Boundary Conditions and Estimates for the Steady Stokes Equations on Staggered Grids

Bertil Gustafsson and Jonas Nilsson
Department of Scientific Computing
Uppsala University
Box 120
SE-75104 Uppsala
Sweden

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Abstract

We consider the steady state Stokes equations, describing low speed flow and derive estimates of the solution for various types of boundary conditions. We formulate the boundary conditions in a new way, such that the boundary value problem becomes non-singular. By using a difference approximation on a staggered grid we are able to derive a non-singular approximation in a direct way. Furthermore, we derive the same type of estimates as for the continuous case. Numerical experiments confirm the theoretical results.

1 Introduction

In this paper we consider the steady state Stokes equations, describing low speed flow. The original equations are stated with the continuity equation explicitly given, leading to a divergence-free velocity field. Let $\mathbf{v}$ be the velocity components in the three coordinate directions and p the pressure. Then the Stokes equations in their most general form have the form

$$\begin{aligned} \nabla p &= \nu \Delta \mathbf{v}\,, \\ \mathbf{div}\ \mathbf{v} &= 0 \end{aligned} \tag{1.1}$$

We consider here the 2D-case. In Cartesian coordinates (x, y), we use the notation $\mathbf{U} = (u, v, p)^T$ for the velocity components in the x- and y-direction and the pressure, respectively. Then the equations are in component form

$$p_x = \nu(u_{xx} + u_{yy})\,, \tag{1.2}$$

$$p_y = \nu(v_{xx} + v_{yy})\,, \tag{1.3}$$

$$u_x + v_y = 0\,. \tag{1.4}$$

We shall refer to this form as the *divergence form* of the Stokes equations.

Nevertheless, for computational purpose most methods are based on another form of the equations. The continuity equation (1.4) is substituted by an equation for the pressure, obtained by differentiating (1.2), (1.3) and using (1.4):

$$p_{xx} + p_{yy} = 0\,. \tag{1.5}$$

The combination (1.2),(1.3),(1.5) will be refered to as the *pressure form* of the Stokes equations.

In both cases, the question is how the boundary conditions should be formulated. Most often, it is assumed that the velocity field $\mathbf{v} = \mathbf{v}_0$ is given at the whole boundary $\partial\Omega$. However, in some applications, more flexibility is obtained if the pressure is prescribed, at least at some part of the boundary.

In the first part of this paper, we will analyze the well-posedness and derive estimates of the solution for various types of boundary conditions. The divergence condition introduces a restriction on the boundary data

$$\int_{\partial\Omega} \mathbf{v}_0 \cdot \mathbf{n}\, dS = 0\,, \tag{1.6}$$

where $\mathbf{n}$ is the outer normal to $\partial\Omega$. This restriction can be seen as an effect of the fact that the boundary value problem is singular. This leads to the troublesome difficulty that the algebraic system to be solved for any consistent approximation, necessarily is singular as well. There are ways to modify this algebraic system such that it becomes non-singular. One way is to expand the system with the null vector of the adjoint matrix. This new system is non-singular, although the original system is singular. The drawback is that this introduces an error in the solution. In a similar technique proposed by Strikwerda [4] the discrete approximation for the divergence equations $D_h^x u + D_h^y v = 0$ is replaced by $D_h^x u + D_h^y v = \delta_h$, where δ_h is a constant chosen to guarantee a solution and must be determined as part of the solution. For some numerical schemes the discrete compatibility condition corresponding to (1.6) is automatically satisfied, which is often true for staggered grid methods. Abdallah [1] showed how to design a scheme for the pressure Poisson equation (1.5) on non-staggered grid such that the discrete compatibility condition is exactly satisfied. However, the problem with a singular algebraic system still remains.

In this paper we take another and cleaner approach. By formulating the boundary conditions for the Stokes equations in a different way, such that the boundary value problem becomes non-singular, we are able to derive a non-singular approximation in a direct way.

The approximation is based on a second order difference approximation on a staggered grid. This type of approximation has been used by others, see for example [2], [5], [3] and [6]. However, in our case we apply the boundary conditions in a way such that the analysis of the discrete case is very close to the analysis for the continuous case, and we are able to derive the same type of estimates. The reason for this is that no extra numerical boundary conditions are required, and no parasitic solutions are present.

We limit ourselves to the two-dimensional case, and for the theoretical analysis we assume periodic solutions in one direction. The boundary conditions are modified in a straightforward way for the general non-periodic case, and numerical experiments confirm the theoretical analysis.

2 Boundary Conditions for the Divergence Form

We shall first consider the case with periodic solutions in the y-direction: $\mathbf{U}(x,y) = \mathbf{U}(x, y+2\pi)$. The equations are defined in a rectangular domain $\Omega = \{0 \le x \le 1,\ 0 \le y \le 2\pi\}$.

The common type of boundary conditions is to specify the velocity u, v on both sides $x = 0, 1$. However, there is a restriction on the data. By integrating the divergence condition (1.4) over the whole domain, one obtains the condition

$$\int_0^{2\pi} u(0,y)dy = \int_0^{2\pi} u(1,y)dy\,,$$

where the periodicity has been taken into account. This leads to a difficulty for any discretization procedure leading to a system of algebraic equations. Since the right hand side of this system will contain the given velocity data, and these data are not arbitrary, the system necessarily contains a redundancy, which must be eliminated in some way. In order to get a well conditioned form of the discrete system, it is important to formulate the boundary conditions for the differential equations in the proper way.

