t}_{n+1} &= \hat{t}_n + \hat{T}_n, \\ \hat{\lambda}_n &= F(\hat{z}(t_n)), \end{aligned} \quad (7)$$

and

$$\hat{T}_n = \Phi(\hat{z}(\hat{t}_n) + K_f(\hat{y}(\hat{t}_n) - y(\hat{t}_n))). \quad (8)$$

The observer offers two design degrees of freedom represented by the gain matrices: The gain matrix $K_f \in \mathbb{R}^{1 \times 2}$ directly impacts the discrete dynamics

of (6) while $K \in \mathbb{R}^{3 \times 2}$ is responsible for continuous linear output error feedback to the estimates $\hat{x}(t)$. Notice that, in the discrete part of the observer, only the timing of the impulses is adjusted by an output error feedback, i.e. (8), but not the amplitudes, cf. (6).

From observer equations (6)–(8), it can be concluded that since the values \hat{T}_n are generated at time instants \hat{t}_n , the convergence of the estimated firing times \hat{t}_n to t_n depends on two quantities:

- (a) the difference between $z(t_n)$ and $\hat{z}(\hat{t}_n)$,
- (b) the feedback term $K_f(\hat{y}(\hat{t}_n) - y(\hat{t}_n))$.

While the discrete gain K_f influences mostly a), the continuous gain K strikes a balance between a) and b) as they are interconnected through the observer dynamics.

A high gain K emphasizes the role of a) and, at the same time, plays down the role of b). Indeed, on an interval of continuity, a high value of K results in fast convergence of $\hat{x}(t)$ to $x(t)$, and thus implies a small output error $\hat{y}(\hat{t}_n) - y(\hat{t}_n)$. Since the continuous output is the only plant variable available to the observer, excessively fast convergence in the continuous state estimate $\hat{x}(t)$ is adverse to the convergence of the discrete state \hat{t}_n . Small and even zero values of K are feasible since A is Hurwitz. Yet, lack of continuous feedback would be a problem in a more sophisticated plant modeling setting, when a measurement disturbance in $y(t)$ is taken into account, because of poor filtering of the estimate $\hat{x}(t)$ through the nominal continuous plant dynamics.

A high value of K_f is not either desirable as it, according (8), would cause saturation in the frequency modulation function of the observer and generate a pulse at $\hat{t}_{n+1} \approx \hat{t}_n + \Phi_1 + \Phi_2$ that is unrelated to the firings in the observed plant. By setting the discrete gain to zero, i.e. $K_f = 0$, a previously considered in [4] observer is obtained. It is completely functional but suffers from slow convergence for low-multiplicity periodic solutions in system (4)–(5). As it was established in [19], this shortcoming can be recuperated by a non-zero K_f while the impact of K_f on the observer convergence is negligible for high multiplicity solutions. Thus, an design procedure for observer (6)–(8) should focus on 1-periodic solutions (1-cycles) as the worst-case option.

To start with, select the continuous feedback gain as

$$K = \begin{bmatrix} k_1 & 0 \\ k_3 & 0 \\ g_2 & k_6 \end{bmatrix}, \quad (9)$$

where $k_1, k_3, k_6 \in \mathbb{R}$ and g_2 is a sub-diagonal element of A .

The following proposition shows that selecting K as in (9) preserves full control over the observer continuous dynamics.

Proposition 1. For a given set of complex numbers $\Lambda = \{\mu_j, j = 1, 2, 3\}$, where Λ may include a complex number μ_j only together with its complex conjugate number $\bar{\mu}_j$, the following gains

$$k_1 = \frac{1}{g_1}[b_1^2 + b_1(\mu_1 + \mu_2) + \mu_1\mu_2],$$

$$k_3 = -(b_1 + b_2 + (\mu_1 + \mu_2)), \quad k_6 = -\mu_3 - b_3$$

assign the spectrum of $D = A - KL$ to Λ .

Proof. Omitted for brevity. □

The observer design described below is based on a local approach of assigning, through the output error feedback gains, a guaranteed convergence rate to the local dynamics of a zero solution of the hybrid state estimation error.

Synchronous mode and its local stability

To completely characterize the hybrid system dynamics, not only the continuous state vector $x(t)$, but also the sequence of time instants t_n should be taken into account.

Let $(x(t), t_n)$ be a solution of plant equations (4), (5) with the parameters λ_k, T_k , and $x_k = x(t_k^-)$. Suppose that the hybrid state $(x(t), t_n)$ is reconstructed by means of observer (6)–(8). Without loss of generality, assume that $t_0 \leq \hat{t}_0 < t_1$.

Obviously, the solution $(\hat{x}(t), \hat{t}_n)$ of observer equations (6)–(8) subject to the initial conditions $\hat{t}_0 = t_0, \hat{x}(\hat{t}_0^-) = x(t_0^-)$, yields $\hat{x}(t) = x(t)$ for all $t \geq t_0$. Such a solution $(\hat{x}(t), \hat{t}_n)$ is called a *synchronous mode* of observer (6)–(8) with respect to $(x(t), t_n)$, see [4].

A synchronous mode with respect to $(x(t), t_n)$ will be called *locally asymptotically stable* (see [4]) if, for any solution $(\hat{x}(t), \hat{t}_n)$ of (6)–(8) such that the initial estimation errors $|\hat{t}_0 - t_0|$ and $\|\hat{x}(\hat{t}_0^-) - x(t_0^-)\|$ are sufficiently small, it follows that $\hat{t}_n - t_n \rightarrow 0$ and $\|\hat{x}(\hat{t}_n^-) - x(t_n^-)\| \rightarrow 0$ as $n \rightarrow \infty$. The latter implies $\hat{\lambda}_n - \lambda_n \rightarrow 0$ as $n \rightarrow \infty$.

Proposition 2. The following statements are equivalent

1. $(\hat{x}(t), \hat{t}_n)$ is a synchronous mode with respect to $(x(t), t_n)$;
2. $\hat{y}(t) \equiv y(t)$ for all $t \geq t_0$.

Proof. See [19]. □

Recall from [19] the Poincaré mapping capturing the propagation of the continuous plant and observer states through the discrete cumulative sequence of the feedback firing instants

$$\begin{bmatrix} \hat{x}_{n+1} \\ \hat{t}_{n+1} \end{bmatrix} = Q(\hat{x}_n, \hat{t}_n) = \begin{bmatrix} P(\hat{x}_n, \hat{t}_n) \\ \hat{t}_n + \Phi(\alpha(\hat{x}_n, \hat{t}_n)) \end{bmatrix}, \quad (10)$$

where

$$\begin{aligned} P(\hat{x}_n, \hat{t}_n) &= e^{A(\hat{t}_n + \Phi(\alpha(\hat{x}_n, \hat{t}_n)) - t_n)} x(t_n^+) \\ &+ e^{D\Phi(\alpha(\hat{x}_n, \hat{t}_n))} \left[\hat{x}_n + F(C\hat{x}_n)B - e^{A(\hat{t}_n - t_n)} x(t_n^+) \right] \\ &- \lambda_n e^{D(\hat{t}_n + \Phi(\alpha(\hat{x}_n, \hat{t}_n)) - t_n)} B, \end{aligned} \quad (11)$$

$R = C + K_f L$, $\alpha(\hat{x}_n, \hat{t}_n) = R\hat{x}_n - K_f y(\hat{t}_n)$ for any $n = 0, 1, \dots$

The synchronous mode with respect to the hybrid solution $(x(t), t_k)$ is completely characterized by the vector sequence (x_k, t_k) . It is known from Theorem 2 in [20] that the mapping $Q(\hat{x}_n, \hat{t}_n)$ is smooth, which property allows for a linearization of the mapping Q in vicinity of (x_k, t_k) . Then the Jacobian of Q at the point (x_k, t_k) is calculated as

$$Q'(x_k, t_k) = J_k = \begin{bmatrix} (J_k)_{11} & (J_k)_{12} \\ (J_k)_{21} & (J_k)_{22} \end{bmatrix}, \quad (12)$$

where

$$\begin{aligned} (J_k)_{11} &= \Phi'_k A x_{k+1} R + e^{D\Phi(Cx_k)} (I + F'_k B C), \\ (J_k)_{12} &= A x_{k+1} (1 - \Phi'_k K_f L A x_k) - e^{D\Phi(Cx_k)} (A(x_k + \lambda_k B)), \\ (J_k)_{21} &= \Phi'_k R, \quad (J_k)_{22} = 1 - \Phi'_k K_f L A x_k. \end{aligned}$$

Here Φ'_k and F'_k are used to denote the derivatives $\Phi'(\alpha(x_k, t_k))$ and $F'(\alpha(x_k, t_k))$, respectively.

Consider a solution $(x(t), t_k)$ of system (4)–(5) satisfying $x_{k+1} \equiv x_k$, $\lambda_{k+1} \equiv \lambda_k$, $T_{k+1} \equiv T_k$ (1-cycle) and a synchronous mode of observer (6)–(8) with respect to $(x(t), t_k)$. Since $J_{i+1} \equiv J_i$, all the matrices in the sequence $\{J_i\}_{i=0}^\infty$ satisfy $J_i = J_0$. The following theorem provides a simple tool for checking local stability of the observer with respect to the solution in question.

Theorem 1 ([19]). *Let the matrix J_0 be Schur stable, i.e. all the eigenvalues of this matrix lie strictly inside the unit circle. Then the synchronous mode with respect to $(x(t), t_k)$ is locally asymptotically stable.*

Theorem 1 formulates a necessary and sufficient stability condition guiding the choice of the observer gains K and K_f . As pointed out above, the condition is local and depends not only on the coefficients of the system, but also on the parameters of the observed periodic mode. The spectral radius of the Jacobian matrix in (12) reflects the local convergence rate of the linearized observer dynamics. To achieve the fastest possible (local) convergence, the static gains K and K_f can be chosen to fulfill the conditions of Theorem 1 while minimizing the spectral radius of the Jacobian.

Denote for brevity

$$\begin{aligned} r_1 &= \Phi'_0 C A x_0 + 1, \\ R_2 &= F'_0 B C A x_0 - \lambda_0 A B, \end{aligned}$$

and consider the following coefficients

$$\begin{aligned} d_0 &= r^4 + r^3 a_1 + r^2 a_2 + r a_3 + a_4, \\ d_1 &= 4r^4 + 2r^3 a_1 - 2r a_3 - 4a_4, \\ d_2 &= 6r^4 - 2r^2 a_2 + 6a_4, \\ d_3 &= 4r^4 - 2r^3 a_1 + 2r a_3 - 4a_4, \\ d_4 &= r^4 - r^3 a_1 + r^2 a_2 - r a_3 + a_4, \end{aligned} \tag{13}$$

where

$$\begin{aligned} a_1 &= -e^{\mu_1 T_0} - e^{\mu_2 T_0} - e^{\mu_3 T_0} - r_1, \\ a_2 &= e^{(\mu_1 + \mu_2 + \mu_3) T_0} \left[e^{-\mu_1 T_0} + e^{-\mu_2 T_0} + e^{-\mu_3 T_0} \right. \\ &\quad \left. - \Phi'_0 R \left[-e^{-D T_0} (e^{-\mu_1 T_0} + e^{-\mu_2 T_0} + e^{-\mu_3 T_0}) \right. \right. \\ &\quad \left. \left. + e^{-2D T_0} \right] R_2 \right] + (e^{\mu_1 T_0} + e^{\mu_2 T_0} + e^{\mu_3 T_0}) \\ &\quad \times (r_1 + \Phi'_0 \lambda_0 K_f L A B), \\ a_3 &= e^{(\mu_1 + \mu_2 + \mu_3) T_0} \left[-1 - r_1 (e^{-\mu_1 T_0} + e^{-\mu_2 T_0} + e^{-\mu_3 T_0}) \right. \\ &\quad \left. + \Phi'_0 K_f L \left((e^{-\mu_1 T_0} + e^{-\mu_2 T_0} + e^{-\mu_3 T_0}) \lambda_0 A B \right. \right. \\ &\quad \left. \left. - e^{-D T_0} R_2 \right) \right], \\ a_4 &= e^{(\mu_1 + \mu_2 + \mu_3) T_0} (r_1 + \Phi'_0 \lambda_0 R A B), \end{aligned} \tag{14}$$

and r is a positive parameter, $\Lambda = \{\mu_i, i = 1, 2, 3\}$ is as before the spectrum of the matrix D .

The theorem below gives a necessary and sufficient conditions for the spectral radius of J_0 to be less than r .

Theorem 2. *All the eigenvalues of Jacobian J_0 lie within a circle of the radius r on the complex plane if and only if the following system of inequalities is satisfied:*

$$\begin{aligned} d_0 > 0, \quad d_1 > 0, \quad d_2 > 0, \quad d_3 > 0, \quad d_4 > 0, \\ d_3(d_1d_2 - d_0d_3) - d_4d_1^2 > 0. \end{aligned} \tag{15}$$

Proof. Omitted for brevity. □

Corollary 1. *The synchronous mode with respect to $(x(t), t_k)$ is locally asymptotically stable if and only if system (15) is feasible for $r < 1$.*

Lemma 1. *The eigenvalue $w = e^{\mu_3 T_0} = e^{(-b_3 - k_6)T_0}$ of the matrix e^{DT_0} is also an eigenvalue of the Jacobian J_0 .*

Proof. Omitted for brevity. □

Introduce the notation $c_1 = a_1 + e^{(-b_3 - k_6)T_0}$, $c_2 = a_2 + c_1 e^{(-b_3 - k_6)T_0}$, $\rho_{\min} = \frac{1}{2} (e^{-b_1 T_0} + \Phi'_0 C A x_0 + 1)$, $\mathcal{D} = c_1^2 - 4c_2$.

Define the function $g(\cdot)$ of the gain coefficient k_3 as

$$g(k_3) = \frac{\sqrt{\mathcal{D}}}{2} + \frac{1}{2} e^{(-b_2 - k_3)T_0}.$$

Note that \mathcal{D} is sign-indefinite and the function $g(\cdot)$ can take complex values.

Theorem 3. *Assume that $\rho_{\min} > 0$. Let the gain K_f belong to the hyperplane*

$$CA(x_0 + \lambda_0 B) + \lambda_0 K_f L A B = -\frac{1}{\Phi'_0} \tag{16}$$

and

$$k_1 = 0, \quad k_6 = -b_3 - \frac{\ln(\varkappa)}{T_0}, \tag{17}$$

where $0 < \varkappa \leq \rho_{\min}$, $k_3 = k_3^* = \arg \min |g(k_3)|$, and $|\cdot|$ is the absolute value of a complex number. Then the spectral radius of Jacobian J_0 is given by

$$\rho(J_0) = \rho_{\min} + |g(k_3^*)|.$$

Proof. Omitted for brevity. □

Note that k_1 is chosen as zero to reduce unnecessary variability in \hat{x}_1 caused by the feedback of the output error. The result of Theorem 3 for the case $\rho_{\min} < 0$ can be obtained similarly.

The observer gains defined in Theorem 3 are obviously not optimal in the sense of the least spectral radius of the Jacobian. Without the assumed

structural restrictions, there may exist K and K_f that yield a smaller value of the spectral radius than the ones obtained in Theorem 3. This may, for instance, occur for a high gain K_f , while the continuous gain K is insignificant or renders the matrix D non-Hurwitz. In fact, the performance of such an observer will exhibit high sensitivity to the initial value mismatch of \hat{x}_0 and x_0 . This is generally a consequence of the selected performance criterion being a local characteristic that does not take into account the strongly nonlinear nature of the system under consideration.

The observer gains proposed in Theorem 3 allow to keep a balance between aspects a) and b), as discussed in Section Observer. Although the spectral radius does not achieve a global minimum, local stability condition of Theorem 1 remains valid for the original nonlinearized plant.

Numerical example

Assume the following numerical values in model (4), (5)

$$A = \begin{bmatrix} -0.012 & 0 & 0 \\ 2.8 & -0.15 & 0 \\ 0 & 1.5 & -0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$C = [0 \quad 0 \quad 1],$$

$$\Phi(z) = 40 + 80 \frac{(z/2.7)^2}{1 + (z/2.7)^2}, \quad F(z) = 0.05 + \frac{5}{1 + (z/2.7)^2}.$$

Then the stationary solution in the plant is a stable 1-cycle characterized by the parameters $x_0 = [0.0516 \quad 1.0479 \quad 17.8606]$, $\lambda_0 = 0.1617$, $T_0 = 118.2066$.

Use the observer gains defined by Theorem 3. From (17) for $w = 0.0001$ one obtains

$$K_f = [-10.8137, 0], \quad K = \begin{bmatrix} 0 & k_3^* & g_2 \\ 0 & 0 & -0.0221 \end{bmatrix}^T,$$

and $\rho_{\min} = 0.6001$. A search for k_3 that minimizes $|g(\cdot)|$ yields $k_3^* = -0.0872$ and $g(k_3^*) = 0.007$, see Fig. 1. As a result, the spectral radius of Jacobian J_0 is $\rho(J_0) = 0.6071$.

The transients in the continuous states of the observer due to initial conditions mismatch with respect to those of the plant are depicted in Fig. 2.

Fig. 3 illustrates the transients in the sequence $\hat{\lambda}_n$ produced by the observer relative to the sequence λ_n of the plant, caused by a mismatch in the initial conditions of the plant and those of the observer, namely $\hat{t}_0 - t_0 = -12.5$, $\hat{x}_0 - x_0 = [0.002, 0, 2]$. The black vertical lines of height $-\hat{\lambda}_n$ positioned at \hat{t}_n correspond to the observer firing sequence. The pulse

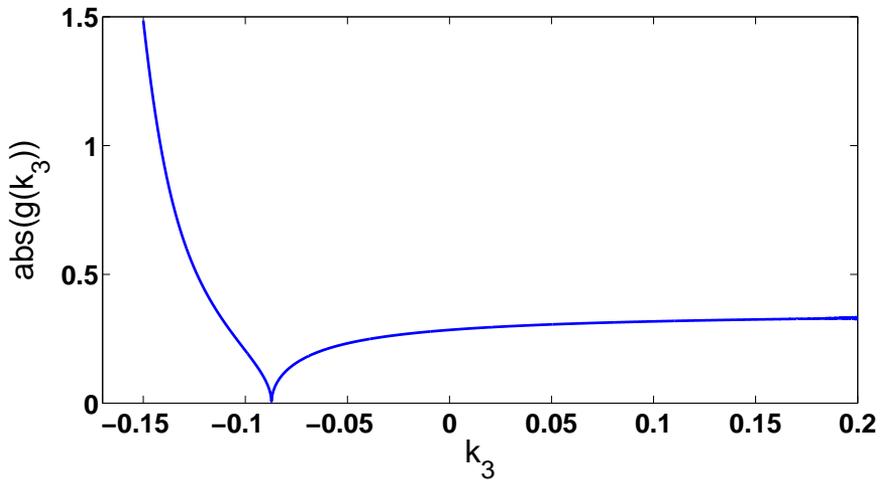


Figure 1: The the dependence of the absolute value of the function $g(\cdot)$ on the observer gain k_3 .

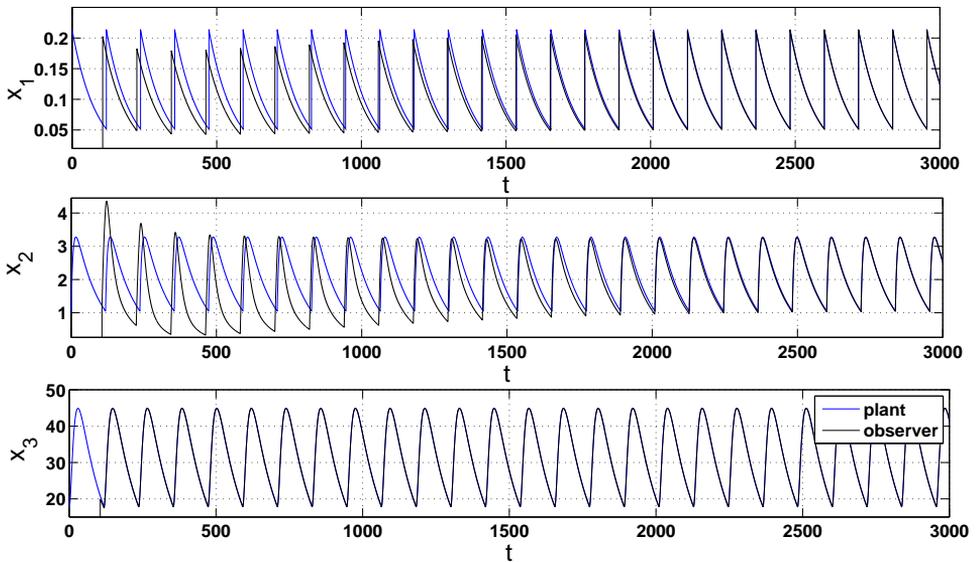


Figure 2: Transients in the continuous states of the observer (black lines) with respect to the corresponding plant states in 1-cycle (blue lines).

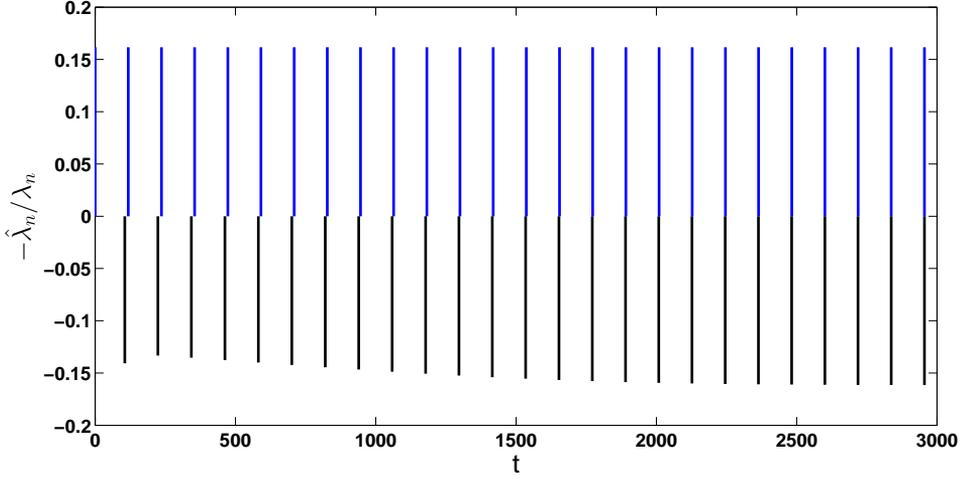


Figure 3: Transients due to non-zero initial conditions in the firing times and weights of the plant and the observer: Blue lines (upper part of the figure) mark the firing times of the plant in 1-cycle with the firing times t_n and the weights λ_n . Black lines (lower part of the figure) correspond to the pulse modulation of the observer \hat{t}_n with the height equal to $-\hat{\lambda}_n$.

modulation of the plant is shown by blue lines of height λ_n positioned at t_n . It can be seen that the firing instants of the observer become synchronized with those of the plant and the impulse weights $\hat{\lambda}_n$ asymptotically converge to λ_n .