Prescription of Both Velocity Components

We suggest the following form of the boundary conditions when prescribing the velocity:

$$\begin{array}{ll} u(0,y) - \frac{1}{2\pi}\int_0^{2\pi} u(0,y)dy = w_0(y)\,, & u(1,y) = u_1(y)\,, \\ v(0,y) = v_0(y)\,, & v(1,y) = v_1(y)\,, \\ \int_0^{2\pi} p(0,y)dy = q_0\,, & \end{array} \tag{2.1}$$

where it is assumed that $\int_0^{2\pi} w_0(y)dy = 0$. We use the notation $\mathbf{v} = (u,v)^T$, $\mathbf{v}_B = (w_0, u_1, v_0, v_1)^T$, and define the norms by

$$||\mathbf{v}(\cdot,\cdot)|| = \{\int_0^{2\pi}\int_0^1 |\mathbf{v}(x,y)|^2 dx\, dy\}^{1/2}\,, \quad |\mathbf{v}|^2 = u^2 + v^2\,,$$

$$||\mathbf{v}_B(\cdot)|| = \{\int_0^{2\pi} |\mathbf{v}_B(y)|^2 dy\}^{1/2}, \quad |\mathbf{v}_B|^2 = |w_0|^2 + |u_1|^2 + |v_0|^2 + |v_1|^2 ,$$

$$||p(\cdot,\cdot)|| = \{\int_0^{2\pi}\int_0^1 |p(x,y)|^2 dx\, dy\}^{1/2} .$$

We have

Theorem 2.1 *Assume that the boundary data $w_0,\ u_1,\ v_0,\ v_1$ are 2π- periodic in y and $\int_0^{2\pi} w_0(y)dy = 0$, but otherwise arbitrary. Then the system (1.2),(1.3),(1.4) with boundary conditions (2.1) has a unique solution, and there is an estimate*

$$||\mathbf{v}(\cdot,\cdot)||^2 \leq const\,(|q_0|^2 + ||\mathbf{v}_B(\cdot)||^2) , \tag{2.2}$$

$$||p(\cdot,\cdot)||^2 \leq const\,(|q_0|^2 + ||\mathbf{v}_B(\cdot)||^2 + ||\frac{\partial \mathbf{v}_B}{\partial y}(\cdot)||^2) , \tag{2.3}$$

Proof: By periodicity we can write the solution as a Fourier series

$$\mathbf{U}(x,y) = \frac{1}{2\pi}\sum_{\omega=-\infty}^{\infty} \hat{\mathbf{U}}(x,\omega)e^{i\omega y} ,$$

where $\hat{\mathbf{U}} = (\hat{u},\hat{v},\hat{p})^T$. When introducing this into the differential equations, we get the Fourier-transformed system

$$\begin{aligned} \hat{p}_x &= \nu(\hat{u}_{xx} - \omega^2\hat{u}) , \\ i\omega\hat{p} &= \nu(\hat{v}_{xx} - \omega^2\hat{v}) , \\ \hat{u}_x + i\omega\hat{v} &= 0 . \end{aligned} \tag{2.4}$$

We now have to distinguish between two cases: $\omega \neq 0$ and $\omega = 0$. It is convenient to distinguish also between the cases $\omega > 0$ and $\omega < 0$, and we exclude first negative values of ω. The general solution has the form
<u>$\omega > 0$:</u>

$$\hat{u}(x,\omega) = (a_\omega + b_\omega x)e^{-\omega x} + (c_\omega + d_\omega(x-1))e^{\omega(x-1)} \tag{2.5}$$

$$\hat{v}(x,\omega) = i(-a_\omega + \frac{1}{\omega}b_\omega - b_\omega x)e^{-\omega x} + i(c_\omega + \frac{1}{\omega}d_\omega + d_\omega(x-1))e^{\omega(x-1)} \tag{2.6}$$

$$\hat{p}(x,\omega) = 2\nu(b_\omega e^{-\omega x} + d_\omega e^{\omega(x-1)}) \tag{2.7}$$

$\underline{\omega = 0}$:

$$\begin{aligned} \hat{u}(x,0) &= a_0 && (2.8) \\ \hat{v}(x,0) &= b_0 + c_0 x && (2.9) \\ \hat{p}(x,0) &= d_0 && (2.10) \end{aligned}$$

Note that there are four undetermined constants $a_\omega,\ b_\omega,\ c_\omega,\ d_\omega$ for each ω including $\omega = 0$. However, the structure is different for $\omega = 0$, and it does not allow for prescribing $\hat{u}$ on both sides. One of the conditions on $\hat{u}$ needs to be substituted by a condition on $\hat{p}$. This is why we have formulated the boundary conditions as above. In Fourier space we get

$\underline{\omega > 0}$:

$$\begin{aligned} \hat{u}(0,\omega) = \hat{w}_0(\omega)\,, \quad \hat{u}(1,\omega) = \hat{u}_1(\omega)\,, \\ \hat{v}(0,\omega) = \hat{v}_0(\omega)\,, \quad \hat{v}(1,\omega) = \hat{v}_1(\omega)\,. \end{aligned} \tag{2.11}$$