Hybrid observer performance can be measured in numerous ways. The convergence to a synchronous mode is characterized here by the first time instant when \hat{t}_n comes into ε_f -neighborhood of t_n and never leaves it:

$$\mathcal{P}(\varepsilon_f) = \hat{t}_{n^*}, \quad n^* = \min\{k : |\hat{t}_N - t_N| < \varepsilon_f \forall N > k\}. \quad (18)$$

This criterion somehow captures the most demanding state estimation error in the hybrid observer since all the information regarding the discrete state of the plant comes from the continuous measurements. The relationship between the value of the threshold in (18) and $\mathcal{P}(\varepsilon_f)$ is depicted in Fig. 4, where the proposed observer is compared with that of $K_f = 0$ and K having the following structure:

$$K = \begin{bmatrix} 0 & k_1 & 0 \\ 0 & 0 & k_2 \end{bmatrix}^T,$$

that has been used in, e.g. [4, 20]. Choose $k_1 = k_2 = 1.2$, rendering the spectral radius of J_0 to be equal 0.9588, where the optimal spectral radius

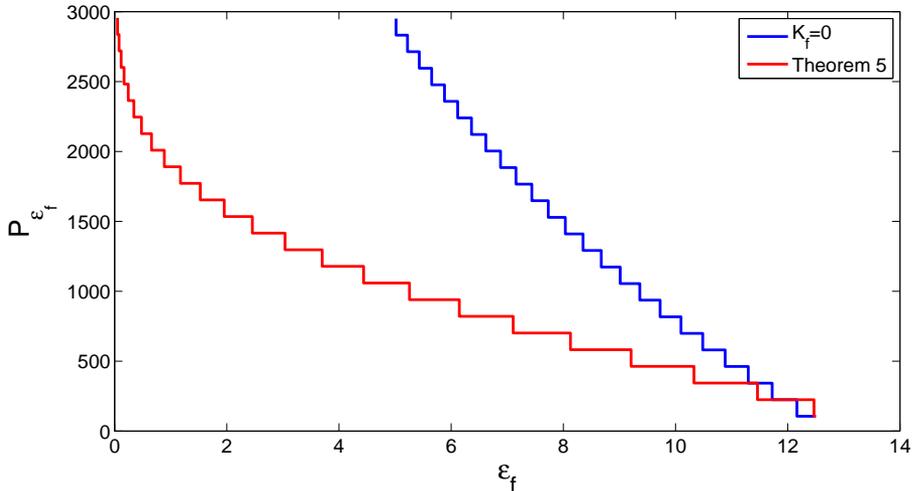


Figure 4: The dependence of settling time $\mathcal{P}(\varepsilon_f)$ on the synchronization threshold ε_f established by simulation.

for $K_f = 0$ is 0.9581. From Fig. 4 and Fig. 5, it is seen that for the observer gains from Theorem 3 the convergence is much faster.

Conclusions

A hybrid observer reconstructing a solution of a linear continuous system subject to an intrinsic pulse-modulated feedback from continuous measurements of the plant output is considered. Due to the impulsive nature of the system, the observer dynamics are highly nonlinear. A Poincaré mapping capturing the propagation of the continuous observer states through the cumulative sequence of the plant and observer impulses is derived. Orbital stability of hybrid solutions to the observer equations is related to the eigenvalues of the Jacobian matrix of the mapping. A practically important special case of third-order continuous dynamics and both amplitude and frequency modulation in the feedback of the plant portraying testosterone regulation in the human male is treated. As a worst-case scenario for observer convergence, the plant solutions exhibiting 1-cycles are analyzed. A new design scheme for selecting the feedback gains of the observer for the impulsive model of testosterone regulation is derived. Numerical simulations and calculations illustrating the observer performance are also provided.

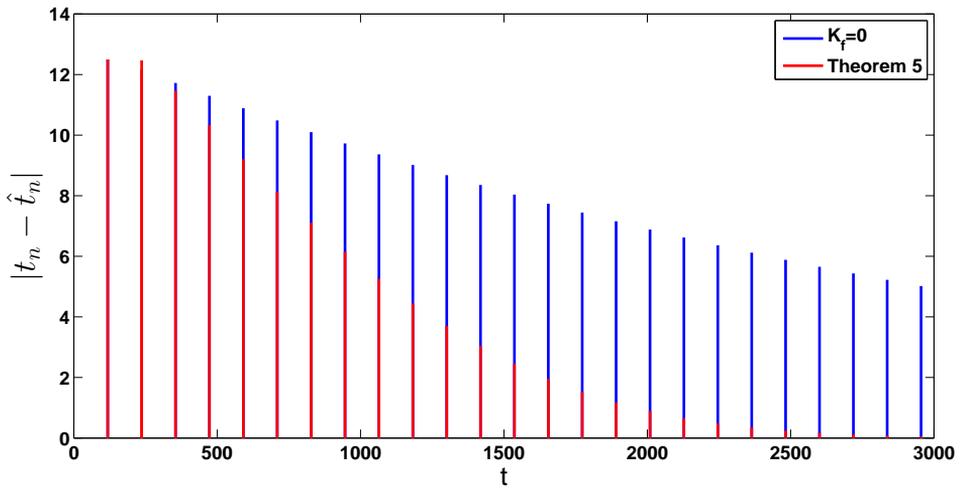


Figure 5: The discrete estimation error $e(t_n) = |t_n - \hat{t}_n|$ at the plant firing times t_n .

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Paper III

Finite-dimensional Hybrid Observer for Delayed Impulsive Model of Testosterone Regulation

Diana Yamalova, Alexander Churilov and Alexander Medvedev

Abstract

The paper deals with the model-based estimation of hormone concentrations that are inaccessible for direct measurement in the blood stream. Previous research demonstrated that the dynamics of non-basal endocrine regulation can be closely captured by linear continuous models with time delays under a pulse-modulated feedback. The presence of continuous time delays is inevitable in such a model due to transport phenomena and the time necessary for an endocrine gland to produce a certain hormone quantity. Yet, thanks to the Finite-Dimensional reducibility of the linear time-delay part of the system, a finite-dimensional model can be used to reconstruct both the continuous and discrete states of the hybrid time-delay plant. A hybrid observer exploiting this possibility is suggested and analyzed by means of a discrete impulse-to-impulse mapping.

Introduction

Hormones mediate communication between organs and tissues through the bloodstream carrying chemical messages that regulate many aspects in the human body, i.e. metabolism, growth as well as the sexual function and the reproductive processes. Hormones are secreted by endocrine glands directly into the bloodstream in continuous (basal) or pulsatile (non-basal) manner. Endocrine glands, interacting via hormone concentrations in blood, build up dynamical control loops characterized by self-sustained oscillations of the involved physiological quantities [23].

The endocrine system of testosterone regulation in the male essentially consists of three hormones, namely gonadotropin-releasing hormone (GnRH), luteinizing hormone (LH), and testosterone (Te). GnRH is produced in the hypothalamus of the brain and released in short pulses. Reaching the pituitary gland, GnRH stimulates production of LH, which in turn stimulates production of Te in the testes. Finally, both the GnRH outflow and the LH secretion are subject to feedback inhibition by Te [36]. However, the inhibition of LH has a relatively small effect on the dynamics of the closed-loop system and therefore not considered in this paper.

An impulsive mathematical model of testosterone regulation was proposed in [8] and is shown to comport with experimental data in [24]. It constitutes an impulsive version of Goodwin oscillator, a mathematical model that is well known in mathematical biology (see, e.g., [17, 16, 18, 32, 12]). The impulsive Goodwin oscillator consists of a continuous and an impulsive part [19], thus possessing hybrid dynamics and presenting a special version of an impulsive differential system [22, 31, 42, 19, 34]. It mathematically portrays the concept of pulsatile hormone regulation described in medical literature (see, e.g., [13]).

More recently, the impulsive Goodwin oscillator was augmented with a time delay in the continuous part of the system [6, 5], making it more aligned with the biological nature as transport phenomena and biosynthesis are omnipresent in endocrine and metabolic systems, [38, 33, 2, 11, 21, 30, 25, 29]. With the time delay taken into account, the pulse-modulated model of endocrine regulation acquires an infinite-dimensional continuous part. The closed-loop dynamics become therefore both hybrid and infinite-dimensional, which combination is mathematically challenging and so far rarely treated. However, the cascade structure of the continuous part, together with the impulsive feedback, allow to apply the concept of finite-dimensional reducibility (FD-reducibility), [6, 5]. In particular, it was shown [5] that the dynamics of an impulsive time-delay system with an FD-reducible continuous part coincide on certain time intervals with the dynamics of a delay-free impulsive system. This idea plays a key role in the present study.

Concentrations of the hormones secreted in human hypothalamus that is located in the lower central part of the brain are not available for direct measurement due to ethical reasons and need to be estimated. It poses an unusual observation problem. A considerable number of papers is devoted to the observability of hybrid systems, e.g. [35, 10, 1]. The discrete states of a system are usually assumed known, while observers for hybrid systems that are able to reconstruct discrete states from only continuous measurements are not so well covered in the literature.

In endocrine systems with pulsatile secretion, the highest degree of uncertainty is associated with the discrete (impulsive) part whose states have to be reconstructed from hormone concentration measurements. Two model-based estimation approaches are currently known. The first one is based on batch deconvolution techniques (blind system identification) [20, 37], while the relatively recent second one employs a state observer, whose estimates are corrected by output estimation error feedback [9, 40]. An extension of the observer scheme proposed in [9] to impulsive systems with time delay in continuous part was considered in [39]. Unlike the case treated in [39], the observer proposed here does not explicitly involve a delay but is rather based on a finite-dimensional plant model. Hence, the main contribution of the paper is in the novel structure and subsequent analysis of a hybrid observer exploiting a finite-dimensional model to reconstruct the states of the time-delay system.

Notice that impulsive feedback in the observer treated below is not contributed by design to achieve a performance objective but rather constitutes an integral and unmeasurable part of the plant model. On the contrary, in the impulsive observers for state estimation of linear and nonlinear continuous systems proposed in [27, 28, 3, 4], the observer state is updated in an impulsive fashion in order to achieve e.g. faster convergence. This distinction results in a major complication in observer design for plants with intrinsic impulsive feedback as the timing and weights of the impulses are unknown and have to be estimated by the observer.

A preliminary version of the present material without proofs of the main statements was presented in [41].

The paper is organized as follows. First, an impulsive time-delay model is summarized and reduced to an equivalent delay-free one. Then, making use of the reduced model, a hybrid observer is proposed and a pointwise (impulse-to-impulse) mapping describing its dynamics is derived. Further, the properties of the mapping pertaining to the observer performance are investigated. Then the impulsive time-delay model of testosterone regulation is described. Numerical simulations and calculations illustrating the observer design performance are also provided.

System equations

Consider an impulsive time-delay model [6] given by the equations

$$\begin{aligned} \frac{d\tilde{x}(t)}{dt} &= A_0 \tilde{x}(t) + A_1 \tilde{x}(t - \tau), \quad \tilde{z}(t) = C\tilde{x}(t), \quad \tilde{y}(t) = L\tilde{x}(t), \\ \tilde{t}_0 &= 0, \quad \tilde{t}_{n+1} = \tilde{t}_n + \tilde{T}_n, \quad \tilde{x}(\tilde{t}_n^+) = \tilde{x}(\tilde{t}_n^-) + \tilde{\lambda}_n \tilde{B}, \\ \tilde{T}_n &= \Phi(\tilde{z}(\tilde{t}_n)), \quad \tilde{\lambda}_n = F(\tilde{z}(\tilde{t}_n)), \end{aligned} \quad (1)$$

where $A_0 \in \mathbb{R}^{n_x \times n_x}$, $A_1 \in \mathbb{R}^{n_x \times n_x}$, $\tilde{B} \in \mathbb{R}^{n_x \times 1}$, $C \in \mathbb{R}^{1 \times n_x}$, $L \in \mathbb{R}^{n_y \times n_x}$ are constant matrices and $C\tilde{B} = 0$.

In (1), \tilde{z} is the scalar controlled output, \tilde{y} is the measurable output vector, \tilde{x} is the state vector and τ is a constant time delay. The amplitude modulation function $F(\cdot)$ and frequency modulation function $\Phi(\cdot)$ are continuous and bounded: $F(\cdot)$ is non-increasing and $\Phi(\cdot)$ is non-decreasing.

System (1) is considered for $t \geq 0$ subject to the initial condition $\tilde{x}(t) = \varphi(t)$, $-\tau \leq t < 0$, where $\varphi(t)$ is a continuous initial vector function. The state vector $\tilde{x}(t)$ of system (1) experiences jumps at the times $t = t_n$, $n = 0, 1, \dots$. The condition $C\tilde{B} = 0$ ensures that the modulating signal $\tilde{z}(t)$ is continuous in time.

Only the time delay values that are less than the minimal distance between two consecutive impulses are considered

$$\inf_{\xi} \Phi(\xi) > \tau, \quad (2)$$

so that $\tilde{T}_n > \tau$ for all n . This condition implies that only one firing of the pulse-modulated feedback in (1) is possible within a time interval whose length is equal to the time-delay value.

Suppose that the linear part of the system possesses the property of finite dimensional (FD) reducibility [6, 5], implying that

$$A_1 A_0^k A_1 = 0 \quad \text{for } k = 0, 1, \dots, n_x.$$

The notion of FD-reducibility is a formalization of the so-called ‘‘linear chain trick’’ originating from [14, 15] for the system in question.

Reduction to a delay-free impulsive system

Define the matrices $A = A_0 + A_1 e^{-A_0 \tau}$, $B = e^{-A \tau} e^{A_0 \tau} \tilde{B}$. Introduce a delay-free impulsive system

$$\begin{aligned} \frac{dx(t)}{dt} &= Ax(t), \quad z(t) = Cx(t), \quad y(t) = Lx(t), \\ t_{n+1} &= t_n + T_n, \quad x(t_n^+) = x(t_n^-) + \lambda_n B, \\ T_n &= \Phi(z(t_n)), \quad \lambda_n = F(z(t_n)). \end{aligned} \quad (3)$$

The following lemma obtained in [5] reveals the relationship between the solutions of system (1) and those of system (3).

Lemma 1. *Consider solutions $\tilde{x}(t)$, $x(t)$ of systems (1), (3), respectively. Assume that $t_1 = \tilde{t}_1$ and $x(t_1^-) = \tilde{x}(\tilde{t}_1^-)$. Then it holds that $t_n = \tilde{t}_n$, $\lambda_n = \tilde{\lambda}_n$ and $x(t_n^-) = \tilde{x}(\tilde{t}_n^-)$ for all $n \geq 1$. Moreover,*

$$x(t) = \tilde{x}(t), \quad \tilde{t}_n + \tau \leq t < \tilde{t}_{n+1}, \quad n = 0, 1, \dots$$

At the same time, generally, the solutions do not coincide entirely

$$x(t) \neq \tilde{x}(t), \quad \tilde{t}_n \leq t < \tilde{t}_n + \tau.$$

The result above will be exploited further in the paper to design a finite-dimensional observer for the infinite-dimensional hybrid system in (1). Note that the value of the time delay in the delay-free impulsive system still influences the system dynamics as τ effects the matrix coefficients A , B of (3).

A hybrid observer

The purpose of state observation in hybrid closed-loop system (1) is to produce estimates $(\hat{t}_n, \hat{\lambda}_n)$ of the impulse parameters $(\tilde{t}_n, \tilde{\lambda}_n)$. Notice that, unlike in the conventionally treated hybrid state estimation problem formulations, the jump times \tilde{t}_n are considered to be unmeasurable in (1) and require estimation. In fact, the problem solved by the proposed observer is synchronization of the firings in the feedback of the plant representing its discrete state and those of the observer.

From Lemma 1, it follows that one can produce estimates $(\hat{t}_n, \hat{\lambda}_n)$ of the impulse parameters $(\tilde{t}_n, \tilde{\lambda}_n)$ of (1) by exploiting the delay-free model in (3). To evaluate $(\hat{t}_n, \hat{\lambda}_n)$, an estimate of the continuous state vector of (3), i.e. $x(t)$ is produced by the hybrid observer:

$$\begin{aligned} \frac{d\hat{x}(t)}{d\hat{t}} &= A\hat{x}(t) + \mathcal{K}(t)(y(t) - \hat{y}(t)), \quad \hat{y}(t) = L\hat{x}(t), \quad \hat{z}(t) = C\hat{x}(t), \\ \hat{x}(\hat{t}_n^+) &= \hat{x}(\hat{t}_n^-) + \hat{\lambda}_n B, \quad \hat{t}_{n+1} = \hat{t}_n + \hat{T}_n, \\ \hat{T}_n &= \Phi(\hat{z}(\hat{t}_n)), \quad \hat{\lambda}_n = F(\hat{z}(\hat{t}_n)), \\ \mathcal{K}(t) &= \begin{cases} 0, & \hat{t}_n < t < \hat{t}_n + \tau, \\ K = \text{const}, & \hat{t}_n + \tau \leq t \leq \hat{t}_{n+1}. \end{cases} \end{aligned} \quad (4)$$

Notice that $\hat{z}(t)$, $\hat{y}(t)$ are generally discontinuous in time.

The switched feedback gain \mathcal{K} is zero in the time intervals where the solutions of system (1) and those of system (2) do not coincide, while the static feedback gain $K \in \mathbb{R}^{n_x \times n_y}$ is chosen to satisfy the stability conditions derived in Section 8.

Synchronous mode

Keeping in mind that the purpose of the hybrid estimation here is essentially synchronization, and following [9], introduce the notion of a synchronous mode for the plant—observer system (3), (4). Let $(x(t), t_n)$ be a solution of plant equations (3) with the parameters λ_n , T_n , and $x_n = x(t_n^-)$. Suppose that the plant is already running at the moment when the observer is initiated, i.e. $t_a \leq \hat{t}_0 < t_{a+1}$, for some integer $a \geq 1$.

Consider the solution $(\hat{x}(t), \hat{t}_n)$ of observer equations (4) subject to the initial conditions $\hat{t}_0 = t_a$, $\hat{x}(\hat{t}_0^-) = x(t_a^-)$, that yields $\hat{x}_n = x_{n+a}$, $\hat{t}_n = t_{n+a}$, $\hat{\lambda}_n = \lambda_{n+a}$, $n = 0, 1, 2, \dots$, and $\hat{x}(t) = x(t)$ for $t \geq t_a$. Such a solution $(\hat{x}(t), \hat{t}_n)$ will be called a *synchronous mode* with respect to $(x(t), t_n)$. Thus, a synchronous mode is a null solution of the state estimation error dynamics of hybrid observer (4) on $t \in [0, \infty)$.

Following [9], a synchronous mode will be called *locally asymptotically stable* if for any solution $(\hat{x}(t), \hat{t}_n)$ of (4) such that the initial estimation errors $|\hat{t}_0 - t_a|$ and $\|\hat{x}(\hat{t}_0^-) - x(t_a^-)\|$ are sufficiently small, it follows that $\hat{t}_n - t_{n+a} \rightarrow 0$ and $\|\hat{x}(\hat{t}_n^-) - x(t_{n+a}^-)\| \rightarrow 0$ as $n \rightarrow \infty$. The latter implies $\hat{\lambda}_n - \lambda_{n+a} \rightarrow 0$ as $n \rightarrow \infty$. In the definition above, the operator $\|\cdot\|$ stands for any vector norm.

To ensure practical usefulness of the observer, stability properties of the synchronous mode have to be investigated. By choice of K , the synchronous mode has to be rendered asymptotically stable with a suitable convergence rate and domain of attraction.

Pointwise mapping and its properties

Consider the pointwise mapping describing the evolution of the observer hybrid state from one firing of the impulsive part in (4) to the next one:

$$\begin{bmatrix} \hat{x}(\hat{t}_n^-) \\ \hat{t}_n \end{bmatrix} \mapsto \begin{bmatrix} \hat{x}(\hat{t}_{n+1}^-) \\ \hat{t}_{n+1} \end{bmatrix}. \quad (5)$$

For any integer numbers k and s , $0 \leq k \leq s$, define the sets

$$S_{k,s} = \{(\zeta, \theta) : \theta \in \mathbb{R}, \zeta \in \mathbb{R}^{n_x}, t_k \leq \theta < t_{k+1}, t_s \leq \theta + \Phi(C\zeta) < t_{s+1}\}.$$

Hence, each point (\hat{x}_n, \hat{t}_n) of the observer hybrid state belongs to one of the sets $S_{k,s}$, i.e. to each (\hat{x}_n, \hat{t}_n) one can uniquely match two points (x_k, t_k) and (x_s, t_s) of the observed system (if $k = s$, these points coincide) such that $t_k \leq \hat{t}_n < t_k + \Phi(Cx_k)$, $t_s \leq \hat{t}_n + \Phi(C\hat{x}_n) < t_s + \Phi(Cx_s)$.

Introduce

$$G_k(\zeta, \theta) = \begin{cases} e^{D(\theta + \Phi(C\zeta) - t_{k+1})}, & \text{if } \theta \leq t_{k+1} - \tau, \\ e^{D(\Phi(C\zeta) - \tau)} e^{A(\theta + \tau - t_{k+1})}, & \text{if } t_{k+1} - \tau < \theta, \end{cases}$$

where $D = A - KL$. Note that the functions $G_k(\zeta, \theta)$ are piecewise continuous due to the definition of \mathcal{K} .

Define $P(\zeta, \theta) = P_{k,s}(\zeta, \theta)$ at $(\zeta, \theta) \in S_{k,s}$ with

$$P_{k,s}(\zeta, \theta) = e^{A(\theta + \Phi(C\zeta) - t_s)} x(t_s^+) - e^{D(\Phi(C\zeta) - \tau)} e^{A\tau} \left(e^{A(\theta - t_k)} x(t_k^+) - \zeta - F(C\zeta)B \right) - \sum_{j=k+1}^s \lambda_j G_{j-1}(\zeta, \theta) B.$$

For brevity sake, denote $\hat{x}_n = \hat{x}(\hat{t}_n^-)$.

Theorem 1. *Pointwise mapping (5) is given by the equations*

$$\hat{x}_{n+1} = P(\hat{x}_n, \hat{t}_n), \quad \hat{t}_{n+1} = \hat{t}_n + \Phi(C\hat{x}_n). \quad (6)$$

Proof. See Appendix. □

Theorem 2. *The mapping $P(\zeta, \theta)$ is continuous.*

Proof. See Appendix. □

It will be shown in the next section that the mapping $P(\zeta, \theta)$ is not continuously differentiable in the whole state space. However, due to its local differentiability, local stability properties of mapping (6) characterizing the dynamics of the observer state can be investigated via linearization.

Linearization of the discrete-time map

The behaviors of pointwise mapping (6) in vicinity of the points (x_k, t_k) will be studied with respect to local stability of a synchronous mode.

To show the smoothness of the mapping $P(\zeta, \theta)$ introduced below at the points (x_k, t_k) , divide each set $S_{i,j}$ for all $i \neq j$ into two subsets $S_{i,j}^{left}$ and

$S_{i,j}^{right}$ by the hyperplanes $\theta = t_{i+1} - \tau$, i.e. $S_{i,j} = S_{j,j}^{left} \cup S_{i,j}^{right}$, where

$$\begin{aligned} S_{i,j}^{left} &= \{(\zeta, \theta) : \theta \in \mathbb{R}, \zeta \in \mathbb{R}^{n_x}, \\ &\quad t_i \leq \theta \leq t_{i+1} - \tau, t_j \leq \theta + \Phi(C\zeta) < t_{j+1}\}, \\ S_{i,j}^{right} &= \{(\zeta, \theta) : \theta \in \mathbb{R}, \zeta \in \mathbb{R}^{n_x}, \\ &\quad t_{i+1} - \tau < \theta < t_{i+1}, t_j \leq \theta + \Phi(C\zeta) < t_{j+1}\}. \end{aligned}$$

Note that the set $S_{i,i}$ ($i = j$) is not divided into subsets $S_{i,i}^{left}$ and $S_{i,i}^{right}$ due to the assumption on delay in (2) and thus implying $S_{i,i}^{right} = \emptyset$ and $S_{i,i}^{left} = S_{i,i}$.