$\underline{\omega = 0}$:

$$\begin{aligned} & \qquad\qquad\quad \hat{u}(1,0) = \hat{u}_1(0)\,, \\ & \hat{v}(0,0) = \hat{v}_0(0)\,, \quad \hat{v}(1,0) = \hat{v}_1(0)\,, \\ & \hat{p}(0,0) = \hat{q}_0\,. \end{aligned} \tag{2.12}$$

There is still one difficulty left when determining the constants $a_\omega,\ b_\omega,\ c_\omega,\ d_\omega$ from the boundary conditions. For $\omega \to \infty$, the determinant of the coefficient matrix approaches zero. To avoid this, we rescale the coefficients as

$$\begin{aligned} b_\omega &= \omega \tilde{b}_\omega\,, \\ d_\omega &= \omega \tilde{d}_\omega\,, \end{aligned}$$

and obtain for $\omega > 0$

$$\begin{bmatrix} 1 & 0 & e^{-\omega} & -\omega e^{-\omega} \\ e^{-\omega} & \omega e^{-\omega} & 1 & 0 \\ -i & i & i\,e^{-\omega} & i(1-\omega)e^{-\omega} \\ -ie^{-\omega} & i(1-\omega)e^{-\omega} & i & i \end{bmatrix} \begin{bmatrix} a_\omega \\ \tilde{b}_\omega \\ c_\omega \\ \tilde{d}_\omega \end{bmatrix} = \begin{bmatrix} \hat{w}_0(\omega) \\ \hat{u}_1(\omega) \\ \hat{v}_0(\omega) \\ \hat{v}_1(\omega) \end{bmatrix}. \tag{2.13}$$

For $\omega \to \infty$, the determinant is -1, and since all elements are bounded, the condition number is bounded independent of ω for $1 \leq \omega < \infty$. Therefore, the constants are bounded in terms of the right hand side.

By symmetry we get exactly the same result for $\omega < 0$. For $\omega = 0$, the determination of the constants is obviously well conditioned. Hence, the final estimate for the original constants is
<u>$\omega \neq 0$:</u>

$$\begin{aligned} |a_\omega|^2 + |c_\omega|^2 &\le \text{const}(|\hat{w}_0(\omega)|^2 + |\hat{u}_1(\omega)|^2 + |\hat{v}_0(\omega)|^2 + |\hat{v}_1(\omega)|^2)\,, \\ |b_\omega|^2 + |d_\omega|^2 &\le \text{const}\,|\omega|^2(|\hat{w}_0(\omega)|^2 + |\hat{u}_1(\omega)|^2 + |\hat{v}_0(\omega)|^2 + |\hat{v}_1(\omega)|^2)\,, \end{aligned}$$

<u>$\omega = 0$:</u>

$$|a_0|^2 + |b_0|^2 + |c_0|^2 + |d_0|^2 \le \text{const}(|\hat{u}_1(0)|^2 + |\hat{v}_0(0)|^2 + |\hat{v}_1(0)|^2 + |\hat{q}_0|^2)\,. \tag{2.14}$$

Next, we take a closer look at the influence of the extra factor $|\omega|$ occuring in the estimate for $b_\omega,\ d_\omega$. For $\hat{p}$ we obviously get $\hat{p}(0,\omega) \sim \hat{p}(1,\omega) \sim |\omega|$ corresponding to the introduction of y-derivatives in physical space, as shown in (2.3). At the inner part of the domain, the growth doesn't become worse.

We next turn to the estimate for u and v. Consider the case $\omega > 0$ and the expressions (2.5), (2.6) for $\hat{u},\ \hat{v}$ near $x = 0$. The constants $b_\omega,\ d_\omega$ are multiplied either by $e^{-\omega x}/\omega$ or by $xe^{-\omega x}$. In the first case, the extra factor ω is obviously canceled. In the second case, we have terms of the form $\omega x e^{-\omega x}$, when including the extra factor ω. But

$$\max_{\omega x \ge 0}(\omega x e^{-\omega x}) = e^{-1} \quad \text{for } \omega x = 1\,.$$

By symmetry, the corresponding result holds near $x = 1$. Hence the estimate for $\hat{u},\ \hat{v}$ does not deteriorate because of the extra factor ω. All these arguments apply for negative ω as well. Hence, by Parseval's relation, the estimate (2.2) holds, and the theorem is proven.

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At this point we make a remark about the significance of the form of the boundary conditions. Assume that the condition on u

$$u(0,y) - \frac{1}{2\pi}\int_0^{2\pi} u(0,y)dy = w_0(y)\ , \quad u(1,y) = u_1(y) \tag{2.15}$$

under the restriction

$$\int_0^{2\pi} w_0(y)dy = 0 \tag{2.16}$$

is substituted by the more common form

$$u(0,y) = u_0(y)\ ,\quad u(1,y) = u_1(y) \tag{2.17}$$

under the restriction

$$\int_0^{2\pi} u_0(y)dy = \int_0^{2\pi} u_1(y)dy\,. \tag{2.18}$$

Let us introduce a perturbation in the data u_0 such that the restriction (2.18) is not fully satisfied:

$$\int_0^{2\pi} u_0(y)dy = \int_0^{2\pi} u_1(y)dy + \delta\,.$$

For the Fourier coefficient $\hat{u}$ we now have the boundary value problem for $\omega = 0$

$$\hat{u}_x(x,0) = 0\ ,$$

$$\hat{u}(0,0) = \hat{u}_1(0) + \delta\ ,\quad \hat{u}(1,0) = \hat{u}_1(0)\,,$$

which has no solution.