Consider a point (x_k, t_k) for some $k \geq 1$. It can be seen that the closures of the four sets $S_{k,k}$, $S_{k-1,k+1}$, $S_{k-1,k}$, $S_{k,k+1}$ intersect at the point (x_k, t_k) . Moreover, $(x_k, t_k) \in S_{k,k+1}$. For a sufficiently small neighbor-

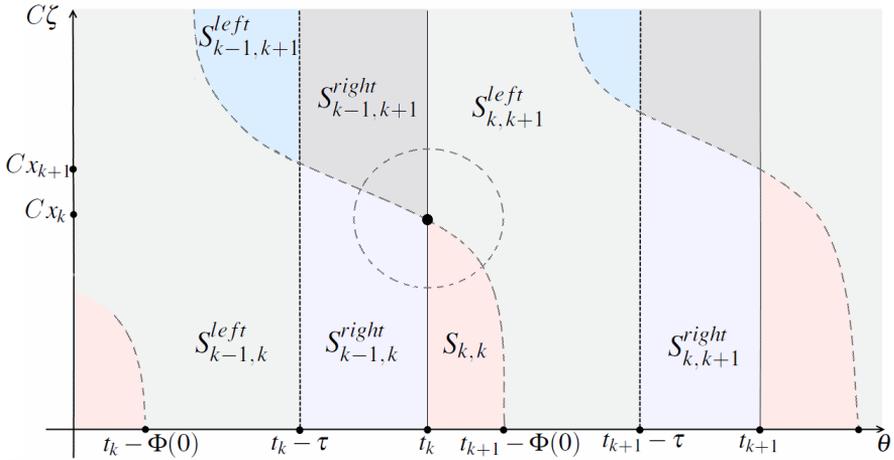


Figure 1: A neighborhood of the point (x_k, t_k) in the axes θ and $C\zeta$.

hood of the point (x_k, t_k) , the mapping $P(\zeta, \theta)$ can only take the values $P(\zeta, \theta) = P_{i,j}(\zeta, \theta)$, where $(\zeta, \theta) \in S_{i,j}$ and $S_{i,j}$ is one of the four sets $S_{k,k}$, $S_{k-1,k+1}^{right}$, $S_{k-1,k}^{right}$, $S_{k,k+1}^{left}$, see Fig. 1.

Denote for brevity $\Phi'_k = \Phi'(Cx_k)$, $F'_k = F'(Cx_k)$.

Theorem 3. *If $LB = 0$ and the scalar functions $F(\cdot)$, $\Phi(\cdot)$ have continuous derivatives, then the partial derivatives of $P(\zeta, \theta)$ are continuous at the points (x_k, t_k) , $k \geq 1$, and given by the following expression*

$$\begin{aligned} \frac{\partial}{\partial \zeta} P(x_k, t_k) &= \Phi'_k Ax_{k+1} C + e^{D(T_k - \tau)} e^{A\tau} [I + F'_k BC], \\ \frac{\partial}{\partial \theta} P(x_k, t_k) &= Ax_{k+1} - e^{D(T_k - \tau)} e^{A\tau} A(x_k + \lambda_k B). \end{aligned} \tag{7}$$

Proof. See Appendix. □

Introduce additional notation referring to mapping (5) and augmenting the continuous state vector with the discrete state. Define the function

$$Q_{k,s}(q) = \begin{bmatrix} P_{k,s}(q) \\ \theta + \Phi(C\zeta) \end{bmatrix}, \text{ where } q = \begin{bmatrix} \zeta \\ \theta \end{bmatrix}.$$

Set $Q(q) = Q_{k,s}(q)$ for $q \in S_{k,s}$. Then by the definition of $P(\zeta, \theta)$ in Section one has $Q(q) = \begin{bmatrix} P(q) \\ \theta + \Phi(C\zeta) \end{bmatrix}$, and from Theorem 1 it follows that $\hat{q}_{n+1} = Q(\hat{q}_n)$, where $\hat{q}_n = \begin{bmatrix} \hat{x}_n \\ \hat{t}_n \end{bmatrix}$.

Propagation of the observer hybrid dynamics through the firing times is described by iterations of the operator Q . The m -th iteration is defined as

$$Q^{(m)}(q) = Q(Q(\dots(Q(q)))).$$

Theorem 3 implies that the operator Q can be linearized in a vicinity of (x_k, t_k) .

From Section , it follows that the synchronous mode with respect to $x(t)$ is completely characterized by the vector sequence

$$\hat{q}_n^0 = \begin{bmatrix} x_k \\ t_k \end{bmatrix}, \text{ where } n = k + a, \quad (8)$$

a is a number from the definition of a synchronous mode.

Then the Jacobian of Q at the point \hat{q}_n^0 is calculated as

$$Q'(\hat{q}_n^0) = J_k(x_k, t_k) = \begin{bmatrix} (J_k)_{11} & (J_k)_{12} \\ (J_k)_{21} & (J_k)_{22} \end{bmatrix}, \quad (9)$$

where

$$\begin{aligned} (J_k)_{11} &= \Phi'_k A x_{k+1} C + e^{D(T_k - \tau)} e^{A\tau} [I + F'_k B C], \\ (J_k)_{12} &= A x_{k+1} - e^{D(T_k - \tau)} e^{A\tau} A(x_k + \lambda_k B), \\ (J_k)_{21} &= \Phi'_k C, \quad (J_k)_{22} = 1. \end{aligned}$$

By the chain rule, it follows that, for any $m \geq 1$, the Jacobian of the m -th iteration of the mapping is given by the expression

$$\left(Q^{(m)}\right)'(\hat{q}_n^0) = J_{k+m-1} J_{k+m-2} \dots J_{k+1} J_k.$$

Local stability of a synchronous mode with respect to an m -cycle

A solution of (3) is called m -cycle if it is periodic with exactly m pulse modulation instants in the least period. The existence conditions of an m -cycle in pulse-modulated time-delay system (3) were studied in [6],[5].

Let $(x(t), t_n)$ be an m -cycle of plant (3), where m is some integer, $m \geq 1$. The existence conditions of an m -cycle of pulse-modulated system with time delay are readily derived in [5]. Then $x_{n+m} \equiv x_n$, $\lambda_{n+m} \equiv \lambda_n$, $T_{n+m} \equiv T_n$. Consider a synchronous mode of observer (4) with respect to $(x(t), t_n)$ and let \hat{q}_n^0 be the corresponding vector sequence as in (8), such that $\hat{q}_{n+1}^0 = Q(\hat{q}_n^0)$ is satisfied.

Consider previously defined matrices J_i . Since $J_{i+m} \equiv J_i$, the sequence $\{J_i\}_{i=0}^{\infty}$ contains at most m distinct matrices, namely J_0, \dots, J_{m-1} . The theorem below provides a simple tool for checking local stability of observer (4).

Theorem 4. *Let the matrix product $J_0 \dots J_{m-1}$ be Schur stable, i.e., all the eigenvalues of this matrix lie strictly inside the unit circle. Then the synchronous mode with respect to $(x(t), t_n)$ is locally asymptotically stable.*

Proof. The result can be proved along the lines of Theorem 3 in [9]. \square

Theorem 4 formulates a stability condition guiding the choice of the observer gain K that appears in the matrix $D = A - KL$. As pointed out above, the condition is local and depends not only on the coefficients of the system, but also on the parameters of the observed periodic mode. In particular, the multiplicity of the periodical solution in the plant has to be known. The spectral radius of Jacobian (9) reflects the local convergence rate of the linearized observer dynamics. To optimize the observer performance, the static gain K can be chosen numerically to fulfill the conditions of Theorem 4 while minimizing the spectral radius of the Jacobian.

Mathematical model of testosterone regulation

To model testosterone regulation in the human male [5], the case of a third-order system (1) with the matrices

$$A_0 = \begin{bmatrix} -b_1 & 0 & 0 \\ g_1 & -b_2 & 0 \\ 0 & 0 & -b_3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & g_2 & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad (10)$$

$$L = [0 \quad 1 \quad 0], \quad C = [0 \quad 0 \quad 1]$$

is considered. Here b_1, b_2, b_3, g_1, g_2 are given positive parameters reflecting the kinetics of the involved hormones. From the biology of the system, one has $b_i \neq b_j$ for $i \neq j$. The elements of x correspond to the concentrations of GnRH (x_1), LH (x_2), and of Te (x_3).

The presence of a constant time delay τ in closed loop relates to a delay in the hormone transport in the blood stream and a delay occurring in hormone synthesis prior to secretion [26]. The contribution of the transport delays is relatively smaller than that due to synthesis of testosterone. In the simulations, the delay value is selected so that the minimal distance between two consecutive impulses does not exceed the sum of the testosterone synthesis and hormone transport delays, i.e $\tau < 40$. This is in line with the data provided in [2, 38].

The concentrations of Te and LH can be measured in the blood, while the concentration of GnRH is typically not available in humans and has to be estimated. Nonetheless, the level of testosterone is usually overly more noisy than the level of LH (see, e.g., [24]), and it is difficult to distinguish between the basal and pulsatile components. Thus the structure of the output row vector L is chosen so that only the measurement of LH concentration is taken into account.

Within a feedback construct, pulsatile secretion of a hormone gives rise to a dynamic system where amplitude and frequency modulation is employed to control the concentrations of other hormones, ostensibly in order to induce sustained oscillations in the closed-loop system.

As the amplitude modulation function $F(\cdot)$ and frequency modulation function $\Phi(\cdot)$ Hill functions with the following (continuous) parameterizations are chosen

$$\Phi(\xi) = \varkappa_1 + \varkappa_2 \frac{(\xi/h)^q}{1 + (\xi/h)^q}, \quad F(\xi) = \varkappa_3 + \frac{\varkappa_4}{1 + (\xi/h)^q},$$

where $\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4, h, q$ are positive parameters, q is integer. It is easy to check that the functions $F(\cdot)$ and $\Phi(\cdot)$ are smooth, strictly monotonic and bounded.

Since $A_1 A_0^k A_1 = 0$ for $k = 0, 1, 2$, system (1) with matrix coefficients (10) is FD-reducible, and, hence, can be represented in the form of delay-free system (3) with $A = A_0 + A_1 e^{-A_0 \tau}$, $B = e^{-A \tau} e^{A_0 \tau} \hat{B}$.

The matrix exponentials are given by

$$e^{A_0 t} = \begin{bmatrix} e^{-b_1 t} & 0 & 0 \\ E_{21}(t) & e^{-b_2 t} & 0 \\ 0 & 0 & e^{-b_3 t} \end{bmatrix}, \quad e^{A t} = e^{A_0 t} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ E_{31}(t) & E_{32}(t) & 0 \end{bmatrix},$$

where

$$E_{21}(t) = -\frac{g_1}{b_1 - b_2} \left(e^{-b_1 t} - e^{-b_2 t} \right), \quad E_{32}(t) = -\frac{g_2 e^{b_2 t}}{b_2 - b_3} \left(e^{-b_2 t} - e^{-b_3 t} \right),$$

$$E_{31}(t) = \frac{g_1 g_2}{b_1 - b_2} \left[\frac{e^{b_1 t}}{b_1 - b_3} \left(e^{-b_1 t} - e^{-b_3 t} \right) - \frac{e^{b_2 t}}{b_2 - b_3} \left(e^{-b_2 t} - e^{-b_3 t} \right) \right].$$

Introduce the numbers

$$\alpha_i = \prod_{j=1, j \neq i}^3 \frac{1}{b_i - b_j}, \quad i = 1, 2, 3.$$

Then it can be easily seen that

$$B = e^{-A\tau} e^{A_0\tau} \tilde{B} = \begin{bmatrix} 1 \\ 0 \\ B_0 \end{bmatrix}, \quad B_0 = g_1 g_2 \sum_{i=0}^3 \alpha_i e^{b_i \tau}.$$

The state vector $\tilde{x}(t)$ of system (1) with matrix coefficients (10) experiences jumps at the times $t = t_n$, portraying non-basal (episodic) release of GnRH. However, because of the matrix relationship $LB = 0$, the assumptions of Theorem 3 are valid and the impulse-to-impulse mapping is smooth.

Below, the observer design and performance are exemplified by two cases of periodical solutions in the plant arising for different values of the time delay within the considered interval. Notice that for the numerical values in question, the multiplicity of the periodical solutions in the plant decreases with increasing delay.

Numerical examples

Assume the following values in model (1): $h = 2.7$, $b_1 = 0.02$, $b_2 = 0.15$, $b_3 = 0.1$, $g_1 = 0.6$, $g_2 = 1.5$ and

$$\Phi(\xi) = 40 + 80 \frac{(\xi/h)^2}{1 + (z/h)^2}, \quad F(z) = 0.05 + \frac{5}{1 + (\xi/h)^2}.$$

Since $\inf_{\xi} \Phi(\xi) = 40$, then the time-delay value is within $0 \leq \tau < 40$, to make the analysis of this paper applicable.

Observation of a 4-cycle

Let $\tau = 5$. Then, the plant has a stable 4-cycle with

$$x_0^T = [0.0334 \quad 0.1543 \quad 3.1980]^T,$$

$$x_1^\top = [0.3821 \quad 1.7635 \quad 36.4028]^\top,$$

$$x_3^\top = [0.0420 \quad 0.1941 \quad 4.0212]^\top,$$

$$x_4^\top = [0.2455 \quad 0.1329 \quad 23.4306]^\top,$$

where \cdot^\top denotes transpose.

Choose the observer feedback gain in the form

$$K = [k_1 \quad k_2 \quad k_3]^\top$$

with $k_1 \geq 0$, $k_2 \geq 0$, $k_3 = 0$. Then

$$D = \begin{bmatrix} -b_1 & -k_1 & 0 \\ g_1 & -b_2 - k_2 & 0 \\ g_2 E_{21}(-\tau) & g_2 e^{b_2 \tau} & -b_3 \end{bmatrix}.$$

Hence, the characteristic polynomial of D is independent of τ and equal to

$$p_D(s) = (s + b_3)(s^2 + (b_1 + b_2 + k_2)s + k_1 g_1 + b_1(b_2 + k_2)).$$

Since b_1, b_2, b_3, g_1 are positive and k_1, k_2 are non-negative, D is Hurwitz stable.

To ensure the (locally) fastest convergence rate, find k_1, k_2 for which the synchronous mode is locally asymptotically stable and the spectral radius of $J_0 J_1 J_2 J_3$ is minimal. By inspection of Fig. 2, Fig. 3, and Fig. 4, such values of k_1, k_2 are

$$k_1 = 0.47, \quad k_2 = 6.15, \quad \rho(J_0 J_1 J_2 J_3) = 0.00058.$$

Hybrid observer performance can be measured in numerous ways. The convergence to a synchronous mode is characterized here by the first time instant when \hat{t}_n comes into ε_f -neighborhood of t_n and never leaves it:

$$\mathcal{P}(\varepsilon_f) = \hat{t}_{n^*}, \quad n^* = \min\{k : |\hat{t}_N - t_N| < \varepsilon_f \quad \forall N > k\}. \quad (11)$$

This criterion somehow captures the most demanding state estimation error in the hybrid observer since all the information regarding the discrete state in (1) comes from the continuous measurements. The relationship between the value of the threshold in (11) and $\mathcal{P}(\varepsilon_f)$ is depicted in Fig. 5.

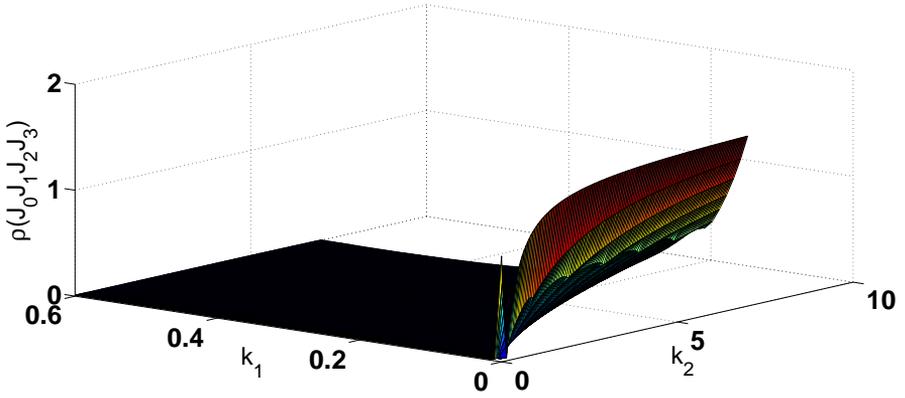


Figure 2: 4-cycle with $\tau = 5$. The spectral radius of the product $J_0 J_1 J_2 J_3$ as a function of k_1, k_2 . The values less than one correspond to stability.

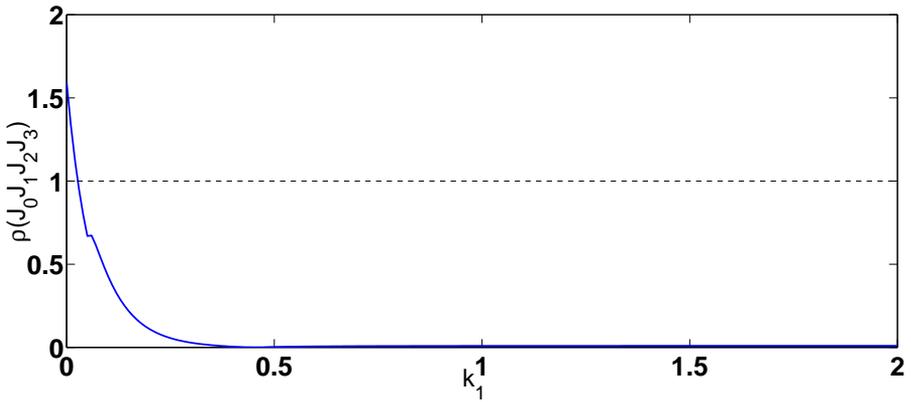


Figure 3: 4-cycle with $\tau = 5$. The spectral radius of the product $J_0 J_1 J_2 J_3$ as a function of k_1 ($k_2 = 6.85$). The values less than one correspond to stability.

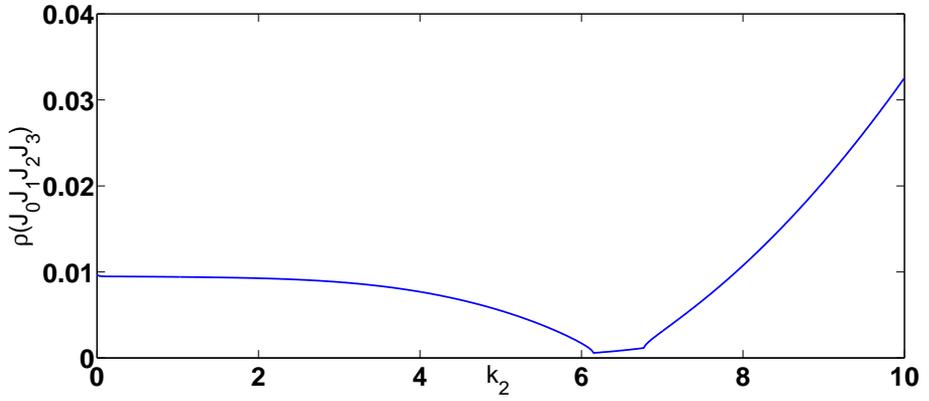


Figure 4: 4-cycle with $\tau = 5$. The spectral radius of the product $J_0 J_1 J_2 J_3$ as a function of k_2 ($k_1 = 0.47$). The values less than one correspond to stability.

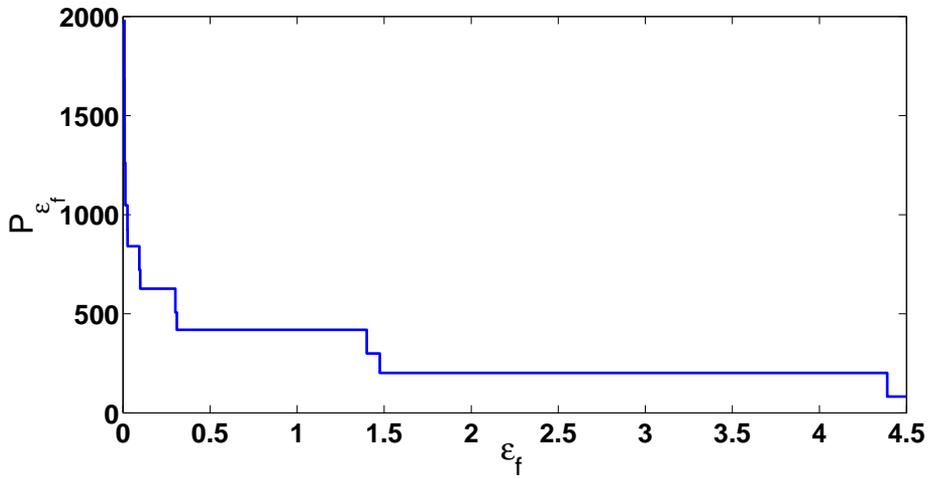


Figure 5: 4-cycle with $\tau = 5$. The dependence of settling time $\mathcal{P}(\varepsilon_f)$ on the synchronization threshold ε_f for $k_1 = 0.47$, $k_2 = 6.15$.

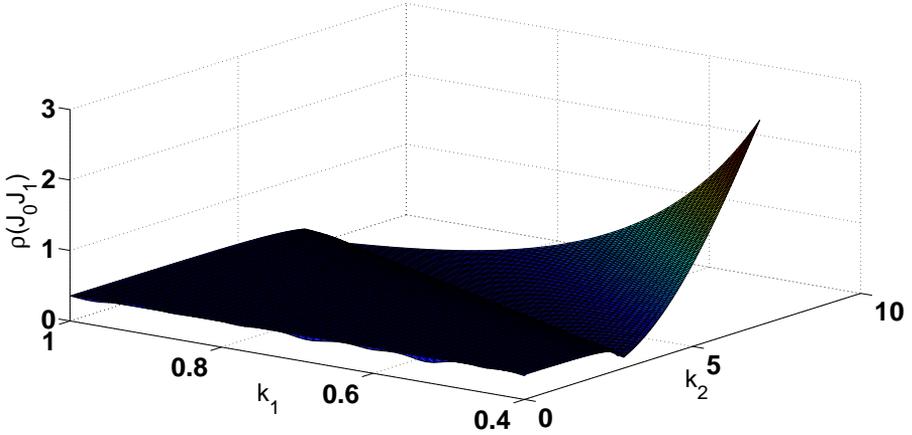


Figure 6: 2-cycle with $\tau = 30$. The dependence of the spectral radius of the product $J_0 J_1$ on k_1, k_2 . The values less than one correspond to stability.

Observation of a 2-cycle

Increasing the time delay to $\tau = 30$ yields a stable 2-cycle with

$$x_0^T = [0.0272 \quad 0.1255 \quad 4.2853]^T,$$

$$x_1^T = [0.2141 \quad 0.9883 \quad 33.3766]^T.$$

A search for k_1, k_2 that render a locally asymptotically stable synchronous mode and minimal spectral radius of $J_0 J_1$ gives (see Figs. 6, 7, 8):

$$k_1 = 0.93, k_2 = 6.85, \rho(J_0 J_1) = 0.149.$$

The relationship between the value of the threshold in (11) and $\mathcal{P}(\varepsilon_f)$ for a certain stabilizing observer gain is depicted in Fig. 9.

Conclusions

A state estimation problem motivated by unmeasured hormone concentrations in the system of non-basal testosterone regulation in the human male is considered. The system dynamics are modeled by a linear continuous time-delay system under intrinsic pulse-modulated feedback. The continuous part of the model is known to possess the property of finite-dimensional reducibility that opens up for the use of a finite-dimensional (delay-free) model for the reconstruction of the discrete and continuous states of the process model. A hybrid observer exploiting this possibility is introduced and analyzed by means of a discrete impulse-to-impulse mapping.

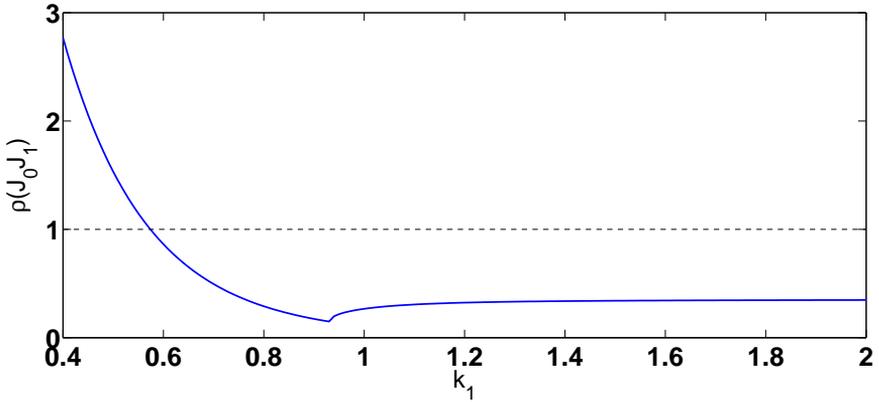


Figure 7: 2-cycle with $\tau = 30$. The dependence of the spectral radius of the product J_0J_1 on k_1 ($k_2 = 6.85$). The values less than one correspond to stability.

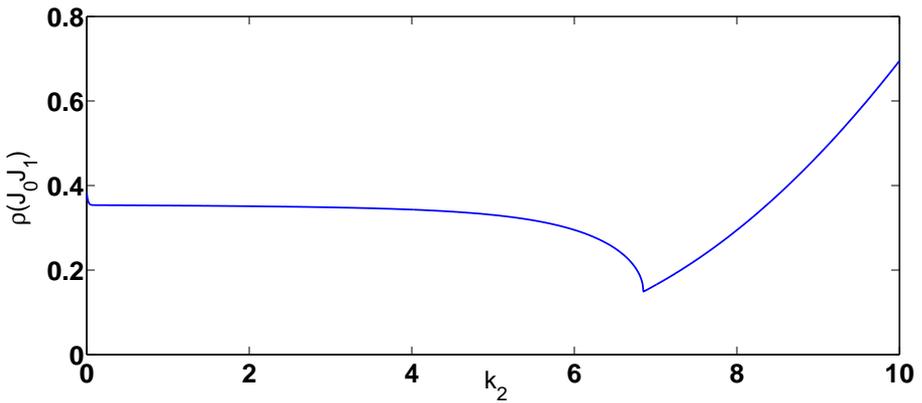


Figure 8: 2-cycle with $\tau = 30$. The dependence of the spectral radius of the product J_0J_1 on k_2 ($k_1 = 0.47$). The values less than one correspond to stability.

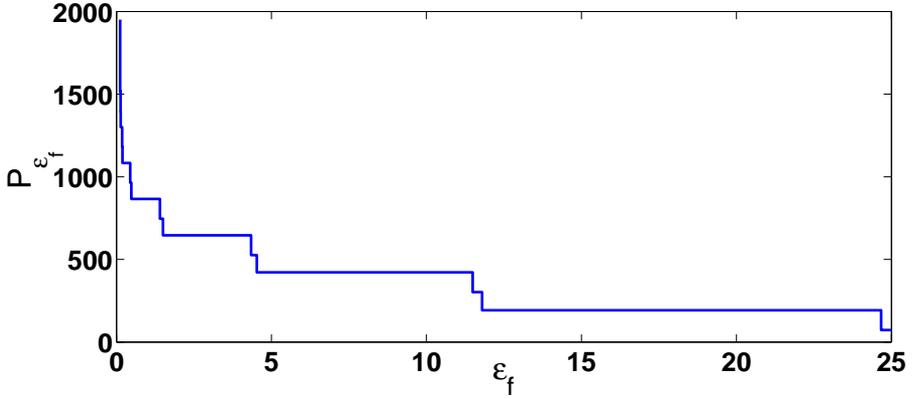


Figure 9: 2-cycle with $\tau = 30$. The dependence of settling time $\mathcal{P}(\varepsilon_f)$ on accuracy ε_f for $k_1 = 0.93$, $k_2 = 6.85$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Appendix

Proof of Theorem 1

Consider the difference $r(t) = x(t) - \hat{x}(t)$ in the interval $(\hat{t}_n, \hat{t}_{n+1})$ and suppose that

$$t_k \leq \hat{t}_n < t_{k+1}, \quad t_s \leq \hat{t}_n + \Phi(C\hat{x}_n) < t_{s+1}$$

for some k and s , such that $s \geq k$ (Fig.1). Obviously, $r(t)$ satisfies a differential equation

$$\frac{dr(t)}{dt} = \mathcal{D}(t)r(t),$$

where

$$\mathcal{D}(t) = \begin{cases} A, & \text{if } \hat{t}_n \leq t < \hat{t}_n + \tau, \\ D, & \text{if } \hat{t}_n + \tau \leq t < \hat{t}_{n+1} \end{cases}$$

at all the points t , where $r(t)$ has no jumps.

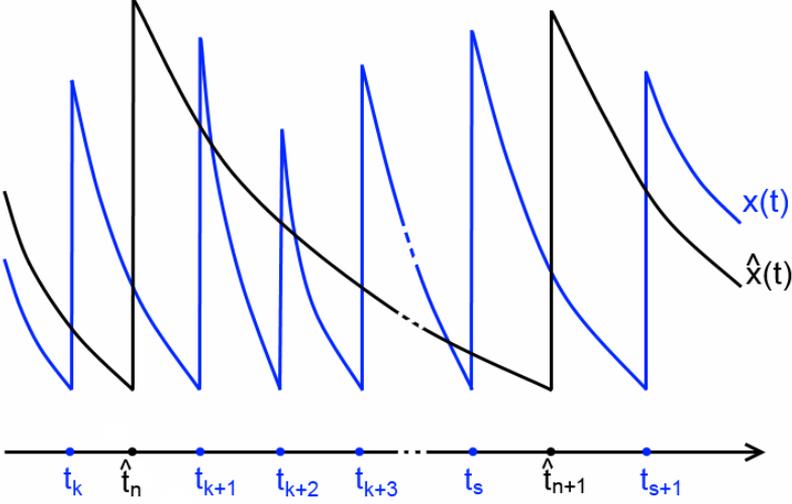


Figure 10: The firing times of the plant and the observer

Derive explicit formulas for the map (5). Introduce a number $m \geq 0$ such that $s = k + m$.

For $m = 0$ (i.e. $s = k$) the vector function $r(t)$ has no jumps in the interval $(\hat{t}_n, \hat{t}_{n+1})$. Hence,

$$\begin{aligned} \hat{x}_{n+1} &= x(\hat{t}_{n+1}^-) - r(\hat{t}_{n+1}^-) = e^{A(\hat{t}_{n+1}-t_s)}x(t_s^+) - e^{D(\hat{t}_{n+1}-\hat{t}_n-\tau)}r(\hat{t}_n + \tau) \\ &= e^{A(\hat{t}_{n+1}-t_s)}x(t_s^+) - e^{D(\hat{t}_{n+1}-\hat{t}_n-\tau)}e^{A\tau}r(\hat{t}_n^+) = e^{A(\hat{t}_{n+1}-t_s)}x(t_s^+) \\ &\quad - e^{D(\hat{t}_{n+1}-\hat{t}_n-\tau)}e^{A\tau} \left(e^{A(\hat{t}_n-t_k)}x(t_k^+) - \hat{x}(\hat{t}_n^-) - \hat{\lambda}_n B \right), \end{aligned}$$

that implies (6) for $s = k$.

For $m \geq 1$ the function $r(t)$ has jumps $r(t^+) - r(t^-) = \lambda_i B$ at the points $t = t_i, k + 1 \leq i \leq s$. The assumption $\inf_{\tilde{z}} \Phi(\tilde{z}) > \tau$ guarantees that $\hat{t}_n + \tau < t_i$ for $i > k + 1$, while the point t_{k+1} may lie either in the interval $(\hat{t}_n, \hat{t}_n + \tau)$, or in the interval $(\hat{t}_n + \tau, \hat{t}_{n+1})$, so these two cases should be considered separately.

Suppose that $m = 1$ (i.e. $s = k + 1$). Then $\hat{x}_{n+1} = x(\hat{t}_{n+1}^-) - r(\hat{t}_{n+1}^-) = e^{A(\hat{t}_{n+1}-t_s)}x(t_s^+) - r(\hat{t}_{n+1}^-)$.

Find $r(\hat{t}_{n+1}^-)$.

i) If $\hat{t}_n + \tau \leq t_s$, one has

$$\begin{aligned} r(\hat{t}_{n+1}^-) &= e^{D(\hat{t}_{n+1}-t_s)} r(t_s^+) = e^{D(\hat{t}_{n+1}-t_s)} \left(e^{D(t_s-\hat{t}_n-\tau)} r(\hat{t}_n + \tau) + \lambda_s B \right) \\ &= e^{D(\hat{t}_{n+1}-\hat{t}_n-\tau)} e^{A\tau} r(\hat{t}_n^+) + \lambda_s e^{D(\hat{t}_{n+1}-t_s)} B. \end{aligned}$$

ii) If $t_s < \hat{t}_n + \tau$, one has

$$\begin{aligned} r(\hat{t}_{n+1}^-) &= e^{D(\hat{t}_{n+1}-\hat{t}_n-\tau)} r(\hat{t}_n + \tau) = e^{D(\hat{t}_{n+1}-\hat{t}_n-\tau)} e^{A(\hat{t}_n+\tau-t_s)} r(t_s^+) \\ &= e^{D(\hat{t}_{n+1}-\hat{t}_n-\tau)} e^{A\tau} r(\hat{t}_n^+) + \lambda_s e^{D(\hat{t}_{n+1}-\hat{t}_n-\tau)} e^{A(\hat{t}_n+\tau-t_s)} B. \end{aligned}$$

Hence $r(\hat{t}_{n+1}^-) = e^{D(\hat{t}_{n+1}-\hat{t}_n-\tau)} e^{A\tau} r(\hat{t}_n^+) + \lambda_s G_k(\hat{x}(\hat{t}_n^-), \hat{t}_n) B$.

Thus,

$$\begin{aligned} \hat{x}_{n+1} &= e^{A(\hat{t}_{n+1}-t_s)} x(t_s^+) - e^{D(\hat{t}_{n+1}-\hat{t}_n-\tau)} e^{A\tau} \left(e^{A(\hat{t}_n-t_k)} x(t_k^+) - \hat{x}(\hat{t}_n^-) - \hat{\lambda}_n B \right) \\ &\quad - \lambda_s G_k(\hat{x}(\hat{t}_n^-), \hat{t}_n) B, \end{aligned}$$

what implies (6) if $s = k + 1$.

Suppose that $m \geq 2$. Prove that $r(t_{k+2}^+) = e^{D(t_{k+2}-\hat{t}_n-\tau)} e^{A\tau} r(\hat{t}_n^+) + \lambda_{k+1} G_{k,k+2}(\hat{t}_n) B + \lambda_{k+2} B$, where

$$G_{i,j}(\theta) = \begin{cases} e^{D(t_j-t_{i+1})}, & \text{if } \theta + \tau \leq t_{i+1} \\ e^{D(t_j-\theta-\tau)} e^{A(\theta+\tau-t_{i+1})}, & \text{if } t_{i+1} < \theta + \tau \end{cases}$$

for some i, j such that $k \leq i < j \leq s$.

i) If $\hat{t}_n + \tau \leq t_{k+1}$, then

$$\begin{aligned} r(t_{k+2}^+) &= e^{D(t_{k+2}-t_{k+1})} r(t_{k+1}^+) + \lambda_{k+2} B \\ &= e^{D(t_{k+2}-t_{k+1})} \left(e^{D(t_{k+1}-\hat{t}_n-\tau)} r(\hat{t}_n + \tau) + \lambda_{k+1} B \right) + \lambda_{k+2} B \\ &= e^{D(t_{k+2}-\hat{t}_n-\tau)} e^{A\tau} r(\hat{t}_n^+) + \lambda_{k+1} e^{D(t_{k+2}-t_{k+1})} B + \lambda_{k+2} B. \end{aligned}$$

ii) If $\hat{t}_n < t_{k+1} < \hat{t}_n + \tau < t_{k+2}$, then

$$\begin{aligned} r(t_{k+2}^+) &= e^{D(t_{k+2}-\hat{t}_n-\tau)} r(\hat{t}_n + \tau) + \lambda_{k+2} B \\ &= e^{D(t_{k+2}-\hat{t}_n-\tau)} e^{A(\hat{t}_n+\tau-t_{k+1})} r(t_{k+1}^+) + \lambda_{k+2} B \\ &= e^{D(t_{k+2}-\hat{t}_n-\tau)} e^{A(\hat{t}_n+\tau-t_{k+1})} \left(e^{A(t_{k+1}-\hat{t}_n)} r(\hat{t}_n^+) + \lambda_{k+1} B \right) + \lambda_{k+2} B. \end{aligned}$$

For $m \geq 3$ one has

$$r(t_s^+) = e^{D(t_s-\hat{t}_n-\tau)} e^{A\tau} r(\hat{t}_n^+) + \lambda_{k+1} G_{k,s}(\hat{t}_n) B + \lambda_{k+2} e^{D(t_s-t_{k+2})} B + \dots + \lambda_s B.$$

Thus one conclude that

$$r(\hat{t}_{n+1}^-) = e^{D(\hat{T}_n - \tau)} e^{A\tau} r(\hat{t}_n^+) + r_{k,s}, \quad (12)$$

where

$$r_{k,s} = \sum_{j=k+1}^s \lambda_j e^{D(\hat{t}_{n+1} - t_s)} G_{j-1,s}(\hat{t}_n) B.$$

Then (12) can be rewritten as

$$\hat{x}_{n+1} = x(\hat{t}_{n+1}^-) + e^{D(\hat{T}_n - \tau)} e^{A\tau} \left(\hat{x}_n + \hat{\lambda}_n B - x(\hat{t}_n^+) \right) - r_{k,s}, \quad (13)$$

Since

$$x(\hat{t}_{n+1}^-) = e^{A(\hat{t}_{n+1} - t_s)} x(t_s^+), \quad x(\hat{t}_n^+) = e^{A(\hat{t}_n - t_k)} x(t_k^+)$$

and

$$G_k(\hat{x}_n, \hat{t}_n) = e^{D(\hat{t}_n + \Phi(C\hat{x}_n) - t_s)} G_{k,s}(\hat{t}_n),$$

equality (13) implies (6).

Proof of Theorem 2

Since

$$x(t_k^+) = e^{A(t_k - t_{k-1})} x(t_{k-1}^+) + \lambda_k B, \quad k \geq 1,$$

it is straightforward to see that

$$P_{k,s}(\zeta, \theta) - P_{k-1,s}(\zeta, \theta) = -\lambda_k \left[e^{D(\Phi(C\zeta) - \tau)} e^{A(\theta + \tau - t_k)} - G_{k-1}(\zeta, \theta) \right] B, \quad (14)$$

$$P_{k,s}(\zeta, \theta) - P_{k,s-1}(\zeta, \theta) = \lambda_s \left[e^{A(\Phi(C\zeta) + \theta - t_s)} - G_{s-1}(\zeta, \theta) \right] B \quad (15)$$

for $k \geq 1, s \geq 1$. Since the functions $G_k(\cdot, \theta)$ are continuous for all k , then the function $P(\zeta, \theta)$ can have gaps only on the surfaces in space (ζ, θ) , where either $\theta = t_k$ or $\theta + \Phi(C\zeta) = t_s$ for some k, s .

Yet, from (14), (15) and because $G_{k-1}(\zeta, t_k) = e^{D(\Phi(C\zeta) - \tau)} e^{A\tau}$ and $G_{s-1}(\zeta, t_s - \Phi(C\zeta)) = I$, it follows that

$$P_{k,s} - P_{k-1,s}|_{\theta=t_k} = 0, \quad P_{k,s} - P_{k,s-1}|_{\theta+\Phi(C\zeta)=t_s} = 0,$$

and the function P is continuous everywhere.

Proof of Theorem 3

Since

$$\frac{\partial G_k(\zeta, \theta)}{\partial \theta} = \begin{cases} e^{D(\theta + \Phi(C\zeta) - t_{k+1})} D, & \text{if } \theta \leq t_{k+1} - \tau, \\ e^{D(\Phi(C\zeta) - \tau)} e^{A(\theta + \tau - t_{k+1})} A, & \text{if } t_{k+1} - \tau < \theta \end{cases}$$

has a gap on the surface $M_{k,\tau} = \{(\zeta, \theta) : \theta = t_{k+1} - \tau\}$, then $\frac{\partial P_{k,s}(\zeta, \theta)}{\partial \theta}$ is continuous everywhere on $S_{k,s}$, except $M_{k,\tau}$ for $s > k$ (if $s = k$ then $\frac{\partial P_{k,s}}{\partial \theta}$ is continuous everywhere on $S_{k,s}$).

It is easy to see that for any $0 < \tau < \inf(\Phi(\cdot))$ there exists a sufficiently small neighborhood W_k of a point (x_k, t_k) such that $W_k \cap M_{k-1,\tau} = W_k \cap M_{k,\tau} = \emptyset$. Hence, the partial derivatives of $P(\zeta, \theta)$ can have gaps either on the surface $M_k = \{(\zeta, \theta) : \theta = t_k\}$ or on the surface $N_{k+1} = \{(\zeta, \theta) : \theta + \Phi(C\zeta) = t_{k+1}\}$.

Let $(\zeta, \theta) \in M_k = \{(\zeta, \theta) : \theta = t_k\}$. From (14) it follows that $\frac{\partial u_k(\zeta, t_k)}{\partial \zeta} = \frac{\partial u_{k-1}(\zeta, t_k)}{\partial \zeta}$ and $\frac{\partial u_k(\zeta, t_k)}{\partial \theta} = \frac{\partial u_{k-1}(\zeta, t_k)}{\partial \theta}$. Hence $\frac{\partial P(\zeta, \theta)}{\partial \zeta}$ and $\frac{\partial P(\zeta, \theta)}{\partial \theta}$ have no gaps on M_k .

Let $(\zeta, \theta) \in N_{k+1}$. From (15) it follows that

$$\begin{aligned} \left. \frac{\partial(v_{k+1}(\zeta, \theta) - v_k(\zeta, \theta))}{\partial \zeta} \right|_{\theta + \Phi(C\zeta) = t_{k+1}} &= -\lambda_{k+1} \Phi'(C\zeta) KLBC = 0, \\ \left. \frac{\partial(v_{k+1}(\zeta, \theta) - v_k(\zeta, \theta))}{\partial \theta} \right|_{\theta + \Phi(C\zeta) = t_{k+1}} &= -\lambda_{k+1} KLB = 0. \end{aligned}$$

Hence $\frac{\partial P(\zeta, \theta)}{\partial \zeta}$ and $\frac{\partial P(\zeta, \theta)}{\partial \theta}$ have no gaps on N_{k+1} .

Formula (7) follows by direct calculations along the lines of Theorem 2 in [7].

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Paper IV

Hybrid State Observer for Time-Delay Systems under Intrinsic Impulsive Feedback

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Abstract

A hybrid static gain observer for systems described by a linear time-delay continuous part under impulsive feedback is suggested. The purpose of the observer is to asymptotically drive the state estimation error in the continuous states to zero and synchronize the sequence of modulated jump instants estimated by the observer with that of the plant. Conditions on the observer gain matrix to locally stabilize the observer error along an arbitrary periodic plant solution are obtained and the observer performance is illustrated by numerical simulations.

Introduction

Systems where continuous dynamics interact with discrete events in closed loop are ubiquitous in biology and medicine, [19]. A prominent example of such hybrid interaction comes from the field of neuroendocrinology [13]. The brain, especially the hypothalamus, controls the secretion of pituitary gland hormones and exerts episodic (event-based) feedback action on the relatively slow dynamics of the hormone kinetics. The feedback mechanism implemented by the neurons can be modeled by frequency and amplitude modulation, as described e.g. in [29]. Then, general behavior of the neuroendocrine feedback system can be captured by linear time-invariant models with pulse modulation, [8], [6].

The presence of time delays in closed loop is an essential phenomenon in endocrinology. Time delays occur in endocrine systems mainly due to two circumstances. First, there is a delay due to the transport of the hormones in the blood stream from the secretion site to the site where the hormone molecules bind to the target receptors. Second, the time needed to synthesize the hormone before secretion when releasable pools of it are lacking also results in a delay [29]. Time delays were introduced in numerous mathematical models of endocrine feedback [28, 1, 12, 18, 25, 23, 24] and pointed out as the main reason of sustained closed-loop oscillations in those systems, see e.g. [13]. However, the well-known pulsatile mechanism of non-basal regulation was not included in the models.

An impulsive model of testosterone regulation was proposed in [8] and analyzed in detail in [7]. Following the biological evidence, the model was based on the principles of pulse modulation [14]. It demonstrated a good agreement with clinical data [15, 21] and provided an explanation to the experimentally observed complex dynamical phenomena in endocrine systems, including deterministic chaos [31]. In fact, even without a time delay, the model lacks equilibria and appears to exhibit only periodical or chaotic behaviors. Yet, in order to improve the biological fidelity of the model, it was modified in [5, 3, 4] by introducing a time-delay into its continuous part. In further papers [32, 10, 2] these results were extended to the case of “a large delay”, when the time delay can be greater than the time interval between two consecutive pulse modulation instants. Only “small delays” are considered in the present paper. Yet the theory developed for handling “large delays” comes in handy in the observer analysis below. It was also observed that the cascade structure of the continuous part, together with the impulsive feedback, allow for a significant simplification of the closed-loop hybrid dynamics by applying the notion of finite-dimensional (FD) reducibility.

Due to the intrinsic nature of the neuroendocrine feedback, the signal impacting the continuous part of the model (hormone kinetics) is not avail-

able for measurement and has to be estimated. A standard in endocrinology procedure for that is based on deconvolution cast in the form nonlinear least-squares optimization [17]. To incorporate the knowledge of the feedback law, a hybrid state observer for continuous oscillating systems under intrinsic pulse-modulated feedback was proposed in [6], without taking into account the time delay. The present paper extends the results of [6] to the case of a time delay in the continuous part of the plant and brings the observer closer to an application to biological data.

The paper is composed as follows. First the equations of the time-delay plant with a pulse modulated feedback portraying an endocrine system with non-basal regulation are summarized and a hybrid state observer for it is introduced. The existing theory is then generalized to the models whose continuous part is comprised by FD-reducible time-delay dynamics. A pointwise discrete mapping that governs the evolution of the observer state is derived and its properties are studied. Simulation results are provided to illustrate the performance of the proposed observer.

System equations

Consider the impulsive time-delay model of non-basal endocrine regulation treated in [5, 3]:

$$\begin{aligned} \frac{dx(t)}{dt} &= A_0x(t) + A_1x(t - \tau), \quad z(t) = Cx(t), \\ y(t) &= Lx(t), \quad t_{n+1} = t_n + T_n, \quad x(t_n^+) = x(t_n^-) + \lambda_n B, \\ T_n &= \Phi(z(t_n)), \quad \lambda_n = F(z(t_n)), \end{aligned} \quad (1)$$

where $A_0 \in \mathbb{R}^{n_x \times n_x}$, $A_1 \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times 1}$, $C \in \mathbb{R}^{1 \times n_x}$, $L \in \mathbb{R}^{n_y \times n_x}$ are constant matrices, z is the scalar controlled output, y is the vector measurable output, x is the state vector (1), and τ is a constant time delay. Here $x(t_n^-)$, $x(t_n^+)$ are one-sided limits of $x(t)$ (left and right, correspondingly) at the time instant t_n .

System (1) is considered for $t \geq 0$ and subject to the initial condition $x(t) = \varphi(t)$, $-\tau \leq t < 0$, where $\varphi(t)$ is a continuous initial vector function. The matrix relationships

$$CB = 0, \quad LB = 0 \quad (2)$$

apply to (1) and are essential for further analysis. Suppose that the matrix A_0 is Hurwitz stable. Let the modulation functions of the feedback $\Phi(\cdot)$ and $F(\cdot)$ be continuous, strictly monotonic and bounded with strictly positive lower bounds. The latter condition implies that system (1) has no equilibria.