Next we introduce a perturbation in the data w_0 in our form of the boundary conditions such that the data $w_0(y)$ is substituted by $w_0(y)+\delta(y)$, which means that the restriction (2.16) is not fully satisfied:

$$\int_0^{2\pi} w_0(y)dy = \int_0^{2\pi} \delta(y)dy\,.$$

In this case we get for the Fourier coefficients the same ODE-system and boundary conditions as for the unperturbed case, except for the boundary data where $\hat{w}_0(\omega)$ is substituted by $\hat{w}_0(\omega) + \hat{\delta}(\omega)$ for $\omega > 0$. For $\omega = 0$ the coefficient $\hat{w}_0(0)$ does not enter the system at all. In particular, at $x = 0$ the coefficient $\hat{u}(0,0)$ is determined by

$$\hat{u}_x(x,0) = 0\ ,$$

$$\hat{u}(1,0) = \hat{u}_1(0)\,,$$

which does not lead to any contradiction.

The corresponding conclusion can be made for discretizations used for computation. With our boundary conditions, we have a basis for obtaining a non-singular and well conditioned algebraic system of equations.

Prescription of One Velocity Component and the Pressure

Next we consider the case where the pressure is prescribed on the boundary instead of one of the velocity components. It turns out that it is not possible to leave out one of the velocity components completely. The conditions are

$$\begin{aligned} u(0,y) - \frac{1}{2\pi}\int_0^{2\pi} u(0,y)dy = w_0(y) \quad &, \quad u(1,y) = u_1(y)\,, \\ p(0,y) - \frac{1}{2\pi}\int_0^{2\pi} p(0,y)dy = q_0(y) \quad &, \quad p(1,y) = p_1(y)\,, \\ \int_0^{2\pi} v(0,y)dy = z_0 \quad &, \quad \int_0^{2\pi} v(1,y)dy = z_1\,. \end{aligned} \tag{2.19}$$

In this case the derivatives of the boundary data are not necessary in the estimate of the solution. We have

Theorem 2.2 *Assume that the boundary data* $w_0,\ u_1,\ q_0,\ p_1$ *are* 2π*- periodic in* y *and* $\int_0^{2\pi} w_0(y)dy = \int_0^{2\pi} q_0(y)dy = 0$*, but otherwise arbitrary. Then the system (1.2),(1.3),(1.4) with boundary conditions (2.19) has a unique solution, and there is an estimate*

$$||\mathbf{U}(\cdot,\cdot)||^2 \leq const\,(|z_0|^2 + |z_1|^2 + \int_0^{2\pi}(|w_0(y)|^2 + |u_1(y)|^2 + |q_0(y)|^2 + |p_1(y)|^2)dy) \tag{2.20}$$

Proof: As in Theorem 2.1 we Fourier transform and arrive at (2.4) with the general form of the solution given by (2.5), (2.6), (2.7),(2.8), (2.9),(2.10). The boundary conditions are
$\underline{\omega > 0}$:

$$\begin{aligned} \hat{u}(0,\omega) = \hat{w}_0(\omega)\,, \quad \hat{u}(1,\omega) = \hat{u}_1(\omega)\,, \\ \hat{p}(0,\omega) = \hat{p}_0(\omega)\,, \quad \hat{p}(1,\omega) = \hat{p}_1(\omega)\,. \end{aligned} \tag{2.21}$$

$\underline{\omega = 0}$:

$$\begin{aligned} & \hat{u}(1,0) = \hat{u}_1(0)\,, \\ \hat{v}(0,0) = z_0\,, \quad & \hat{v}(1,0) = z_1\,, \\ & \hat{p}(1,0) = \hat{p}_1(0)\,. \end{aligned} \tag{2.22}$$

For $\omega > 0$ we get

$$\begin{aligned}
a_\omega + e^{-\omega}c_\omega - e^{-\omega}d_\omega &= \hat{u}_0(\omega)\,, \\
e^{-\omega}a_\omega + e^{-\omega}b_\omega + c_\omega &= \hat{u}_1(\omega)\,, \\
2\nu b_\omega + 2\nu e^{-\omega}d_\omega &= \hat{p}_0(\omega)\,, \\
2\nu e^{-\omega}b_\omega + 2\nu d_\omega &= \hat{p}_1(\omega)\,,
\end{aligned}$$

which is a well conditioned system independent of ω, even as $\omega \to \infty$. For $\omega = 0$ we get a simple and well conditioned system for the constants a_0, b_0, c_0, d_0 from the boundary conditions (2.22). This shows that all the constants $a_\omega, b_\omega, c_\omega, d_\omega$ can be estimated in terms of the given data independent of ω. Since the x-dependent functions occuring in (2.5),(2.6), (2.7),(2.8), (2.9),(2.10) are all bounded, the estimate (2.20) follows.

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A third set of boundary conditions is

$$\begin{aligned}
v(0,y) &= v_0(y) \quad , \qquad v(1,y) = v_1(y)\,, \\
p(0,y) - \frac{1}{2\pi}\int_0^{2\pi} p(0,y)dy &= q_0(y) \quad , \qquad p(1,y) = p_1(y)\,, \\
\int_0^{2\pi} u(0,y)dy &= w_0 \quad .
\end{aligned} \tag{2.23}$$

In the same way as above we can prove

Theorem 2.3 *Assume that the boundary data $v_0,\ v_1,\ q_0,\ p_1$ are 2π- periodic in y and $\int_0^{2\pi} q_0(y)dy = 0$, but otherwise arbitrary. Then the system (1.2),(1.3),(1.4) with boundary conditions (2.23) has a unique solution, and there is an estimate*

$$||\mathbf{U}(\cdot,\cdot)||^2 \leq const\,(|w_0|^2 + \int_0^{2\pi} (v_0(y)|^2 + |v_1(y)|^2 + |q_0(y)|^2 + |p_1(y)|^2)dy) \tag{2.24}$$

■

3 Boundary Conditions for the Pressure Form

In this section we consider the pressure form (1.2),(1.3),(1.5)of the equations, and assume as above that the solutions are 2π-periodic in the y-direction. The Fourier-transformed system corresponding to (2.4) is

$$\begin{aligned} \hat{p}_x &= \nu(\hat{u}_{xx} - \omega^2 \hat{u})\,, \\ i\omega\hat{p} &= \nu(\hat{v}_{xx} - \omega^2 \hat{v})\,, \\ \hat{p}_{xx} - \omega^2 \hat{p} &= 0\,. \end{aligned} \tag{3.1}$$

As in the previous section we first exclude negative values of ω, and obtain the general form of the solution as
<u>$\omega > 0$:</u>

$$\begin{aligned} \hat{u}(x,\omega) &= (a_\omega + b_\omega x)e^{-\omega x} + (c_\omega + d_\omega(x-1))e^{\omega(x-1)} \\ \hat{v}(x,\omega) &= (\alpha_\omega - b_\omega i x)e^{-\omega x} + (\beta_\omega + i d_\omega(x-1))e^{\omega(x-1)} \\ \hat{p}(x,\omega) &= 2\nu(b_\omega e^{-\omega x} + d_\omega e^{\omega(x-1)}) \end{aligned}$$

<u>$\omega = 0$:</u>

$$\hat{u}(x,0) = a_0 + \alpha_0 x + \frac{1}{\nu}\beta_0 \frac{x^2}{2}\,, \tag{3.2}$$

$$\hat{v}(x,0) = b_0 + c_0 x\,, \tag{3.3}$$

$$\hat{p}(x,0) = d_0 + \beta_0 x\,. \tag{3.4}$$

In addition to the constants $a_\omega, b_\omega, c_\omega, d_\omega$ that we had for the divergence form of the equations, we now have the extra constants $\alpha_\omega, \beta_\omega$. The extra boundary conditions now required are given by imposing the divergence condition at $x = 0, 1$. The complete set of conditions are

$$\begin{array}{ll} u(0,y) - \frac{1}{2\pi}\int_0^{2\pi} u(0,y)dy = w_0(y)\,, & u(1,y) = u_1(y)\,, \\ v(0,y) = v_0(y)\,, & v(1,y) = v_1(y)\,, \\ \int_0^{2\pi} p(0,y)dy = q_0\,, & \\ u_x(0,y) + v_y(0,y) = 0\,, & u_x(1,y) + v_y(1,y) = 0\,, \end{array} \tag{3.5}$$

where it is assumed that $\int_0^{2\pi} w_0(y)dy = 0$. By imposing the divergence boundary conditions first, we get

$$\hat{u}_x(x,\omega) + i\omega\hat{v}(x,\omega) = 0\ , \quad x = 0,1$$

in Fourier space, which leads to

$$\begin{aligned} \alpha_\omega &= i(\frac{b_\omega}{\omega} - a_\omega)\,, \\ \beta_\omega &= i(\frac{d_\omega}{\omega} + c_\omega)\,, \end{aligned}$$

for $\omega \neq 0$, and

$$\alpha_0 = \beta_0 = 0$$

for $\omega = 0$. We then have exactly the form (2.5)-(2.10) of the solution, and we can apply the same arguments as in *Theorem 2.1* to prove

Theorem 3.1 *Assume that the boundary data $w_0,\ u_1,\ v_0,\ v_1$ are 2π- periodic in y and $\int_0^{2\pi} w_0(y)dy = 0$, but otherwise arbitrary. Then the pressure form of the system (1.2),(1.3),(1.5) with boundary conditions (3.5) has a unique solution, and there is an estimate*

$$||\mathbf{v}(\cdot,\cdot)||^2 \leq const\,(|q_0|^2 + ||\mathbf{v}_B(\cdot)||^2)\,, \tag{3.6}$$

$$||p(\cdot,\cdot)||^2 \leq const\,(|q_0|^2 + ||\mathbf{v}_B(\cdot)||^2 + ||\frac{\partial \mathbf{v}_B}{\partial y}(\cdot)||^2)\,, \tag{3.7}$$

■

For the type of boundary conditions (2.19) and (2.23), where the pressure is prescribed instead of one of the velocity components, we add the divergence condition as a boundary condition on both sides, and obtain the theorems 2.2 and 2.3 without further modification.