The states $x(t)$ of system (1) experience jumps at times $t = t_n$. However, because of the imposed conditions on the system matrices expressed by (2), the outputs $y(t)$, $z(t)$ are continuous.

Time delay values that are less than the minimal time interval between two consecutive pulse modulation instants are considered

$$\inf_z \Phi(z) > \tau,$$

so that $T_n > \tau$ for all n . Therefore, only "small delays" in the continuous part of (1), compared to the jump period of the pulse-modulated feedback, are treated below, cf. [32, 10, 2].

Henceforth, the continuous (linear) part of system (1) is assumed to be finite dimension (FD) reducible [5, 3], i.e.

$$A_1 A_0^k A_1 = 0 \quad \text{for } k = 0, 1, \dots, n_x - 1. \quad (3)$$

Any solution $x(t)$ of an FD-reducible linear time-delay equation

$$\frac{dx(t)}{dt} = A_0 x(t) + A_1 x(t - \tau) \quad (4)$$

defined for $t \geq 0$ with some initial function $\varphi(t)$, $-\tau \leq t \leq 0$, satisfies a delay-free linear equation

$$\frac{dx(t)}{dt} = D x(t)$$

for $t \geq \tau$, where $D = A_0 + A_1 e^{-A_0 \tau}$ (see [5, 3, 4]). For an FD-reducible system, the eigenvalue spectrum of the matrix A_0 coincides with that of D and, thus, the spectrum of D is independent of τ , i.e.

$$\det(sI_{n_x} - A_0 - A_1 e^{-A_0 \tau}) = \det(sI_{n_x} - A_0)$$

for all complex s and any τ . Additionally, time-delay linear system (4) has a finite (pole) spectrum, since

$$\det(sI_{n_x} - A_0 - A_1 e^{-\tau s}) = \det(sI_{n_x} - A_0),$$

for all complex s . However, FD-reducibility cannot be reduced to the finite spectrum property of system (4) but also involves the structure of the system matrices, as illustrated by the following example.

Example Consider (4) with the system matrices

$$A_0 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \quad (5)$$

Obviously, the characteristic polynomial $\det(sI_{n_x} - A_0 - A_1 e^{-\tau s})$ has a finite set of roots, namely $\{-1, -1, -1\}$. However,

$$A_1 A_0^k A_1 = (-1)^k A_1^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (-1)^k & 0 & 0 \end{bmatrix},$$

for any integer k , so system (4) is not FD-reducible.

The property of FD-reducibility can be characterized as follows.

Lemma 1 ([4]). *System (4) is FD-reducible iff there exists an invertible $n_x \times n_x$ matrix S such that*

$$S^{-1} A_0 S = \begin{bmatrix} U & 0 \\ W & V \end{bmatrix}, \quad S^{-1} A_1 S = \begin{bmatrix} 0 & 0 \\ \bar{W} & 0 \end{bmatrix}, \quad (6)$$

where the blocks U, V are square, the blocks W, \bar{W} are of the same dimension.

In the matrices of (5), a partition described by (6) cannot be achieved since the position of the nonzero elements of A_1 do not allow for two square blocks (U, V) on the main diagonal of A_0 .

Introduce a new notion of observability that is more restrictive than conventional spectral observability (see, e.g., [26, 27]) and requires, in addition, FD-reducibility .

Definition 1. *The linear part of system (1) will be called spectrally FD-observable if for any given complex self-conjugate set of numbers $\Lambda = \{\lambda_j, j = 1, \dots, n_x\}$ there exists a matrix K such that the eigenvalue spectrum of $A_0 - KL$ coincides with Λ , and, moreover,*

$$A_1 (A_0 - KL)^k A_1 = 0 \quad \text{for } k = 0, 1, \dots, n_x - 1. \quad (7)$$

In other words, (7) parallels condition (3) but for A_0 replaced with $A_0 - KL$. Relationships (7) imply that

$$\det(sI_{n_x} - D + KL) = \det(sI_{n_x} - A_0 + KL)$$

for all complex s . Further on, this property is supposed to hold with respect to the continuous part of (1).

Notice that since FD-reducibility is a more limiting property than finite spectrum, spectral FD-observability cannot be reduced to finite spectrum assignability (see, e.g. [20, 30, 22, 16]) that is guaranteed by spectral controllability or observability, depending on the considered design problem. Besides, the notion of FD-reducibility is applicable as defined to only autonomous systems (4) while spectral controllability (observability) covers input-to-state (state-to-output) properties. Sufficient conditions for spectral FD-observability are given by the following lemma.

Lemma 2. *For system (4) represented in the block form of (6), write L in terms of the matrix blocks*

$$L = [L_1 \quad L_2],$$

where the sizes of the blocks L_1, L_2 correspond to those of the blocks in (6). Let the matrix pair (L_2, V) be observable and assume that there exists a matrix L_0 such that $L_0 L_2 = 0$ and the matrix pair $(L_1 L_0, U)$ is also observable. Then system (4) is spectrally FD-observable.

Proof. Consider a matrix

$$K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}, \quad K_1 = \tilde{K}_1 L_0,$$

where matrices \tilde{K}_1, K_2 are to be determined. By the choice of L_0 , one has $K_1 L_2 = 0$ and

$$A_0 - KL = \begin{bmatrix} U - \tilde{K}_1 L_0 L_1 & 0 \\ W - K_2 L_1 & V - K_2 L_2 \end{bmatrix}.$$

From Lemma 1, it follows that (7) is valid for any \tilde{K}_1, K_2 . By the observability assumptions, arbitrary eigenvalues can be assigned to the matrices $U - \tilde{K}_1 L_0 L_1, V - K_2 L_2$ by the choice of \tilde{K}_1, K_2 . \square

The purpose of observation in hybrid closed-loop system (1) is to produce estimates $(\hat{t}_n, \hat{\lambda}_n)$ of the pulse modulation parameters (t_n, λ_n) . Obviously, given the sequence $(t_n, \lambda_n), n = 0, \dots, \infty$, estimates of the state vector x of the continuous part can be obtained by any of the conventional state estimation techniques.

Notice that the results of [6] cannot be directly applied to the present case, even under the assumption of the linear part of (1) being FD-reducible. Although the time-delay linear plant can indeed be reduced to a finite dimensional one, for the time-delay hybrid system such a reduction is only valid on certain time intervals. Thus, an observer for the infinite dimensional system given by (1) requires special consideration.

In order to estimate the state vector of (1), a hybrid observer is formulated as:

$$\begin{aligned} \frac{d\hat{x}}{dt} &= A_0 \hat{x}(t) + A_1 \hat{x}(t - \tau) + K(y(t) - \hat{y}(t)), \\ \hat{y}(t) &= L\hat{x}(t), \quad \hat{z}(t) = C\hat{x}(t), \quad \hat{x}(\hat{t}_n^+) = \hat{x}(\hat{t}_n^-) + \hat{\lambda}_n B, \\ \hat{t}_{n+1} &= \hat{t}_n + \hat{T}_n, \quad \hat{T}_n = \Phi(\hat{z}(\hat{t}_n)), \quad \hat{\lambda}_n = F(\hat{z}(\hat{t}_n)). \end{aligned} \tag{8}$$

With $\tau = 0$, the observer above is equivalent to the one treated in [6].

Introduce $D_K = A_0 - KL$, where K is a static feedback gain. If the linear part of (1) is spectrally FD-observable, the matrix K can be chosen in such a way that the matrix D_K is Hurwitz stable and fulfills (7).

Pointwise mapping and its properties

Consider the pointwise mapping describing the evolution of the observer (hybrid) state:

$$(\hat{x}(t_n^-), \hat{t}_n) \mapsto (\hat{x}(t_{n+1}^-), \hat{t}_{n+1}). \quad (9)$$

For any integer numbers k and s , $0 \leq k \leq s$, define the set

$$S_{k,s} = \{(\zeta, \theta) : \theta \in \mathbb{R}, \zeta \in \mathbb{R}^{n_x}, t_k \leq \theta < t_{k+1}, \\ t_s \leq \theta + \Phi(C\zeta) < t_{s+1}\}.$$

Denote

$$G(\theta) = \begin{cases} e^{A_0\theta}, & 0 \leq \theta \leq \tau, \\ e^{D(\theta-\tau)}e^{A_0\tau}, & \tau \leq \theta, \end{cases}$$

$$\tilde{G}(\theta) = \begin{cases} e^{D_K\theta}, & 0 \leq \theta \leq \tau, \\ e^{\tilde{D}_K(\theta-\tau)}e^{D_K\tau}, & \tau \leq \theta, \end{cases}$$

and

$$\tilde{R}(\theta_1, \theta_2) = \begin{cases} e^{\tilde{D}_K\theta_2} [e^{\tilde{D}_K(\theta_1-\tau)}e^{D_K\tau} - e^{D_K\theta_1}], & 0 \leq \theta_1 \leq \tau, \\ 0, & \tau \leq \theta_1, \end{cases}$$

where $\tilde{D}_K = D_K + A_1e^{-D_K\tau}$.

Define $P(\zeta, \theta) = P_{k,s}(\zeta, \theta)$ at $(\zeta, \theta) \in S_{k,s}$, with

$$P_{k,s}(\zeta, \theta) \\ = e^{D(\theta+\Phi(C\zeta)-t_s)}x(t_s^-) - e^{\tilde{D}_K\Phi(C\zeta)} \left(e^{D(\theta-t_k)}x(t_k^-) - \zeta \right) \\ - \lambda_k \left(e^{\tilde{D}_K\Phi(C\zeta)}G(\theta-t_k) + \tilde{R}(\theta-t_k, \Phi(C\zeta)) \right) B \\ + F(C\zeta)\tilde{G}(\Phi(C\zeta))B \\ - \sum_{j=k+1}^s \lambda_j \tilde{G}(\theta + \Phi(C\zeta) - t_j)B + \lambda_s G(\theta + \Phi(C\zeta) - t_s)B.$$

Apply the for brevity the shorthand notation $x_k = x(t_k^-)$, $\hat{x}_n = \hat{x}(t_n^-)$. The mapping defined in the proposition below completely describes the evolution of the observer (hybrid) state (i.e (9)) from one jump time in the intrinsic feedback to another, thus reducing the hybrid dynamics of (1) to a discrete (asynchronous) system.

Theorem 1. *Pointwise mapping (9) is given by the equations*

$$\hat{x}_{n+1} = P(\hat{x}_n, \hat{t}_n), \quad \hat{t}_{n+1} = \hat{t}_n + \Phi(C\hat{x}_n). \quad (10)$$

Proof. Omitted.

Theorem 2. *The mapping $P(\zeta, \theta)$ is continuous.*

Proof. See Appendix A.

Theorem 3. *If the scalar functions $F(\cdot)$, $\Phi(\cdot)$ have continuous derivatives, then the partial derivatives*

$$P'_\zeta = \frac{\partial P}{\partial \zeta}, \quad P'_\theta = \frac{\partial P}{\partial \theta}$$

are continuous everywhere.

Proof. See Appendix B.

Notice that the property proved by Theorem 3 is crucial for the observer design and does not generally apply. For instance, as demonstrated in [9, 11], for the time-delay-free and minimal-dimension case (i.e. $\tau = 0$ and $n_x = 1$), (2) is not satisfied and, thus, continuity of the mapping is not guaranteed. The latter phenomenon results in conditional stability of the observer, i.e. the state estimation error converges for some observer initial conditions but not for other.

Introduce additional notation referring to mapping (9). Define a function

$$Q_{k,s}(q) = \begin{bmatrix} P_{k,s}(\zeta, \theta) \\ \theta + \Phi(C\zeta) \end{bmatrix}, \quad \text{where } q = \begin{bmatrix} \zeta \\ \theta \end{bmatrix}.$$

Set $Q(q) = Q_{k,s}(q)$ for $t_k \leq \theta < t_{k+1}$, $t_s \leq \theta + \Phi(C\zeta) < t_{s+1}$. Then $\hat{q}_{n+1} = Q(\hat{q}_n)$, where

$$\hat{q}_n = \begin{bmatrix} \hat{x}_n \\ \hat{t}_n \end{bmatrix}, \quad Q(q) = \begin{bmatrix} P(\zeta, \theta) \\ \theta + \Phi(C\zeta) \end{bmatrix}.$$

Iterations of the operator Q will be also considered and defined as

$$Q^{(m)}(q) = \underbrace{Q(Q(\dots(Q(q))\dots))}_m.$$

According to the definition, P'_ζ is a $n_x \times n_x$ -matrix, and P'_θ is a n_x -dimensional column. Then the Jacobian matrix of $Q(q)$ is calculated as

$$Q'(q) = \begin{bmatrix} P'_\zeta(\zeta, \theta) & P'_\theta(\zeta, \theta) \\ \Phi'(C\zeta)C & 1 \end{bmatrix}.$$

By the chain rule, the Jacobian matrix of the m -th iteration of the mapping is given by the expression

$$\left(Q^{(m)}\right)'(q) = Q' \left(Q^{(m-1)}(q)\right) Q' \left(Q^{(m-2)}(q)\right) Q' \left(Q(q)\right) Q'(q). \quad (11)$$

Synchronous mode

Let $(x(t), t_n)$ be a solution of plant equations (1) with the parameters λ_k, T_k , and $x_k = x(t_k^-)$. Suppose that the plant is already running at the moment when the observer is initiated, i.e. $t_a \leq \hat{t}_0 < t_{a+1}$, for some $a \geq 1$.

Considering the solution $(\hat{x}(t), \hat{t}_n)$ of observer equations (8) subject to the initial conditions

$$\hat{t}_0 = t_a, \quad \hat{x}(\hat{t}_0^-) = x(t_a^-),$$

yields

$$\hat{x}_n = x_{n+a}, \quad \hat{t}_n = t_{n+a}, \quad \hat{\lambda}_n = \lambda_{n+a}, \quad n = 0, 1, 2, \dots,$$

and $\hat{x}(t) = x(t)$ for $t \geq t_a$. Such a solution $(\hat{x}(t), \hat{t}_n)$ will be called a *synchronous mode* of the observer with respect to $(x(t), t_n)$ (see [6] for a more detailed discussion).

A synchronous mode will be called *locally asymptotically stable* if for any solution $(\hat{x}(t), \hat{t}_n)$ of (8) such that the initial estimation errors $|\hat{t}_0 - t_a|$ and $\|\hat{x}(\hat{t}_0^-) - x(t_a^-)\|$ are sufficiently small, it follows that $\hat{t}_n - t_{n+a} \rightarrow 0$ and $\|\hat{x}(\hat{t}_n^-) - x(t_{n+a}^-)\| \rightarrow 0$ as $n \rightarrow \infty$. The latter implies $\hat{\lambda}_n - \lambda_{n+a} \rightarrow 0$ as $n \rightarrow \infty$.

For brevity, denote $n_a = n + a$, $\Phi'_k = \Phi'(Cx_k)$, $F'_k = F'(Cx_k)$. The synchronous mode with respect to $x(t)$ is completely characterized by the vector sequence

$$\hat{q}_n^0 = \begin{bmatrix} x_{n_a} \\ t_{n_a} \end{bmatrix}. \quad (12)$$

For for all $k \geq 0$, define matrices J_k comprised of the following matrix blocks

$$\begin{aligned} (J_k)_{11} &= \Phi'_k D x_{k+1} C + e^{\tilde{D}_K T_k} \left(I_{n_x} + F'_k e^{-\tilde{D}_K \tau} e^{D_K \tau} B C \right), \\ (J_k)_{12} &= D x_{k+1} - e^{\tilde{D}_K T_k} \left(D x_k + \lambda_k \tilde{D}_K e^{-\tilde{D}_K \tau} e^{D_K \tau} B \right), \\ (J_k)_{21} &= \Phi'_k C, \quad (J_k)_{22} = 1. \end{aligned}$$

Recall the mapping $Q(q)$ introduced in the previous section.

Theorem 4. *For any $n \geq 0$, the Jacobian of $Q(\cdot)$ at \hat{q}_n^0 is calculated as*

$$Q'(\hat{q}_n^0) = J_{n+a}. \quad (13)$$

Proof. Omitted.

From (11) it follows that for any $m \geq 1$

$$\left(Q^{(m)} \right)' (\hat{q}_n^0) = J_{n_a+m-1} J_{n_a+m-2} \dots J_{n_a+1} J_{n_a}. \quad (14)$$

Local stability of a synchronous mode with respect to an m -cycle

A solution of (1) is called m -cycle if it is periodic with exactly m pulse modulation instants in the least period. The existence conditions of an m -cycle in pulse-modulated time-delay system (1) were studied in [5, 3].

Let $(x(t), t_n)$ be an m -cycle of plant (1), where m is some integer, $m \geq 1$. Then $x_{n+m} \equiv x_n$, $\lambda_{n+m} \equiv \lambda_n$, $T_{n+m} \equiv T_n$. Consider a synchronous mode of observer (8) with respect to $(x(t), t_n)$ and let \hat{q}_n^0 be the corresponding vector sequence as in (12), such that $\hat{q}_{n+1}^0 = Q(\hat{q}_n^0)$ is satisfied.

In order to ensure feasibility of the observer, stability properties of the synchronous mode have to be investigated and guaranteed by design. Consider previously defined matrices J_n . Then $J_{n+m} \equiv J_n$, so that the sequence $\{J_n\}_{n=0}^{\infty}$ contains no more than m distinct matrices, namely J_0, \dots, J_{m-1} .

Theorem 5. *Let the matrix product $J_0 \dots J_{m-1}$ be Schur stable, i.e. all the eigenvalues of this matrix lie strictly inside the unit circle. Then the synchronous mode with respect to $(x(t), t_n)$ is locally asymptotically stable.*

Proof. Along the lines of Theorem 3 in [6].

Theorem 6. *Suppose that the linear part of system (1) is spectrally FD-observable. Let $(x(t), t_n)$ be an m -cycle and the notation of Section apply. Suppose*

$$-1 < \prod_{k=0}^{m-1} (\Phi'_k C D x_{k+2} + 1) < 1. \quad (15)$$

Then there is always a matrix K such that D_K is Hurwitz and the matrix product $J_0 \dots J_{m-1}$ is Schur stable.

Proof. Along the lines of Theorem 4 in [6].

Clearly, the stability properties of a synchronous mode depend on the parameters of the observed m -cycle in the plant.

Notice that in (15) $x_m = x_0$ and $x_{m+1} = x_1$. In the case of a 1-cycle, (15) turns into

$$-2 < \Phi'_0 C D x_0 < 0$$

and in the case of a 2-cycle (15) becomes

$$-1 < (\Phi'_0 C D x_0 + 1)(\Phi'_1 C D x_1 + 1) < 1. \quad (16)$$

Numerical examples

Assume the following values in model (1)

$$A_0 = \begin{bmatrix} -b_1 & 0 & 0 \\ g_1 & -b_2 & 0 \\ 0 & 0 & -b_3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & g_2 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = [0 \quad 0 \quad 1].$$

To demonstrate that in this case the linear part of the system is spectrally FD-observable, Lemma 2 is applied. One has

$$U = \begin{bmatrix} -b_1 & 0 \\ g_1 & -b_2 \end{bmatrix}, \quad V = [-b_3], \quad W = [0 \quad 0],$$

$$\bar{W} = [0 \quad g_2], \quad L_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Choose $L_0 = [1 \quad 0]$. Then $L_0 L_1 = 0$ and the matrix pair $(L_0 L_1, U)$ is observable, provided that $g_1 \neq 0$. The matrix pair (L_2, V) is evidently observable. Hence the assumptions of Lemma 2 are fulfilled.

Let $h = 2.7$, $b_1 = 0.02$, $b_2 = 0.15$, $b_3 = 0.1$, $g_1 = 0.6$, $g_2 = 1.5$, and

$$\Phi(z) = 40 + 80 \frac{(z/h)^2}{1 + (z/h)^2}, \quad F(z) = 0.05 + \frac{5}{1 + (z/h)^2}.$$

Since $\inf_z \Phi(z) = 40$, then $0 \leq \tau < 40$. As shown in Fig. 1, the plant has a stable z 4-cycle for $0 \leq \tau \leq 24.7$ and a stable 2-cycle for $24.8 \leq \tau < 40$. Interestingly, the dynamics of the closed-loop system described by (1) are simplified in this case with a higher time-delay value. This is the opposite to what usually happens in a closed-loop dynamical system with an increase of the time delay.

Let $L_m = \prod_{k=0}^{m-1} (\Phi'_k C D x_{k+2} + 1)$. Note that for 2-cycle $L_4 = (L_2)^2$ and, hence, $|L_4| < 1$ implies $|L_2| < 1$. Numerical calculations show, see Fig. 2, that condition (15) is fulfilled. Hence, the existence of a stabilizing observer gain K is guaranteed for the interval $0 \leq \tau < 40$.

Choose the observer feedback gain as

$$K = \begin{bmatrix} 0 & 0 \\ 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}.$$

Then D_K is Hurwitz stable, $(A_0 - KL, A_1)$ is FD-reducible and the synchronous mode is locally asymptotically stable as can be easily checked in Fig. 3.

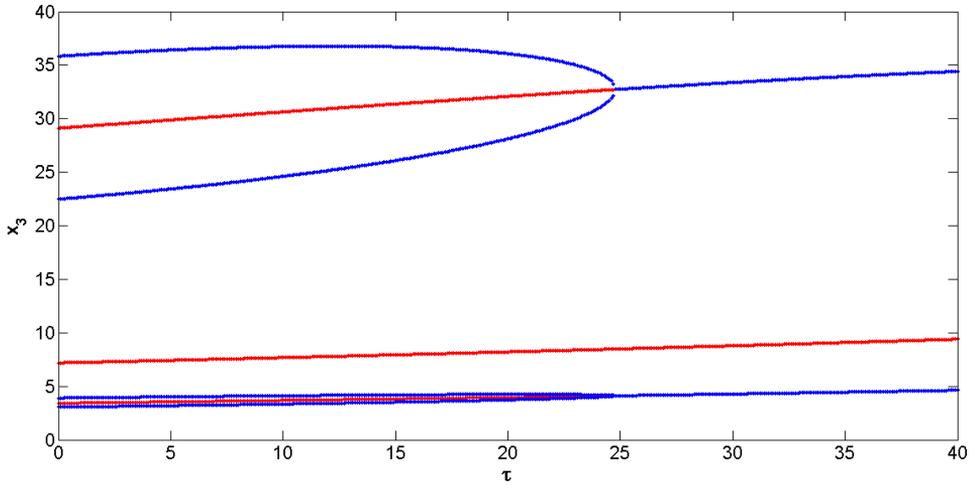


Figure 1: Bifurcation diagram of hybrid plant (1) with respect to τ . The values of x_3 at the feedback jump times t_n are depicted as function of τ . Blue dots — stable solutions, red dots — unstable solutions.

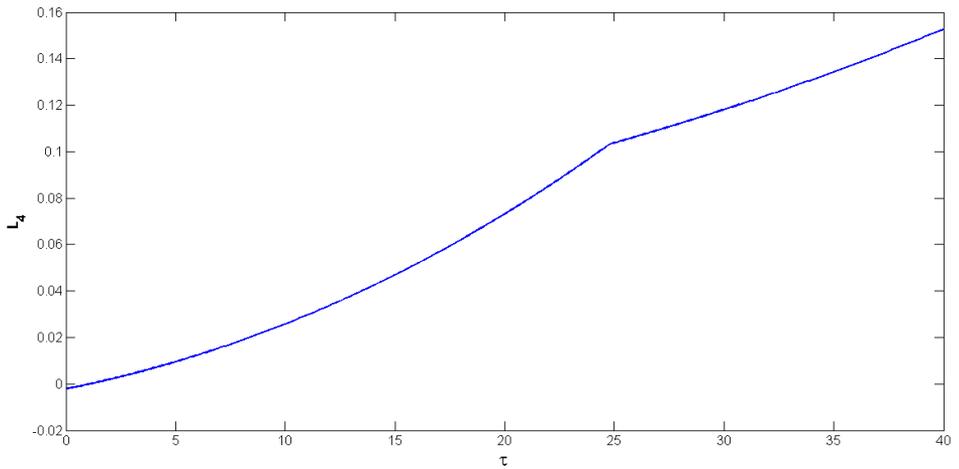


Figure 2: The dependence of L_4 on τ .

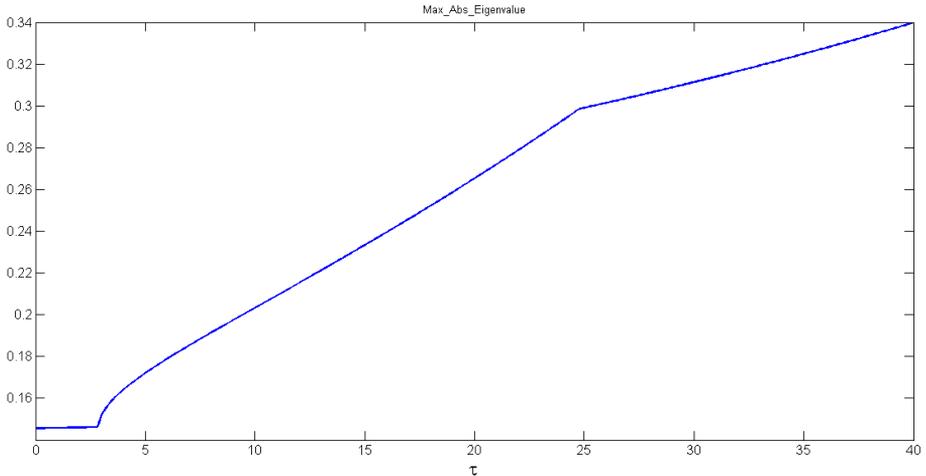


Figure 3: The dependence of the spectral radius of the product $J_0 J_1 J_2 J_3$ on τ . Notice that for $24.8 \leq \tau < 40$, a 4-cycle is reduced to a 2-cycle ($m = 2$) and $J_2 = J_0$, $J_3 = J_1$.

Observation of a 4-cycle

Let $\tau = 20$. Then, the plant has a stable 4-cycle with

$$x_0^\top = [0.0290 \quad 0.1337 \quad 3.7387]^\top,$$

$$x_1^\top = [0.2815 \quad 1.2994 \quad 36.0704]^\top,$$

$$x_3^\top = [0.0329 \quad 0.1518 \quad 4.2459]^\top,$$

$$x_4^\top = [0.2190 \quad 1.0106 \quad 28.1169]^\top,$$

where $(\cdot)^\top$ denotes transpose.

Fig. 4 illustrates the transients in the sequence $\hat{\lambda}_n$ produced by the observer relative to the sequence λ_n of the plant, caused by a mismatch in the initial conditions of the plant and those of the observer.

The red vertical lines of height $-\hat{\lambda}_n$ positioned at \hat{t}_n correspond to the observer jump sequence. The pulse modulation of the plant is shown by blue lines of height λ_n positioned at t_n . It can be seen that the jump instants of the observer become synchronized with those of the plant and the magnitudes of jumps $\hat{\lambda}_n$ asymptotically converge to λ_n .

The transients in the continuous states of the plant and the observer due to initial conditions mismatch are depicted in Fig. 5.

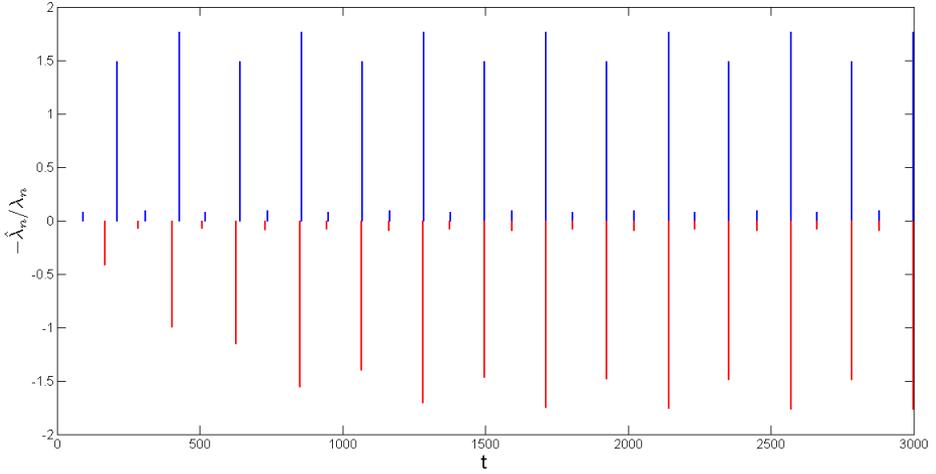


Figure 4: Transients due to non-zero initial conditions in the times and magnitudes of jumps of the plant and the observer ($\tau = 20$): Blue lines (upper part of the figure) mark the jump times of the observer \hat{t}_n with the height equal to $\hat{\lambda}_n$. Red lines (lower part of the figure) correspond to the pulse modulation of the plant in 4-cycle with the jump times t_n and the jumps $-\lambda_n$.

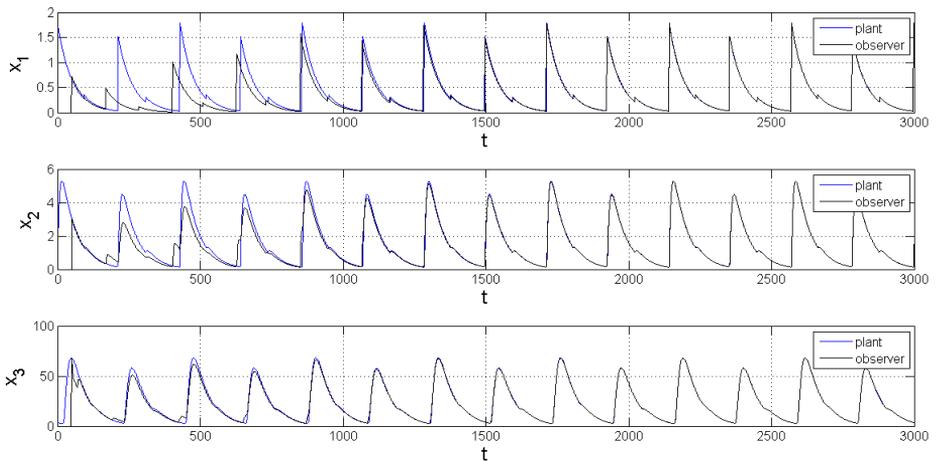


Figure 5: Transients in the continuous components of the plant (blue lines) and the observer (black lines) for $\tau = 20$.

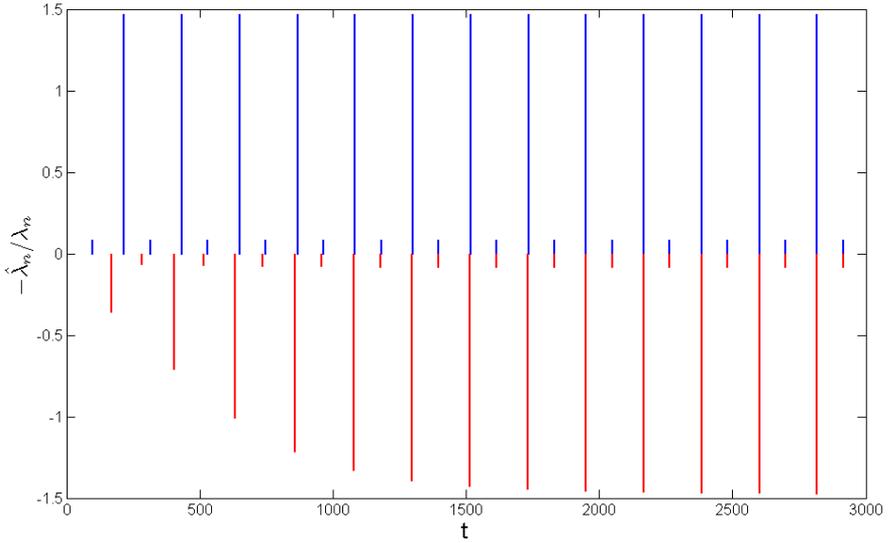


Figure 6: Transients due to non-zero initial conditions in the times and magnitudes of jumps of the plant and the observer ($\tau = 30$).

Observation of a 2-cycle

Let $\tau = 30$ yielding a stable 2-cycle with

$$x_0^\top = [0.0272 \quad 0.1255 \quad 4.2853]^\top,$$

$$x_1^\top = [0.2141 \quad 0.9883 \quad 33.3766]^\top.$$

Fig. 6 illustrates the transients in the sequence $\hat{\lambda}_n$ relative to λ_n , caused by a mismatch between the initial conditions of the plant and those of the observer.

The transients in the continuous states of the plant and the observer due to initial conditions mismatch are depicted in Fig. 7.

Conclusions

An earlier suggested observer structure is adopted for the state observation in hybrid impulsive systems with a time delay in the continuous part of the plant. By local stability analysis of the observer, it is shown that with a proper choice of the observer static gain and knowledge of the plant solution parameters, one can obtain an asymptotically converging estimate of the continuous system states and synchronization between the periodic pulse modulation sequence of the plant and that of the observer.

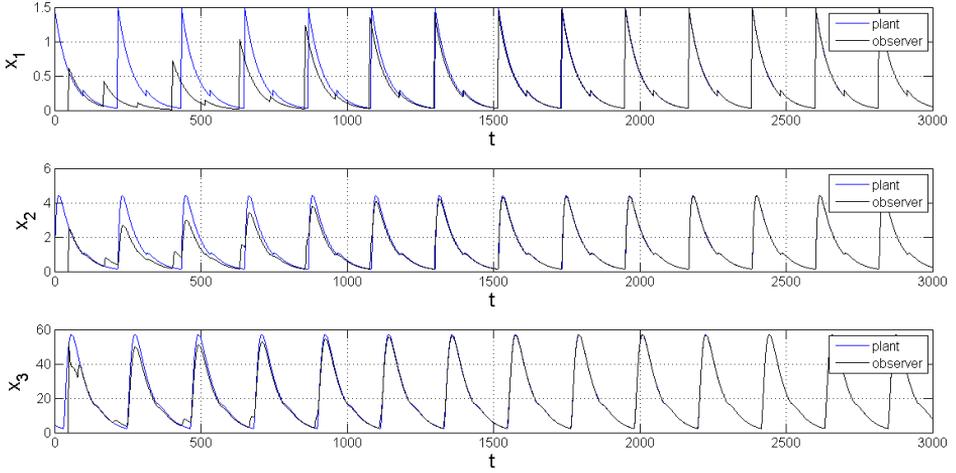


Figure 7: Transients in the continuous components of the plant (blue lines) and the observer (black lines) for $\tau = 30$.

Appendix

Proof of Theorem 2

To prove the result of Theorem, the following property of the functions $P_{k,s}(\zeta, \theta)$ is used.

Lemma 3. *The function $P_{k,s}(\zeta, \theta)$ can be represented as*

$$P_{k,s}(\zeta, \theta) = u_k(\zeta, \theta) + v_s(\zeta, \theta) + w(\zeta), \quad (17)$$

where

$$\begin{aligned} u_k(\zeta, \theta) &= e^{\tilde{D}_K \Phi(C\zeta)} \left[-e^{D(\theta-t_k)} x(t_k^-) - \lambda_k G(\theta - t_k) B \right] \\ &\quad - \lambda_k \tilde{R}(\theta - t_k, \Phi(C\zeta)) B + \sum_{j=1}^k \lambda_j \tilde{G}(\theta + \Phi(C\zeta) - t_j) B, \\ v_s(\zeta, \theta) &= e^{D(\theta + \Phi(C\zeta) - t_s)} x(t_s^-) - \sum_{j=1}^s \lambda_j \tilde{G}(\theta + \Phi(C\zeta) - t_j) B \\ &\quad + \lambda_s G(\theta + \Phi(C\zeta) - t_s) B, \\ w(\zeta) &= e^{\tilde{D}_K \Phi(C\zeta)} \zeta + F(C\zeta) \tilde{G}(\Phi(C\zeta)) B. \end{aligned}$$

Moreover, for $(\zeta, \theta) \in S_{k,s}$ the following recursions hold

$$\begin{aligned} u_k(\zeta, \theta) - u_{k-1}(\zeta, \theta) = \lambda_k \left[-e^{\tilde{D}_K \Phi(C\zeta)} G(\theta - t_k) \right. \\ \left. - \tilde{R}(\theta - t_k, \Phi(C\zeta)) + \tilde{G}(\theta + \Phi(C\zeta) - t_k) \right] B, \end{aligned} \quad (18)$$

$$\begin{aligned} v_s(\zeta, \theta) - v_{s-1}(\zeta, \theta) = \lambda_s \left[G(\theta + \Phi(C\zeta) - t_s) \right. \\ \left. - \tilde{G}(\theta + \Phi(C\zeta) - t_s) \right] B. \end{aligned} \quad (19)$$

Proof. Omitted.

Now Theorem 2 can be proved. Since the functions $G(\theta)$, $\tilde{G}(\theta)$, $\tilde{R}(\theta, \cdot)$ are continuous for $\theta \geq 0$, then the function $P(\zeta, \theta)$ can have gaps only on the surfaces

$$M_k = \{(\zeta, \theta) : \theta = t_k\}, \quad N_s = \{(\zeta, \theta) : \theta + \Phi(C\zeta) = t_s\}.$$

Let $(\zeta, \theta) \in M_k$ for some k . From (18) it follows that $u_k(\zeta, t_k) = u_{k-1}(\zeta, t_k)$, because

$$G(0) = I, \quad \tilde{R}(0, \Phi(C\zeta)) = e^{\tilde{D}_K(\Phi(C\zeta) - \tau)} e^{D_K \tau} - e^{\tilde{D}_K \Phi(C\zeta)}$$

and $\tilde{G}(\Phi(C\zeta)) = e^{\tilde{D}_K(\Phi(C\zeta) - \tau)} e^{D_K \tau}$. Hence, $P(\zeta, \theta)$ has no gaps on M_k .

Let $(\zeta, \theta) \in N_s$ for some s . From (19) it follows that

$$v_s(\zeta, t_s - \Phi(C\zeta)) = v_{s-1}(\zeta, t_s - \Phi(C\zeta)),$$

because $G(0) - \tilde{G}(0) = 0$. Hence, $P(\zeta, \theta)$ has no gaps on N_s .

If $(\zeta, \theta) \in M_k \cap N_s$ for some k and s , then $t_k = t_s - \Phi(C\zeta)$, hence, $P_{k,s}(\zeta, t_k) = P_{k-1,s-1}(\zeta, t_k)$.

Proof of Theorem 3

The following fact is needed for the proof.

Lemma 4. *The partial derivatives*

$$\frac{\partial P_{k,s}(\zeta, \theta)}{\partial \zeta}, \quad \frac{\partial P_{k,s}(\zeta, \theta)}{\partial \theta}$$

are continuous for $(\zeta, \theta) \in S_{k,s}$ whenever the scalar functions $F(\cdot)$, $\Phi(\cdot)$ have continuous derivatives.

Proof. Omitted.

It follows from Lemma 4 that the functions P'_ζ , P'_θ can have gaps either on the surfaces $M_k = \{(\zeta, \theta) : \theta = t_k\}$ or on the surfaces $N_s = \{(\zeta, \theta) : \theta + \Phi(C\zeta) = t_s\}$.

Let $(\zeta, \theta) \in M_k$ for some k . Then

$$\begin{aligned} \left. \frac{\partial(u_k(\zeta, \theta) - u_{k-1}(\zeta, \theta))}{\partial\zeta} \right|_{\theta=t_k} &= \lambda_k \Phi'(C\zeta) \left[-\tilde{D}_K e^{\tilde{D}\Phi(C\zeta)} \right. \\ &\quad \left. - \tilde{D}_K e^{\tilde{D}_K \Phi(C\zeta)} (e^{-D\tau} e^{D_K \tau} - I) + \tilde{D}_K e^{\tilde{D}_K (\Phi(C\zeta) - \tau) e^{D_K \tau}} \right] BC = 0, \end{aligned}$$

$$\begin{aligned} \left. \frac{\partial(u_k(\zeta, \theta) - u_{k-1}(\zeta, \theta))}{\partial\theta} \right|_{\theta=t_k} &= \lambda_k \left[-e^{\tilde{D}_K \Phi(C\zeta)} A_0 \right. \\ &\quad \left. - e^{\tilde{D}_K \Phi(C\zeta)} (\tilde{D}_K e^{-D\tau} e^{D_K \tau} - D_K) + \tilde{D}_K e^{\tilde{D}_K (\Phi(C\zeta) - \tau) e^{D_K \tau}} \right] B = 0, \end{aligned}$$

because $e^{\tilde{D}_K \Phi(C\zeta)} \tilde{D}_K = \tilde{D}_K e^{\tilde{D}_K \Phi(C\zeta)}$ and $(A_0 - D_K)B = KLB = 0$.

Thus

$$\frac{\partial u_k}{\partial\zeta} = \frac{\partial u_{k-1}}{\partial\zeta}, \quad \frac{\partial u_k}{\partial\theta} = \frac{\partial u_{k-1}}{\partial\theta}$$

at a point (ζ, t_k) . Consequently, P'_ζ and P'_θ have no gaps on this surface.

Let $(\zeta, \theta) \in N_s$ for some s .

$$\begin{aligned} \left. \frac{\partial(v_s(\zeta, \theta) - v_{s-1}(\zeta, \theta))}{\partial\zeta} \right|_{\theta+\Phi(C\zeta)=t_s} & \\ &= \lambda_s \Phi'(C\zeta) (A_0 - D_K) BC = 0, \end{aligned}$$

$$\left. \frac{\partial(v_s(\zeta, \theta) - v_{s-1}(\zeta, \theta))}{\partial\theta} \right|_{\theta+\Phi(C\zeta)=t_s} = \lambda_s (A_0 - D_K) B = 0.$$

Thus,

$$\frac{\partial v_s}{\partial\zeta} = \frac{\partial v_{s-1}}{\partial\zeta}, \quad \frac{\partial v_s}{\partial\theta} = \frac{\partial v_{s-1}}{\partial\theta}$$

at a point $(\zeta, t_s - \Phi(C\zeta))$. Hence, P'_ζ and P'_θ have no gaps on this surface.

Clearly, if $(\zeta, \theta) \in M_k \cap N_s$, for some k and s , then $t_k = t_s - \Phi(C\zeta)$, and hence

$$\frac{\partial P_{k,s}}{\partial\zeta} = \frac{\partial P_{k-1,s-1}}{\partial\zeta}, \quad \frac{\partial P_{k,s}}{\partial\theta} = \frac{\partial P_{k-1,s-1}}{\partial\theta}$$

at a point (ζ, t_k) . This means that partial derivatives of $P(\zeta, \theta)$ have no gaps on $(\zeta, \theta) \in M_k \cap N_s$. This completes the proof of Theorem 3.

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Appendix A

This chapter contains detailed proofs omitted from the publication of PAPER II.

Proof of Proposition 1.

With K defined by (9), $D = A - KL$ is a block-diagonal matrix:

$$D = \begin{bmatrix} -b_1 & -k_1 & 0 \\ g_1 & -b_2 - k_3 & 0 \\ 0 & 0 & -b_3 - k_6 \end{bmatrix}.$$

It is straightforward to see $k_6 = -\mu_3 - b_3$. Find now k_1 and k_3 resulting in the desired spectrum of D . Denote

$$A_1 = \begin{bmatrix} -b_1 & 0 \\ g_1 & -b_2 \end{bmatrix}, K_1 = \begin{bmatrix} k_1 \\ k_3 \end{bmatrix}, L_1 = [0 \quad 1], D_1 = \begin{bmatrix} -b_1 & -k_1 \\ g_1 & -b_2 - k_3 \end{bmatrix}.$$

Solve a dual problem: find $K_1^T = [k_1 \quad k_3]$ such that μ_1 and μ_2 are eigenvalues of the matrix $D_1^T = A_1^T - L_1^T K_1^T$. Then the characteristic polynomial of D_1^T is $\Phi_d(s) = s^2 - (\mu_1 + \mu_2)s + \mu_1\mu_2$, the controllability matrix is $M_c = [L_1^T \mid A_1^T L_1^T] = \begin{bmatrix} 0 & g_1 \\ 1 & -b_2 \end{bmatrix}$ and $M_c^{-1} = \begin{bmatrix} b_2/g_1 & 1 \\ 1/g_1 & 0 \end{bmatrix}$.

Now the Ackermann's formula solves the above pole placement problem:

$$K_1^T = [0 \quad 1] M_c^{-1} \Phi_d(A) = \begin{bmatrix} \underbrace{\frac{1}{g_1} [b_1^2 + b_1(\mu_1 + \mu_2) + \mu_1\mu_2]}_{k_1} & \underbrace{-(b_1 + b_2 + (\mu_1 + \mu_2))}_{k_3} \end{bmatrix}.$$

□

Proof of Theorem 2

1. From (12), it is easy to conclude that the matrix J_0 can be decomposed as

$$J_0 = V_0(K_f) + W_0(K), \quad (\text{A.1})$$

where

$$V_0(K_f) = \begin{bmatrix} Ax_0 \\ 1 \end{bmatrix} \begin{bmatrix} \Phi'_0 R & 1 - \Phi'_k K_f L A x_0 \end{bmatrix}$$

and

$$W_0(K) = \begin{bmatrix} e^{DT_0} \\ 0 \end{bmatrix} \begin{bmatrix} I + F'_0 BC & -A(x_0 + \lambda_0 B) \end{bmatrix}.$$

2. Consider a similarity transformation of (A.1)

$$\tilde{J}_0 = T^{-1} J_0 T = \underbrace{T^{-1} V_0(K_f) T}_{\tilde{V}_0(K_f)} + \underbrace{T^{-1} W_0(K) T}_{\tilde{W}_0(K)}, \quad (\text{A.2})$$

where $T = \begin{bmatrix} I & Ax_0 \\ \mathbb{O} & 1 \end{bmatrix}$, $T^{-1} = \begin{bmatrix} I & -Ax_0 \\ \mathbb{O} & 1 \end{bmatrix}$. Then

$$\tilde{V}_0(K_f) = \begin{bmatrix} \mathbb{O} & \mathbb{O} \\ \Phi'_0 R & \Phi'_0 C A x_0 + 1 \end{bmatrix},$$

$$\tilde{W}_0(K) = \begin{bmatrix} e^{DT_0} \\ 0 \end{bmatrix} \begin{bmatrix} I + F'_0 BC & F'_0 BC A x_0 - \lambda_0 AB \end{bmatrix}.$$

3. Obviously, the eigenvalues, determinant and characteristic polynomial of J_0 coincide with those of \tilde{J}_0 . Since $CB = 0$, hence $\det(e^{DT_0}(I + F'_0 BC)) = \det(e^{DT_0}) \neq 0$. Therefore, using Schur complements, the determinant of J_0 is evaluated as follows

$$\begin{aligned} \det(J_0) &= \det(e^{DT_0}) \left[\Phi'_0 C A x_0 + 1 - \Phi'_0 R \right. \\ &\quad \left. \times (e^{DT_0}(I + F'_0 BC))^{-1} e^{DT_0}(F'_0 BC A x_0 - \lambda_0 AB) \right], \end{aligned}$$

or, with the equality $RB = 0$ taken into account,

$$\det(J_0) = \det(e^{DT_0}) (\Phi'_0 C A x_0 + 1 + \Phi'_0 \lambda_0 R A B). \quad (\text{A.3})$$

4. From the Schur complements, it is easy to conclude that the characteristic polynomial of \tilde{J}_0 can be computed as follows:

$$\begin{aligned} p(\lambda) &= \det(\lambda I - J_0) = \det(\lambda I - \tilde{J}_0) = \\ &= \det(\lambda I - e^{DT_0}(I + F'_0 BC)) (\lambda - \Phi'_0 C A x_0 - 1) \\ &\quad - \Phi'_0 R \operatorname{adj}(\lambda I - e^{DT_0}(I + F'_0 BC)) \\ &\quad \times e^{DT_0}(F'_0 BC A x_0 - \lambda_0 AB), \quad (\text{A.4}) \end{aligned}$$

where $\text{adj}(M)$ is the adjugate matrix of M .
 Since $e^{DT_0} = \text{adj}(e^{-DT_0}) \det(e^{DT_0})$, one obtains

$$\begin{aligned} p(\lambda) = \det(e^{DT_0}) & \left[(\lambda - \Phi'_0 C A x_0 - 1) \right. \\ & \times \det(\lambda e^{-DT_0} - (I + F'_0 B C)) \\ & \left. - \Phi'_0 R \text{adj}(\lambda e^{-DT_0} - (I + F'_0 B C))(F'_0 B C A x_0 - \lambda_0 A B) \right]. \quad (\text{A.5}) \end{aligned}$$

The Cayley–Hamilton theorem allows the determinant and the adjugate of $M = (\lambda e^{-DT_0} - (I + F'_0 B C))$ to be represented in terms of traces and powers of \tilde{J}_0 . For the third order case it gives

$$\begin{aligned} \text{adj}(M) &= \frac{1}{2} ((\text{tr } M)^2 - \text{tr } M^2) I_3 - M \text{tr } M + M^2, \\ \det(M) &= \frac{1}{6} ((\text{tr } M)^3 - 3 \text{tr } M \text{tr } M^2 + 2 \text{tr } M^3). \end{aligned}$$

Also notice that with structure (9), the equality $\text{tr}(e^{DT_0} B C) = 0$ holds.