4 The Staggered Grid Method for the Divergence Form

In this section we will do an analogous analysis of the stability for a second order approximation of the Stokes equations, as we did for the continuous case. We will keep the notations and follow the pattern in the previous analysis very closely, and we refer to the continuous case when the details coincide.

The discretization in space is done on a staggered grid for a uniform Cartesian mesh. We assign the velocity component u to grid locations $(x_j - \frac{1}{2}\Delta x, y_k)$ and the velocity component v to $(x_j, y_k - \frac{1}{2}\Delta y)$ and the pressure values p to (x_j, y_k), i.e.

$$\begin{array}{llll} u_{j-\frac{1}{2},k} & = u(x_j - \frac{1}{2}\Delta x, y_k)\,, & j = 0, 1, \dots, N+1, & k = 0, 1, \dots, M, \\ v_{j,k-\frac{1}{2}} & = v(x_j, y_k - \frac{1}{2}\Delta y)\,, & j = 0, 1, \dots, N, & k = 0, 1, \dots, M, \\ p_{j,k} & = p(x_j, y_k)\,, & j = 0, 1, \dots, N, & k = 0, 1, \dots, M, \end{array}$$

where $x_j = j\Delta x$, $y_k = k\Delta y$, $\Delta x = 1/N$ and $\Delta y = 2\pi/M$, see Figure 1.

We use the notation $\mathbf{U}_{j,k} = (u_{j-\frac{1}{2},k}, v_{j,k-\frac{1}{2}}, p_{j,k})^T$, and consider the case with periodic solutions in the y-direction, that is $\mathbf{U}_{j,k} = \mathbf{U}_{j,k+M}$.

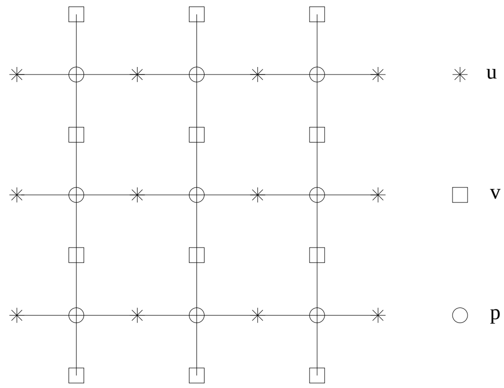

Figure 1: A staggered grid.

For the inner points we discretize the Stokes equations in the divergence form with the second order scheme

$$\begin{aligned} D_{-,x}p_{j,k} &= \nu(D_{+,x}D_{-,x} + D_{+,y}D_{-,y})u_{j-\frac{1}{2},k}\,, \\ D_{-,y}p_{j,k} &= \nu(D_{+,x}D_{-,x} + D_{+,y}D_{-,y})v_{j,k-\frac{1}{2}}\,, \\ D_{+,x}u_{j-\frac{1}{2},k} + D_{+,y}v_{j,k-\frac{1}{2}} &= 0\,, \end{aligned} \tag{4.1}$$

where $D_{+,x}u_{j,k} = (u_{j+1,k} - u_{j,k})/\Delta x$ and $D_{-,x}u_{j,k} = (u_{j,k} - u_{j-1,k})/\Delta x$, and similarly for $D_{\pm,y}$.

The first set of boundary conditions is where the velocity $\mathbf{v} = (u, v)^T$ is specified,

$$\begin{array}{ll}\frac{1}{2}(u_{-\frac{1}{2},k}+u_{\frac{1}{2},k})- & \\ \frac{1}{2\pi}\sum_{k=0}^{M-1}\frac{1}{2}(u_{-\frac{1}{2},k}+u_{\frac{1}{2},k})\Delta y = w_0(y_k)\,, & \frac{1}{2}(u_{N-\frac{1}{2},k}+u_{N+\frac{1}{2},k}) = u_1(y_k)\,, \\ v_{0,k-\frac{1}{2}} = v_0(y_{k-\frac{1}{2}})\,, & v_{N,k-\frac{1}{2}} = v_1(y_{k-\frac{1}{2}})\,, \\ \sum_{k=0}^{M-1} p_{0,k}\Delta y = q_0\,, & \end{array} \tag{4.2}$$

where $\sum_{k=0}^{M-1} w_0(y_k)\Delta y = 0$. We use the notation $\mathbf{v}_B = (w_0, u_1, v_0, v_1)^T$, and define the norms by

$$||\mathbf{U}||_h^2 = \sum_{k=0}^{M-1}\left(\sum_{j=0}^{N+1}|u_{j-\frac{1}{2},k}|^2 + \sum_{j=0}^{N}|v_{j,k-\frac{1}{2}}|^2 + \sum_{j=0}^{N}|p_{j,k}|^2\right)\Delta x\Delta y\,,$$

$$||\mathbf{v}||_h^2 = \sum_{k=0}^{M-1}\left(\sum_{j=0}^{N+1}|u_{j-\frac{1}{2},k}|^2 + \sum_{j=0}^{N}|v_{j,k-\frac{1}{2}}|^2\right)\Delta x\Delta y\,,$$

$$||\mathbf{v}_B||_h^2 = \sum_{k=0}^{M-1}\left(|w_0(y_k)|^2 + |u_1(y_k)|^2 + |v_0(y_{k-\frac{1}{2}})|^2 + |v_1(y_{k-\frac{1}{2}})|^2\right)\Delta y\,,$$