Thus, by direct calculations

$$\begin{aligned} \text{adj}(\lambda e^{-DT_0} - (I + F'_0 B C)) &= \frac{1}{2} \lambda^2 (\text{tr } e^{-DT_0})^2 I_3 \\ &\quad - 3 \lambda \text{tr } e^{-DT_0} I_3 - \frac{1}{2} \lambda^2 \text{tr}(e^{-DT_0})^2 I_3 + \\ &\quad 2 \lambda \text{tr}(e^{-DT_0}) I_3 - \lambda^2 e^{-DT_0} \text{tr } e^{-DT_0} + \lambda e^{-DT_0} \\ &\quad + \lambda F'_0 B C \text{tr } e^{-DT_0} - F'_0 B C + \lambda^2 (e^{-DT_0})^2 \\ &\quad - \lambda F'_0 e^{-DT_0} B C + I_3 - \lambda F'_0 B C e^{-DT_0}, \quad (\text{A.6}) \end{aligned}$$

$$\begin{aligned} \det(\lambda e^{-DT_0} - (I + F'_0 B C)) &= \frac{1}{6} (\lambda^3 (\text{tr } e^{-DT_0})^3 \\ &\quad - 3 \lambda^2 (\text{tr } e^{-DT_0})^2 - 3 \lambda^3 \text{tr } e^{-DT_0} \text{tr}(e^{-DT_0})^2 \\ &\quad + 3 \lambda^2 \text{tr}(e^{-DT_0})^2 - 6 \lambda F'_0 \text{tr}(e^{-DT_0} B C) \\ &\quad + 2 \lambda^3 \text{tr}(e^{-DT_0})^3 + 6 \lambda \text{tr}(e^{-DT_0}) - 6). \quad (\text{A.7}) \end{aligned}$$

Substituting (A.6) and (A.7) into (A.5) one obtains

$$p(\lambda) = \lambda^4 + a_1 \lambda^3 + a_2 \lambda^2 + a_3 \lambda + a_4,$$

with the coefficients a_i , $i = 1, \dots, 4$ given by (14).

Finally, use the bilinear transformation $\lambda = r \frac{\omega + 1}{\omega - 1}$ that maps the left-hand side plane into the inside of the circle of a radius r . With this

transformation, the polynomial $p(\lambda)$ is converted to the polynomial $\mu(\omega) = d_0\omega^4 + d_1\omega^3 + d_2\omega^2 + d_3\omega + d_4$, where the coefficients d_i , $i = 0, \dots, 4$ are given by (13). Therefore, all the roots of the polynomial $p(\lambda)$ lie inside of the circle of radius r if and only if all the roots of the polynomial $\mu(\omega)$ are in the left-hand side. The statement of Theorem 2 follows immediately from the Routh–Hurwitz criterion.

Proof of Lemma 1

From the proof of Theorem 2, it follows that

$$\tilde{J}_0 = \begin{bmatrix} e^{DT_0}(I + F'_0BC) & e^{DT_0}R_2 \\ \Phi'_0R & r_1 \end{bmatrix}.$$

By exploiting the structure of (9) and the definitions of A, B, C , and R_2 , it is easy to see that the matrix $e^{DT_0}(I + F'_0BC)$ and the vector $e^{DT_0}R_2$ have the following structures:

$$e^{DT_0}(I + F'_0BC) = \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & w \end{bmatrix}, \quad e^{DT_0}R_2 = \begin{bmatrix} * \\ * \\ 0 \end{bmatrix}.$$

Hence, the matrix $\tilde{J}_0 - wI$ has a zero third row. Therefore, $\det(\tilde{J}_0 - wI) = \det(J_0 - wI) = 0$ and w is an eigenvalue of \tilde{J}_0 and J_0 .

Proof of Theorem 3

From equality (16), it follows that setting $a_4 = 0$ leads to a zero root of the characteristic polynomial $p(\lambda)$. Therefore, zero is an eigenvalue of J_0 .

From (9), (16), (17) it follows that the matrix e^{DT_0} has the following elements:

$$e^{DT_0} = \begin{bmatrix} e^{-b_1T_0} & 0 & 0 \\ e_{21} & e^{(-b_2-k_3)T_0} & 0 \\ 0 & 0 & \varkappa \end{bmatrix},$$

where

$$e_{21} = \frac{-g_1 (e^{-(b_2+k_3)T_0} - e^{-b_1T_0})}{b_2 - b_1 + k_3}.$$

Since $e^{(-b_3-k_6)T_0} = \varkappa$, one obtains by Lemma 1 that \varkappa is the second eigenvalue of J_0 .

Since $\text{tr}(e^{DT_0}) = e^{-b_1T_0} + e^{(-b_2-k_3)T_0} + \varkappa$, one has

$$a_1 = -e^{(-b_2-k_3)T_0} - [e^{-b_1T_0} + \varkappa + r_1] < 0. \quad (\text{A.8})$$

After factorization and reduction of the roots found above, the characteristic polynomial $p(\lambda)$ of J_0 is transformed to a second-order polynomial:

$$p(\lambda) = \lambda^2 + c_1\lambda + c_2.$$

From (A.8) one obtains $c_1 = -e^{(-b_2-k_3)T_0} - 2\rho_{min} < 0$, hence axis of symmetry of $p(\lambda)$ (vertical line $\lambda = -\frac{c_1}{2}$) is in the right half of the hyperplane $\lambda - p(\lambda)$. Hence, the spectral radius of J_0 is defined by the maximum modulus root of $p(\lambda)$, i.e.

$$\rho(J_0) = \rho_{min} + |g(k_3)|.$$

Theorem 3 follows.

Appendix B

This chapter contains detailed proofs omitted from the publication of PAPER IV.

Proof of Theorem 1.

The state estimation error of the observer $r(t) = x(t) - \hat{x}(t)$ obeys

$$\begin{aligned} \frac{dr}{dt} &= D_0 r(t) + A_1 r(t - \tau), \quad \hat{z}(t) = z(t) - Cr(t), \\ r(t_n^+) &= r(t_n^-) + \lambda_n B, \quad r(\hat{t}_n^+) = r(\hat{t}_n^-) - \hat{\lambda}_n B. \end{aligned} \quad (\text{B.1})$$

Notice that $r(t_n^+) = r(t_n^-) + \lambda_n B - \hat{\lambda}_n B$, when $t_n = \hat{t}_n$.

Pick an arbitrary integer $n \geq 0$ and suppose that

$$t_k \leq \hat{t}_n < t_{k+1}, \quad t_s \leq \hat{t}_n + \Phi(C\hat{x}_n) < t_{s+1}$$

for some k and s , such that $s \geq k$ (Fig.1). Introduce a number $m \geq 0$ such that $s = k + m$.

Now the following steps have to be considered to arrive to the result of Theorem 1.

Lemma 1. *Given the sequences $\{\tilde{t}_n\}_{n=0}^{\infty}$, $\{\tilde{\lambda}_n\}_{n=0}^{\infty}$ of firing times and weights of function $r(t)$. Then solution $r(t)$ at $\tilde{t}_{n+1} - 0$ ($n \geq 1$) is as follows*

$$\begin{aligned} r(\tilde{t}_{n+1}^-) &= e^{\tilde{D}(\tilde{t}_{n+1} - \tilde{t}_n)} r(\tilde{t}_n^-) + \\ &\quad \tilde{\lambda}_n \tilde{G}(\tilde{t}_{n+1} - \tilde{t}_n) B + \tilde{\lambda}_{n-1} \tilde{R}(\tilde{t}_n - \tilde{t}_{n-1}, \tilde{t}_{n+1} - \tilde{t}_n) B. \end{aligned}$$

Proof. The firing times sequence $\{\tilde{t}_n\}_{n=0}^{\infty}$ consists of the union of firing times of the plant and the observer, then

$$\tilde{t}_{n+1} - \tilde{t}_{n-1} \geq \inf_z \Phi(z) > \tau. \quad (\text{B.2})$$

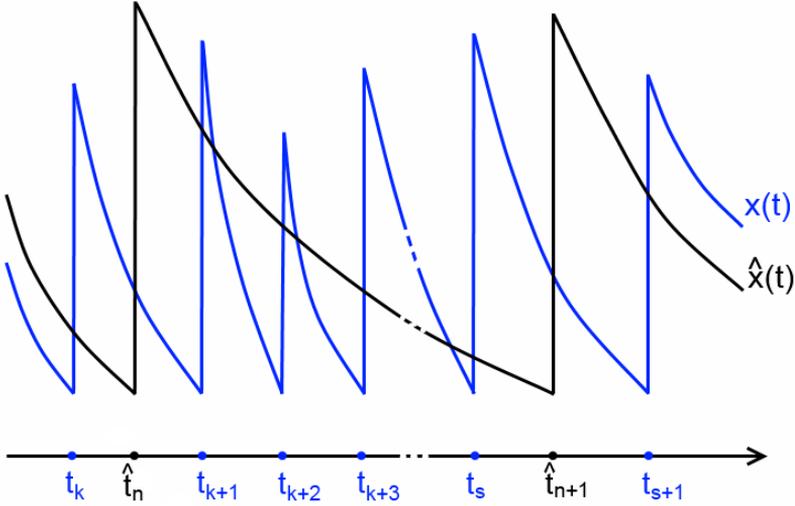


Figure 1: The firing times of the plant and the observer

To prove the statement of lemma use the methodology proposed in [1]. Since the pair (D_0, A_1) is FD-reducible, then matrices D_0 and A_1 can be represented in the block form:

$$D_0 = \begin{bmatrix} U & 0 \\ W & V \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ \bar{W} & 0 \end{bmatrix},$$

and for $\tilde{t}_n < t < \tilde{t}_{n+1}$ the equation

$$\frac{dr}{dt} = D_0 r(t) + A_1 r(t - \tau)$$

is as follows:

$$\dot{u}(t) = Uu(t), \quad \dot{v}(t) = Wu(t) + Vv(t) + \bar{W}u(t - \tau), \quad (\text{B.3})$$

where $r^T = [u^T, v^T]$. Suppose that $B^T = [B_1^T, B_2^T]$, where the sizes of B_1 and B_2 agree with those of u and v . Then

$$u(\tilde{t}_n^+) = u(\tilde{t}_n^-) + \tilde{\lambda}_n B_1, \quad v(\tilde{t}_n^+) = v(\tilde{t}_n^-) + \tilde{\lambda}_n B_2. \quad (\text{B.4})$$

From the first equation of (B.3) one obtains

$$u(t) = \begin{cases} e^{U(t-\tilde{t}_n)} u(\tilde{t}_n^-), & \tilde{t}_{n-1} < t < \tilde{t}_n, \\ e^{U(t-\tilde{t}_n)} u(\tilde{t}_n^+), & \tilde{t}_n < t < \tilde{t}_{n+1}. \end{cases} \quad (\text{B.5})$$

By following [1], rewrite (B.1) as:

$$\dot{r}(t) = \tilde{D}r(t) - (\tilde{D} - D_0) [r(t) - e^{D_0\tau}r(t - \tau)]. \quad (\text{B.6})$$

Since

$$\tilde{D} - D_0 = \begin{bmatrix} 0 & 0 \\ \tilde{W}e^{-U\tau} & 0 \end{bmatrix},$$

(B.1) is equivalent to

$$\dot{r}(t) = \tilde{D}r(t) - (\tilde{D} - D_0)\eta(t), \quad \eta(t) = \begin{bmatrix} u_d(t) \\ * \end{bmatrix}, \quad (\text{B.7})$$

where $u_d(t) = u(t) - e^{U\tau}u(t - \tau)$, and $*$ may be replaced by any suitable dimension vector.

Next, consider four possible cases.

1) Suppose that $\tilde{t}_{n-1} + \tau < \tilde{t}_n < \tilde{t}_{n+1} < \tilde{t}_n + \tau$ (Fig. 2). Pick an arbitrary

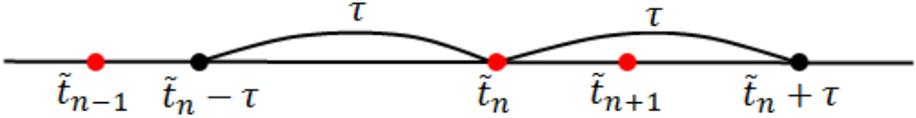


Figure 2: Case 1. The red dots indicate the firing times of the function $r(t)$

number t from the interval $\tilde{t}_n < t < \tilde{t}_{n+1}$. Then $\tilde{t}_{n-1} < t - \tau < \tilde{t}_n$. From (B.5)

$$u(t) = e^{U(t-\tilde{t}_n)}u(\tilde{t}_n^+), \quad u(t - \tau) = e^{U(t-\tau-\tilde{t}_n)}u(\tilde{t}_n^-). \quad (\text{B.8})$$

Hence

$$u_d(t) = e^{U(t-\tilde{t}_n)} [u(\tilde{t}_n^+) - u(\tilde{t}_n^-)] = \tilde{\lambda}_n e^{U(t-\tilde{t}_n)} B_1.$$

Due to suitable choice of $*$, (B.7) suppose that $\eta(t) = \tilde{\lambda}_n e^{D_0(t-\tilde{t}_n)} B$. Notice $\dot{\eta} = D_0\eta$ and the difference $r_d(t) = r(t) - \eta(t)$ satisfies the homogeneous linear equation $\dot{r}_d = \tilde{D}r_d(t)$ for $\tilde{t}_n < t < \tilde{t}_{n+1}$. Then

$$r(\tilde{t}_{n+1}^-) = \eta(\tilde{t}_{n+1}^-) + e^{\tilde{D}(\tilde{t}_{n+1}-\tilde{t}_n)} [r(\tilde{t}_n^+) - \eta(\tilde{t}_n^+)].$$

Since

$$r(\tilde{t}_n^+) - \eta(\tilde{t}_n^+) = r(\tilde{t}_n^-) + \tilde{\lambda}_n B - \tilde{\lambda}_n B = r(\tilde{t}_n^-), \quad \eta(\tilde{t}_{n+1}^-) = \tilde{\lambda}_n e^{D_0(\tilde{t}_{n+1}-\tilde{t}_n)} B,$$

then $r(\tilde{t}_{n+1}^-) = e^{\tilde{D}(\tilde{t}_{n+1}-\tilde{t}_n)} r(\tilde{t}_n^-) + \tilde{\lambda}_n e^{D_0(\tilde{t}_{n+1}-\tilde{t}_n)} B$.

2) Suppose that $\tilde{t}_{n+1} - \tilde{t}_n > \tau$ and $\tilde{t}_n - \tilde{t}_{n-1} > \tau$ (Fig. 3).

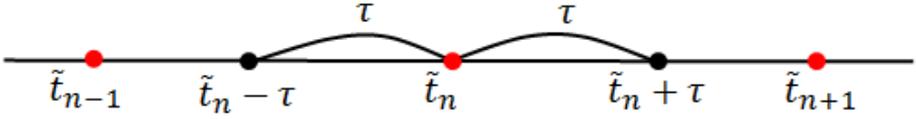


Figure 3: Case 2. The red dots indicate the firing times of the function $r(t)$

From Theorem 1 in [2] it follows that

$$r(\tilde{t}_{n+1}^-) = e^{\tilde{D}(\tilde{t}_{n+1} - \tilde{t}_n - \tau)} r(\tilde{t}_n + \tau).$$

Similarly to the previous case (replacing \tilde{t}_{n+1} by $\tilde{t}_n + \tau$) one has

$$r(\tilde{t}_n + \tau - 0) = r(\tilde{t}_n + \tau) = e^{\tilde{D}\tau} r(\tilde{t}_n^-) + \tilde{\lambda}_n e^{D_0\tau} B,$$

where $f(x^* - 0)$ and $f(x^* + 0)$ denote one-sided limits of the function $f(x)$ as x approaches x^* from below or from above, respectively.

Hence, in case 2 $r(\tilde{t}_{n+1}^-) = e^{\tilde{D}(\tilde{t}_{n+1} - \tilde{t}_n)} r(\tilde{t}_n^-) + \tilde{\lambda}_n e^{\tilde{D}(\tilde{t}_{n+1} - \tilde{t}_n - \tau)} e^{D_0\tau} B$.

3) Suppose that $\tilde{t}_{n+1} - \tilde{t}_n < \tau$ and $\tilde{t}_n - \tilde{t}_{n-1} < \tau$ (Fig. 4).

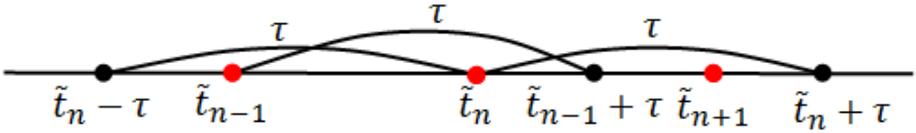


Figure 4: Case 3. The red dots indicate the firing times of the function $r(t)$

Since $\tilde{t}_{n-1} + \tau < \tilde{t}_{n+1} < \tilde{t}_n + \tau$, then $\tilde{t}_{n-1} < \tilde{t}_{n+1} - \tau < \tilde{t}_n$.

Pick an arbitrary number t from the interval $\tilde{t}_{n-1} + \tau < t < \tilde{t}_{n+1}$. Then $\tilde{t}_{n-1} < t - \tau < \tilde{t}_n$, and from (B.5)

$$u(t) = e^{U(t - \tilde{t}_{n-1} - \tau)} u(\tilde{t}_{n-1} + \tau) = e^{U(t - \tilde{t}_n)} u(\tilde{t}_n^+), \quad u(t - \tau) = e^{U(t - \tau - \tilde{t}_n)} u(\tilde{t}_n^-).$$

Hence, for $\tilde{t}_{n-1} + \tau < t < \tilde{t}_{n+1}$, as in the case 1, obtain

$$u(t) - e^{U\tau} u(t - \tau) = e^{U(t - \tilde{t}_n)} [u(\tilde{t}_n^+) - u(\tilde{t}_n^-)] = \tilde{\lambda}_n e^{U(t - \tilde{t}_n)} B_1,$$

consequently, in (B.7) $\eta(t) = \tilde{\lambda}_n e^{D_0(t - \tilde{t}_n)} B$, and the following equalities hold: $\dot{\eta}(t) = D_0\eta(t)$ and $\dot{r}_d(t) = \tilde{D}r_d(t)$. Then

$$r(\tilde{t}_{n+1}^-) = \eta(\tilde{t}_{n+1}^-) + e^{\tilde{D}(\tilde{t}_{n+1} - \tilde{t}_{n-1} - \tau)} [r(\tilde{t}_{n-1} + \tau) - \eta(\tilde{t}_{n-1} + \tau + 0)],$$

and

$$\eta(\tilde{t}_{n+1}^-) = \tilde{\lambda}_n e^{D_0(\tilde{t}_{n+1} - \tilde{t}_n)} B, \quad \eta(\tilde{t}_{n-1} + \tau + 0) = \tilde{\lambda}_n e^{D_0(\tilde{t}_{n-1} + \tau - \tilde{t}_n)} B.$$

Find now $r(\tilde{t}_{n-1} + \tau)$. For $\tilde{t}_n < t < \tilde{t}_{n-1} + \tau$ and, consequently, $\tilde{t}_n - \tau < t - \tau < \tilde{t}_{n-1}$ one has

$$u(t) = e^{U(t-\tilde{t}_n)} u(\tilde{t}_n^+),$$

$$\begin{aligned} u(t - \tau) &= e^{U(t-\tau-\tilde{t}_{n-1})} u(\tilde{t}_{n-1}^-) = e^{U(t-\tau-\tilde{t}_{n-1})} u(\tilde{t}_{n-1}^+) - \\ &\quad \tilde{\lambda}_{n-1} e^{U(t-\tau-\tilde{t}_{n-1})} B = e^{U(t-\tau-\tilde{t}_n)} u(\tilde{t}_n^-) - \tilde{\lambda}_{n-1} e^{U(t-\tau-\tilde{t}_{n-1})} B_1. \end{aligned}$$

Thus,

$$\begin{aligned} u(t) - e^{U\tau} u(t - \tau) &= e^{U(t-\tilde{t}_n)} [u(\tilde{t}_n^+) - u(\tilde{t}_n^-)] + \tilde{\lambda}_{n-1} e^{U(t-\tilde{t}_{n-1})} B_1 = \\ &= \tilde{\lambda}_n e^{U(t-\tilde{t}_n)} B_1 + \tilde{\lambda}_{n-1} e^{U(t-\tilde{t}_{n-1})} B_1, \end{aligned}$$

and on that account $\eta(t) = \tilde{\lambda}_n e^{D_0(t-\tilde{t}_n)} B + \tilde{\lambda}_{n-1} e^{D_0(t-\tilde{t}_{n-1})} B$, $\dot{\eta}(t) = D_0 \eta(t)$ and $\dot{r}_d(t) = \tilde{D} r_d(t)$ on the interval $\tilde{t}_n < t < \tilde{t}_{n-1} + \tau$. Then

$$r(\tilde{t}_{n-1} + \tau - 0) = r(\tilde{t}_{n-1} + \tau) = \eta(\tilde{t}_{n-1} + \tau - 0) + e^{\tilde{D}(\tilde{t}_{n-1} + \tau - \tilde{t}_n)} [r(\tilde{t}_n^+) - \eta(\tilde{t}_n^+)].$$

Since

$$\begin{aligned} r(\tilde{t}_n^+) - \eta(\tilde{t}_n^+) &= r(\tilde{t}_n^-) + \tilde{\lambda}_n B - \tilde{\lambda}_n B - \tilde{\lambda}_{n-1} e^{D_0(\tilde{t}_n - \tilde{t}_{n-1})} B = \\ &= r(\tilde{t}_n^-) - \tilde{\lambda}_{n-1} e^{D_0(\tilde{t}_n - \tilde{t}_{n-1})} B, \end{aligned}$$

$$\eta(\tilde{t}_{n-1} + \tau - 0) = \tilde{\lambda}_n e^{D_0(\tilde{t}_{n-1} + \tau - \tilde{t}_n)} B + \tilde{\lambda}_{n-1} e^{D_0\tau} B,$$

this gives

$$\begin{aligned} r(\tilde{t}_{n-1} + \tau) &= e^{\tilde{D}(\tilde{t}_{n-1} + \tau - \tilde{t}_n)} r(\tilde{t}_n^-) + \tilde{\lambda}_n e^{\tilde{D}(\tilde{t}_{n-1} + \tau - \tilde{t}_n)} B - \\ &\quad - \tilde{\lambda}_{n-1} \left(e^{\tilde{D}(\tilde{t}_{n-1} + \tau - \tilde{t}_n)} e^{D_0(\tilde{t}_n - \tilde{t}_{n-1})} - e^{D_0\tau} \right) B. \end{aligned}$$