$$||p||_h^2 = \sum_{k=0}^{M-1}\sum_{j=0}^{N}|p_{j,k}|^2\Delta x\Delta y\,.$$

In analogy with the Theorem 2.1 for the continuous case, we have

Theorem 4.1 *Assume that the boundary data w_0, u_1, v_0, v_1 are 2π-periodic in y and $\sum_{k=0}^{M-1} w_0(y_k)\Delta y = 0$, but otherwise arbitrary. Then the second order scheme (4.1) with boundary conditions (4.2) has a unique solution, and there is an estimate*

$$||\mathbf{v}||_h^2 \leq const\,(|q_0|^2 + ||\mathbf{v}_B||_h^2)\,, \tag{4.3}$$

$$||p||_h^2 \leq const\,(|q_0|^2 + ||\mathbf{v}_B||_h^2 + ||D_{+,y}\mathbf{v}_B||_h^2)\,, \tag{4.4}$$

where the constants are independent of $\Delta x, \Delta y$.

Proof: By periodicity we expand the solution in a discrete Fourier series representation

$$\mathbf{U}_{j,k} = \frac{1}{\sqrt{2\pi}} \sum_{\omega=-M/2}^{M/2-1} \hat{\mathbf{U}}_j(\omega) e^{i\omega y_k} \,, \tag{4.5}$$

where $\hat{\mathbf{U}}_j = (\hat{u}_{j-\frac{1}{2}}, \hat{v}_j, \hat{p}_j)^T$, and substitute it into the system (4.1). We then get the discrete Fourier-transformed system

$$\begin{aligned}
\Delta x(\hat{p}_j - \hat{p}_{j-1}) \; &= \nu(\hat{u}_{j+\frac{1}{2}} - 2\hat{u}_{j-\frac{1}{2}} + \hat{u}_{j-\frac{3}{2}} + 2\lambda^2(\cos\eta - 1)\hat{u}_{j-\frac{1}{2}}) \,, \\
2i\lambda \sin(\eta/2)\Delta x \hat{p}_j \; &= \nu(\hat{v}_{j+1} - 2\hat{v}_j + \hat{v}_{j-1} + 2\lambda^2(\cos\eta - 1)\hat{v}_j) \,, \\
0 \; &= \hat{u}_{j+\frac{1}{2}} - \hat{u}_{j+\frac{1}{2}} + 2i\lambda \sin(\eta/2)\hat{v}_j \,,
\end{aligned} \tag{4.6}$$

where $\lambda = \frac{\Delta x}{\Delta y}$ and $\eta = \omega\Delta y$. We distinguish between the two cases $\omega \neq 0$ and $\omega = 0$. The general solution for $\omega > 0$ has the form

$$\begin{aligned}
\hat{u}_j(\omega) = \; & (a_\omega + b_\omega x_j)\kappa_1^j + (c_\omega + d_\omega(x_j - 1))\kappa_2^{j-N} \,, \\
\hat{v}_j(\omega) = \; & \hat{V}(\omega) \left(\tfrac{\kappa_1 - 1}{\sqrt{\kappa_1}}(a_\omega + b_\omega x_j) + \tfrac{1+\kappa_1}{\sqrt{\kappa_1}} \tfrac{\Delta x}{2} b_\omega \right) \kappa_1^j + \\
& \hat{V}(\omega) \left(\tfrac{\kappa_2 - 1}{\sqrt{\kappa_2}}(c_\omega + d_\omega(x_j - 1)) + \tfrac{1+\kappa_2}{\sqrt{\kappa_2}} \tfrac{\Delta x}{2} d_\omega \right) \kappa_2^{j-N} \,, \\
\hat{p}_j(\omega) = \; & \tfrac{\nu(\kappa_1+1)}{\sqrt{\kappa_1}} b_\omega \kappa_1^j + \tfrac{\nu(\kappa_2+1)}{\sqrt{\kappa_2}} d_\omega \kappa_2^{j-N} \,,
\end{aligned} \tag{4.7}$$

where $\hat{V}(\omega) = -(2i\lambda \sin(\eta/2))^{-1}$ and

$$\begin{aligned}
\kappa_1 &= 1 + \lambda^2(1 - \cos\eta) - \sqrt{(1 + \lambda^2(1 - \cos\eta))^2 - 1} \,, \\
\kappa_2 &= 1 + \lambda^2(1 - \cos\eta) + \sqrt{(1 + \lambda^2(1 - \cos\eta))^2 - 1} \,.
\end{aligned} \tag{4.8}$$

For $\omega = 0$ the general solution has the form

$$\begin{aligned}
\hat{u}_j(0) &= a_0 \,, \\
\hat{v}_j(0) &= b_0 + c_0 x_j \,, \\
\hat{p}_j(0) &= d_0 \,.
\end{aligned} \tag{4.9}$$

We then have for each ω, including $\omega = 0$, four constants $a_\omega, b_\omega, c_\omega, d_\omega$ to determine from the boundary conditions. But the structure is different for