Thus, in case 3

$$\begin{aligned} r(\tilde{t}_{n+1}^-) &= e^{\tilde{D}(\tilde{t}_{n+1} - \tilde{t}_n)} r(\tilde{t}_n^-) + \tilde{\lambda}_n e^{D_0(\tilde{t}_{n+1} - \tilde{t}_n)} B - \\ &\quad - \tilde{\lambda}_{n-1} e^{\tilde{D}(\tilde{t}_{n+1} - \tilde{t}_n)} \left(e^{D_0(\tilde{t}_n - \tilde{t}_{n-1})} - e^{\tilde{D}(\tilde{t}_n - \tilde{t}_{n-1} - \tau)} e^{D_0\tau} \right) B. \end{aligned}$$

4) Suppose that $\tilde{t}_{n+1} - \tilde{t}_n > \tau$ and $\tilde{t}_n - \tilde{t}_{n-1} < \tau$ (Fig. 5). Similarly to the case 2, obtain

$$r(\tilde{t}_{n+1}^-) = e^{\tilde{D}(\tilde{t}_{n+1} - \tilde{t}_n - \tau)} r(\tilde{t}_n + \tau).$$

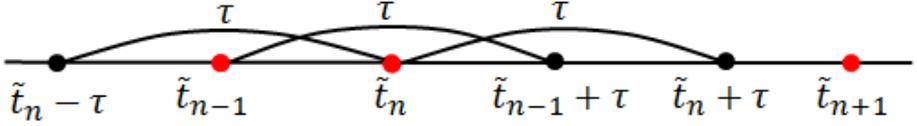


Figure 5: Case 4. The red dots indicate the firing times of the function $r(t)$

From the case 3 (replacing \tilde{t}_{n+1} by $\tilde{t}_n + \tau$) one has

$$r(\tilde{t}_n + \tau) = e^{\tilde{D}\tau} r(\tilde{t}_n^-) + \tilde{\lambda}_n e^{D_0\tau} B - \tilde{\lambda}_{n-1} e^{\tilde{D}\tau} \left(e^{D_0(\tilde{t}_n - \tilde{t}_{n-1})} - e^{\tilde{D}(\tilde{t}_n - \tilde{t}_{n-1} - \tau)} e^{D_0\tau} \right) B.$$

Thus, in case 4

$$r(\tilde{t}_{n+1}^-) = e^{\tilde{D}(\tilde{t}_{n+1} - \tilde{t}_n)} r(\tilde{t}_n^-) + \tilde{\lambda}_n e^{\tilde{D}(\tilde{t}_{n+1} - \tilde{t}_n - \tau)} e^{D_0\tau} B - \tilde{\lambda}_{n-1} e^{\tilde{D}(\tilde{t}_{n+1} - \tilde{t}_n)} \left(e^{D_0(\tilde{t}_n - \tilde{t}_{n-1})} - e^{\tilde{D}(\tilde{t}_n - \tilde{t}_{n-1} - \tau)} e^{D_0\tau} \right) B.$$

Lemma is proved. \square

Continue the proof of the theorem. For $m \geq 1$ on the interval $\hat{t}_n < t < \hat{t}_{n+1}$ the function $r(t)$ experiences jumps $r(t^+) - r(t^-) = \lambda_i B$ at $t = t_i$, $k+1 \leq i \leq s$. The assumption $\inf_z \Phi(z) > \tau$ guarantees, that $\hat{t}_n + \tau < t_{k+2}$, whereas the point t_{k+1} may belong to either the interval $(\hat{t}_n, \hat{t}_n + \tau)$ or the interval $(\hat{t}_n + \tau, \hat{t}_{n+1})$, therefore it is necessary to consideration these two cases.

First consider the case $m \geq 2$, i. e. $s \geq k+2$. Obviously $\hat{x}(\hat{t}_{n+1}^-) = x(\hat{t}_{n+1}^-) - r(\hat{t}_{n+1}^-)$. Note, that $\tilde{R}(t_s - t_{s-1}, \hat{t}_{n+1} - t_s) = 0$, . . . $t_s - t_{s-1} \geq \inf_z \Phi(z) > \tau$. Therefore, from theorem 2 in [2] and lemma 1 it follows

$$\hat{x}(\hat{t}_{n+1}^-) = e^{D(\hat{t}_{n+1} - t_s)} x(t_s^-) + \lambda_s G(\hat{t}_{n+1} - t_s) B - e^{\tilde{D}(\hat{t}_{n+1} - t_s)} r(t_s^-) - \lambda_s \tilde{G}(\hat{t}_{n+1} - t_s). \quad (\text{B.9})$$

Find $r(t_s^-)$.

- By lemma 1 on the interval $\hat{t}_n < t < t_{k+1}$, obtain

$$r(t_{k+1}^-) = e^{\tilde{D}(t_{k+1} - \hat{t}_n)} r(\hat{t}_n^-) - \hat{\lambda}_n \tilde{G}(t_{k+1} - \hat{t}_n) B + \lambda_k \tilde{R}(\hat{t}_n - t_k, t_{k+1} - \hat{t}_n) B.$$

Note, if $t_k < \hat{t}_{n-1}$, the previous formula holds,

as $\hat{t}_n - t_k > \hat{t}_n - \hat{t}_{n-1} \geq \inf_z \Phi(z) > \tau$, therefore, $\tilde{R}(\hat{t}_n - t_k, t_{k+1} - \hat{t}_n) = 0$.

- From lemma 1 on the interval $t_{k+1} < t < t_{k+2}$ follows

$$r(t_{k+2}^-) = e^{\tilde{D}(t_{k+2}-t_{k+1})}r(t_{k+1}^-) + \lambda_{k+1}\tilde{G}(t_{k+2}-t_{k+1})B - \hat{\lambda}_n\tilde{R}(t_{k+1}-\hat{t}_n, t_{k+2}-t_{k+1})B.$$

Note that

$$e^{\tilde{D}(t_{k+2}-t_{k+1})}\tilde{G}(t_{k+1}-\hat{t}_n) + \tilde{R}(t_{k+1}-\hat{t}_n, t_{k+2}-t_{k+1}) = e^{\tilde{D}(t_{k+2}-\hat{t}_n-\tau)}e^{D_0\tau},$$

Substitute $r(t_{k+1}^-)$:

$$r(t_{k+2}^-) = e^{\tilde{D}(t_{k+2}-\hat{t}_n)}r(\hat{t}_n^-) + \lambda_k e^{\tilde{D}(t_{k+2}-t_{k+1})}\tilde{R}(\hat{t}_n-t_k, t_{k+1}-\hat{t}_n)B + \lambda_{k+1}\tilde{G}(t_{k+2}-t_{k+1})B - \hat{\lambda}_n e^{\tilde{D}(t_{k+2}-\hat{t}_n-\tau)}e^{D_0\tau}B.$$

- If $m \geq 3$, lemma 1 on the interval $t_{k+2} < t < t_{k+3}$ gives

$$r(t_{k+3}^-) = e^{\tilde{D}(t_{k+3}-t_{k+2})}r(t_{k+2}^-) + \lambda_{k+1}\tilde{R}(t_{k+2}-t_{k+1}, t_{k+3}-t_{k+2})B + \lambda_{k+2}\tilde{G}(t_{k+3}-t_{k+2})B.$$

Since $t_{k+2}-t_{k+1} > \inf_z \Phi(z) > \tau$, $\tilde{R}(t_{k+2}-t_{k+1}, t_{k+3}-t_{k+2}) = 0$. Hence, $r(t_{k+3}^-) = e^{\tilde{D}(t_{k+3}-t_{k+2})}r(t_{k+2}^-) + \lambda_{k+2}\tilde{G}(t_{k+3}-t_{k+2})B$. Substitute $r(t_{k+2}^-)$:

$$r(t_{k+3}^-) = e^{\tilde{D}(t_{k+3}-\hat{t}_n)}r(\hat{t}_n^-) + \lambda_k e^{\tilde{D}(t_{k+3}-t_{k+1})}\tilde{R}(\hat{t}_n-t_k, t_{k+1}-\hat{t}_n)B + \lambda_{k+1}e^{\tilde{D}(t_{k+3}-t_{k+2})}\tilde{G}(t_{k+2}-t_{k+1})B + \lambda_{k+2}\tilde{G}(t_{k+3}-t_{k+2})B - \hat{\lambda}_n e^{\tilde{D}(t_{k+3}-\hat{t}_n-\tau)}e^{D_0\tau}B.$$

- Analogously,

$$r(t_s^-) = e^{\tilde{D}(t_s-\hat{t}_n)}r(\hat{t}_n^-) + \lambda_k e^{\tilde{D}(t_s-t_{k+1})}\tilde{R}(\hat{t}_n-t_k, t_{k+1}-\hat{t}_n)B - \hat{\lambda}_n e^{\tilde{D}(t_s-\hat{t}_n-\tau)}e^{D_0\tau}B + \sum_{j=k+1}^{s-1} \lambda_j e^{\tilde{D}(t_s-t_{j+1})}\tilde{G}(t_{j+1}-t_j)B.$$

Note, that $\tilde{G}(t_{j+1}-t_j) = e^{\tilde{D}(t_{j+1}-t_j-\tau)}e^{D_0\tau}$, $t_{j+1}-t_j > \tau$. Thus,

$$r(t_s^-) = e^{\tilde{D}(t_s-\hat{t}_n)}r(\hat{t}_n^-) + \lambda_k e^{\tilde{D}(t_s-t_{k+1})}\tilde{R}(\hat{t}_n-t_k, t_{k+1}-\hat{t}_n)B - \hat{\lambda}_n e^{\tilde{D}(t_s-\hat{t}_n-\tau)}e^{D_0\tau}B + r_{k,s}, \quad (\text{B.10})$$

where $r_{k,s} = \sum_{j=k+1}^{s-1} \lambda_j e^{\tilde{D}(t_s - t_j - \tau)} e^{D_0 \tau} B$.

Since $x(\hat{t}_n^-) = e^{D(\hat{t}_n - t_k)} x(t_k^-) + \lambda_k G(\hat{t}_n - t_k) B$,

$$r(\hat{t}_n^-) = x(\hat{t}_n^-) - \hat{x}(\hat{t}_n^-) = e^{D(\hat{t}_n - t_k)} x(t_k^-) + \lambda_k G(\hat{t}_n - t_k) B - \hat{x}(\hat{t}_n^-). \quad (\text{B.11})$$

Finally, by substitution (B.11) into (B.10) and (B.10) into (B.9) obtain

$$\begin{aligned} \hat{x}(\hat{t}_{n+1}^-) &= e^{D(\hat{t}_{n+1} - t_s)} x(t_s^-) - e^{\tilde{D}(\hat{t}_{n+1} - \hat{t}_n)} e^{D(\hat{t}_n - t_k)} x(t_k^-) + e^{\tilde{D}(\hat{t}_{n+1} - \hat{t}_n)} \hat{x}(\hat{t}_n^-) - \\ &- \lambda_k e^{\tilde{D}(\hat{t}_{n+1} - \hat{t}_n)} G(\hat{t}_n - t_k) B - \lambda_k e^{\tilde{D}(\hat{t}_{n+1} - t_{k+1})} \tilde{R}(\hat{t}_n - t_k, t_{k+1} - \hat{t}_n) B + \\ &+ \hat{\lambda}_n e^{\tilde{D}(\hat{t}_{n+1} - \hat{t}_n - \tau)} e^{D_0 \tau} B - r_{k,s} + \lambda_s G(\hat{t}_{n+1} - t_s) B - \lambda_s \tilde{G}(\hat{t}_{n+1} - t_s) B. \end{aligned}$$

Not that $e^{\tilde{D}(\hat{t}_{n+1} - t_{k+1})} \tilde{R}(\hat{t}_n - t_k, t_{k+1} - \hat{t}_n) = \tilde{R}(\hat{t}_n - t_k, t_{k+1} - \hat{t}_n)$ and

$$r_{k,s} = \sum_{j=k+1}^{s-1} \lambda_j \tilde{G}(\hat{t}_{n+1} - t_j) B, \text{ as } \hat{t}_m - t_j > \tau \text{ for all } j = k+1, \dots, s-1.$$

Thus,

$$\begin{aligned} \hat{x}(\hat{t}_{n+1}^-) &= e^{D(\hat{t}_{n+1} - t_s)} x(t_s^-) - e^{\tilde{D}(\hat{t}_{n+1} - \hat{t}_n)} \left(e^{D(\hat{t}_n - t_k)} x(t_k^-) - \hat{x}(\hat{t}_n^-) \right) - \\ &- \lambda_k \left(e^{\tilde{D}(\hat{t}_{n+1} - \hat{t}_n)} G(\hat{t}_n - t_k) + \tilde{R}(\hat{t}_n - t_k, \hat{t}_{n+1} - \hat{t}_n) \right) B + \\ &+ \hat{\lambda}_n \tilde{G}(\hat{t}_{n+1} - \hat{t}_n) B - \sum_{j=k+1}^s \lambda_j \tilde{G}(\hat{t}_{n+1} - t_j) B + \lambda_s G(\hat{t}_{n+1} - t_s) B, \quad (\text{B.12}) \end{aligned}$$

which implies the (6) for $m \geq 2$.

Show that for $m = 0$ and $m = 1$ formula (B.12) is correct.

Suppose that $m = 1$. Then

$$\begin{aligned} \hat{x}(\hat{t}_{n+1}^-) &= x(\hat{t}_{n+1}^-) - r(\hat{t}_{n+1}^-) = e^{D(\hat{t}_{n+1} - t_s)} x(t_s^-) + \lambda_s G(\hat{t}_{n+1} - t_s) B - \\ &e^{\tilde{D}(\hat{t}_{n+1} - t_s)} r(t_s^-) - \lambda_s \tilde{G}(\hat{t}_{n+1} - t_s) B + \hat{\lambda}_n \tilde{R}(t_s - \hat{t}_n, \hat{t}_{n+1} - t_s) B. \end{aligned}$$

Find $r(t_s^-)$:

$$\begin{aligned} r(t_s^-) &= e^{\tilde{D}(t_s - \hat{t}_n)} e^{D(\hat{t}_n - t_k)} x(t_k^-) + \lambda_k e^{\tilde{D}(t_s - \hat{t}_n)} G(\hat{t}_n - t_k) B - \\ &- e^{\tilde{D}(t_s - \hat{t}_n)} \hat{x}(\hat{t}_n^-) - \hat{\lambda}_n \tilde{G}(t_s - \hat{t}_n) B + \lambda_k \tilde{R}(\hat{t}_n - t_k, t_s - \hat{t}_n) B. \end{aligned}$$

Thus,

$$\begin{aligned} \hat{x}(\hat{t}_{n+1}^-) &= e^{D(\hat{t}_{n+1} - t_s)} x(t_s^-) - e^{\tilde{D}(\hat{t}_{n+1} - \hat{t}_n)} \left(e^{D(\hat{t}_n - t_k)} x(t_k^-) - \hat{x}(\hat{t}_n^-) \right) - \\ &\lambda_k \left(e^{\tilde{D}(\hat{t}_{n+1} - \hat{t}_n)} G(\hat{t}_n - t_k) + \tilde{R}(\hat{t}_n - t_k, \hat{t}_{n+1} - \hat{t}_n) \right) B + \\ &\hat{\lambda}_n \tilde{G}(\hat{t}_{n+1} - \hat{t}_n) B + \lambda_s G(\hat{t}_{n+1} - t_s) B - \lambda_s \tilde{G}(\hat{t}_{n+1} - t_s) B, \end{aligned}$$

what coincides with (B.12) if $s = k + 1$.

Suppose that $m = 0$, i. e. $s = k$. Then

$$\begin{aligned} \hat{x}(\hat{t}_{n+1}^-) &= e^{D(\hat{t}_{n+1}-t_k)}x(t_k^-) - e^{\tilde{D}(\hat{t}_{n+1}-\hat{t}_n)}\left(e^{D(\hat{t}_n-t_k)}x(t_k^-) - \hat{x}(\hat{t}_n^-)\right) - \\ &- \lambda_k\left(e^{\tilde{D}(\hat{t}_{n+1}-\hat{t}_n)}G(\hat{t}_n - t_k) + \tilde{R}(\hat{t}_n - t_k, \hat{t}_{n+1} - \hat{t}_n)\right)B + \hat{\lambda}_n\tilde{G}(\hat{t}_{n+1}-\hat{t}_n)B + \\ &\quad + \lambda_kG(\hat{t}_{n+1} - t_k)B, \end{aligned}$$

what coincides with (B.12) if $s = k$. \square

Proof of Theorem 4.

From Theorem 1 and the evident formula

$$\begin{aligned} x(t_{k+1}^-) &= e^{D(t_{k+1}-t_k)}x(t_k^-) + \lambda_kG_k(t_{k+1} - t_k)B = \\ &\quad e^{D(t_{k+1}-t_k)}x(t_k^-) + \lambda_k e^{D(t_{k+1}-t_k-\tau)}e^{A_0\tau}B, \end{aligned}$$

it follows that for $s = k + 1$

$$\begin{aligned} P_{k,k+1}(\zeta, \theta) &= \left[e^{D\Phi(C\zeta)} - e^{\tilde{D}\Phi(C\zeta)}\right]e^{D(\theta-t_k)}x(t_k^-) \\ &- \lambda_k\left[e^{\tilde{D}\Phi(C\zeta)}G(\theta - t_k) + \tilde{R}(\theta - t_k, \Phi(C\zeta)) - e^{D(\theta+\Phi(C\zeta)-t_k-\tau)}e^{A_0\tau}\right]B \\ &\quad - \lambda_{k+1}\left(\tilde{G}(\theta + \Phi(C\zeta) - t_{k+1}) - G(\theta + \Phi(C\zeta) - t_{k+1})\right)B \\ &\quad + e^{\tilde{D}\Phi(C\zeta)}\zeta + F(C\zeta)\tilde{G}(\Phi(C\zeta))B. \end{aligned}$$

Further, the following facts are needed to carry on with the proof.

Lemma 2. *The partial derivatives of $P(\zeta, \theta)$ with respect to its arguments can be calculated at (x_k, t_k) as follows:*

$$\begin{aligned} P'_\theta(x_k, t_k) &= Dx_{k+1} - e^{\tilde{D}T_k}(Dx_k + \lambda_k\tilde{D}e^{-\tilde{D}\tau}e^{D_0\tau}B), \\ P'_\zeta(x_k, t_k) &= \Phi'(Cx_k)Dx_{k+1}C + e^{\tilde{D}T_k}\left[I_{n_x} + F'(Cx_k)e^{-\tilde{D}\tau}e^{D_0\tau}BC\right]. \end{aligned}$$

Proof of Lemma 6. Obviously $P(x_k, t_k) = P_{k,k+1}(x_k, t_k)$. As it was stipulated by Theorem 3, the partial derivatives of $P(\zeta, \theta)$ are continuous, so that

$$\begin{aligned} \frac{\partial}{\partial\zeta}P(x_k, t_k) &= \frac{\partial}{\partial\zeta}P_{k,k+1}(x_k, t_k), \\ \frac{\partial}{\partial\theta}P(x_k, t_k) &= \frac{\partial}{\partial\theta}P_{k,k+1}(x_k, t_k). \end{aligned}$$

By direct calculation

$$\begin{aligned}
\frac{\partial}{\partial \theta} P_{k,k+1}(\zeta, \theta) &= [e^{D\Phi(C\zeta)} - e^{\tilde{D}\Phi(C\zeta)}] De^{D(\theta-t_k)} x(t_k^-) \\
&\quad - \lambda_k \left[e^{\tilde{D}\Phi(C\zeta)} f_1(\theta - t_k) - De^{D(\theta+\Phi(C\zeta)-t_k-\tau)} e^{A_0\tau} \right] B \\
&\quad - \lambda_{k+1} f_2(\theta + \Phi(C\zeta) - t_{k+1}) B, \\
\frac{\partial}{\partial \zeta} P_{k,k+1}(\zeta, \theta) &= \Phi'(C\zeta) [De^{D\Phi(C\zeta)} - \tilde{D}e^{\tilde{D}\Phi(C\zeta)}] e^{D(\theta-t_k)} x(t_k^-) C \\
&\quad - \lambda_k \Phi'(C\zeta) \left[\tilde{D}e^{\tilde{D}\Phi(C\zeta)} G(\theta - t_k) + \tilde{D}\tilde{R}(\theta - t_k, \Phi(C\zeta)) \right. \\
&\quad \quad \quad \left. - De^{D(\theta+\Phi(C\zeta)-t_k-\tau)} e^{A_0\tau} \right] BC \\
&\quad - \lambda_{k+1} \Phi'(C\zeta) f_2(\theta + \Phi(C\zeta) - t_{k+1}) BC + \Phi'(C\zeta) \tilde{D}e^{\tilde{D}\Phi(C\zeta)} \zeta C + \\
&\quad e^{\tilde{D}\Phi(C\zeta)} + F'(C\zeta) \tilde{G}(\Phi(C\zeta)) BC + \Phi'(C\zeta) F(C\zeta) \tilde{D}e^{D(\Phi(C\zeta)-\tau)} e^{D_0\tau} BC,
\end{aligned}$$

where

$$\begin{aligned}
f_1(\theta) &= \begin{cases} A_0 e^{A_0\theta} + \tilde{D}e^{\tilde{D}(\theta-\tau)} e^{D_0\tau} - D_0 e^{D_0\theta}, & \text{if } 0 \leq \theta \leq \tau, \\ De^{D(\theta-\tau)} e^{A_0\tau}, & \text{if } \tau \leq \theta. \end{cases} \\
f_2(\theta) &= \begin{cases} D_0 e^{D_0\theta} - A_0 e^{A_0\theta}, & \text{if } 0 \leq \theta \leq \tau, \\ \tilde{D}e^{\tilde{D}(\theta-\tau)} e^{D_0\tau} - De^{D(\theta-\tau)} e^{A_0\tau}, & \text{if } \tau \leq \theta. \end{cases}
\end{aligned}$$

Since $t_k + \Phi(Cx_k) - t_{k+1} = 0$ and $(A - D)B = 0$, one obtains

$$\begin{aligned}
\frac{\partial}{\partial \theta} P_{k,k+1}(x_k, t_k) &= [e^{D\Phi(Cx_k)} - e^{\tilde{D}\Phi(Cx_k)}] Dx(t_k^-) - \\
&\quad \lambda_k \left[\tilde{D}e^{\tilde{D}(\Phi(Cx_k)-\tau)} e^{D_0\tau} - De^{D(\Phi(Cx_k)-\tau)} e^{A_0\tau} \right] B, \\
\frac{\partial}{\partial \zeta} P_{k,k+1}(x_k, t_k) &= \Phi'(Cx_k) De^{D\Phi(Cx_k)} x_k C + \\
\lambda_k \Phi'(Cx_k) De^{D(\Phi(Cx_k)-\tau)} e^{A_0\tau} BC &+ e^{\tilde{D}\Phi(Cx_k)} [I_{n_x} + F'(Cx_k) e^{-D\tau} e^{D_0\tau} BC].
\end{aligned}$$

Now the statement of Lemma 2 can be derived by taking into account $x_{k+1} = e^{D(t_{k+1}-t_k)} x_k + \lambda_k e^{D(t_{k+1}-t_k-\tau)} e^{A_0\tau} B$. □

Theorem 4 follows directly from Lemma 2. □

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