$\omega = 0$, for which one of the boundary conditions on $\hat{u}$ must be substituted by a condition on $\hat{p}$. In Fourier space we get for $\omega > 0$

$$\begin{aligned}
\tfrac{\kappa_1+1}{2\sqrt{\kappa_1}}a_\omega + \tfrac{\Delta x}{4}\tfrac{\kappa_1-1}{\sqrt{\kappa_1}}b_\omega + \tfrac{\kappa_2+1}{2\sqrt{\kappa_2}}\kappa_2^{-N}c_\omega + \left(\tfrac{\Delta x}{4}\tfrac{\kappa_2-1}{\sqrt{\kappa_2}} - \tfrac{\kappa_2+1}{2\sqrt{\kappa_2}}\right)\kappa_2^{-N}d_\omega &= \hat{w}_0(\omega)\,, \\
\tfrac{\kappa_1+1}{2\sqrt{\kappa_1}}\kappa_1^N a_\omega + \left(\tfrac{\kappa_1+1}{2\sqrt{\kappa_1}} + \tfrac{\Delta x}{4}\tfrac{\kappa_1-1}{\sqrt{\kappa_1}}\right)\kappa_1^N b_\omega + \tfrac{\kappa_2+1}{2\sqrt{\kappa_2}}c_\omega + \tfrac{\Delta x}{4}\tfrac{\kappa_2-1}{\sqrt{\kappa_2}}d_\omega &= \hat{u}_1(\omega)\,, \\
\hat{V}(\omega)\left(\tfrac{\kappa_1-1}{\sqrt{\kappa_1}}a_\omega + \tfrac{1+\kappa_1}{\sqrt{\kappa_1}}\tfrac{\Delta x}{2}b_\omega + \tfrac{\kappa_2-1}{\sqrt{\kappa_2}}\kappa_2^{-N}c_\omega + (\tfrac{1+\kappa_2}{\sqrt{\kappa_2}}\tfrac{\Delta x}{2} - \tfrac{\kappa_2-1}{\sqrt{\kappa_2}})\kappa_2^{-N}d_\omega\right) &= \hat{v}_0(\omega)\,, \\
\hat{V}(\omega)\left(\tfrac{\kappa_1-1}{\sqrt{\kappa_1}}\kappa_1^N a_\omega + (\tfrac{1+\kappa_1}{\sqrt{\kappa_1}}\tfrac{\Delta x}{2} + \tfrac{\kappa_1-1}{\sqrt{\kappa_1}})\kappa_1^N b_\omega + \tfrac{\kappa_2-1}{\sqrt{\kappa_2}}c_\omega + \tfrac{1+\kappa_2}{\sqrt{\kappa_2}}\tfrac{\Delta x}{2}d_\omega\right) &= \hat{v}_1(\omega)\,,
\end{aligned} \tag{4.10}$$

and for $\omega = 0$

$$\begin{aligned}
a_0 &= \hat{u}_1(0)\,, \\
b_0 &= \hat{v}_0(0)\,, \\
b_0 + c_0 &= \hat{v}_1(0)\,, \\
d_0 &= q_0\,.
\end{aligned} \tag{4.11}$$

For the discrete case there are two difficulties when determining the constants $a_\omega, b_\omega, c_\omega, d_\omega$ from the boundary conditions. The first problem is that the determinant of the coefficient matrix approaches zero when $\Delta x, \Delta y$ are fixed and $\omega \to \infty$. To prevent this, we rescale the coefficients as

$$\begin{aligned}
b_\omega &= \tfrac{2\sin(\eta/2)}{\eta}\omega\tilde{b}_\omega\,, \\
d_\omega &= \tfrac{2\sin(\eta/2)}{\eta}\omega\tilde{d}_\omega\,,
\end{aligned} \tag{4.12}$$

which yields the determinant for $\omega > 0$

$$\mathcal{D} = \left(\kappa_1^{2N} - 1\right)^2 - \frac{\omega^2}{\lambda^2\eta^2}\left(\kappa_1 - \kappa_1^{-1}\right)^2\kappa_1^{2N}\,. \tag{4.13}$$

Assume that $\mathcal{D} = 0$, then from the definition of κ_1, κ_2 (4.8), we can rewrite (4.13) as

$$(\kappa_1^{2N} - 1)^2 - b\kappa_1^{2N} = 0\,, \tag{4.14}$$

$$b = 4N^2\left(2a + a^2\right)\,, \tag{4.15}$$

where $a = \lambda^2(1 - \cos\eta)$. But from the equation (4.14) b must fulfill $b = b^*$, where

$$b^* = \left(\left(1 + a - \sqrt{2a + a^2}\right)^N - \left(1 + a + \sqrt{2a + a^2}\right)^N\right)^2\,, \tag{4.16}$$

and that is a contradiction to (4.15). From the binomial theorem we get

$$
\begin{aligned}
b^* - b &= 16N(N-1)(N-\frac{1}{4})a^2 + 8N(N-1)(N-\frac{1}{2})a^3 \\
&+ 2\sum_{k=3}^{2N} \binom{2N}{k} a^k + (4N^2-2N)(2a+a^2)\sum_{k=2}^{2N-2}\binom{2N-2}{k} a^k \\
&+ 2\sum_{k=2}^{N}\binom{2N}{2k}(1+a)^{2N-2k}\left(2a+a^2\right)^k ,
\end{aligned} \tag{4.17}
$$

which shows that $b^* - b$ is a positive monotone increasing function for all $a > 0$ and $N > 1$, since all terms on the right hand side are positive. Hence, the determinant is positive for all $\omega > 0$.

The second problem is that, the determinant of the coefficient matrix approaches zero also when ω is fixed and $\Delta x, \Delta y \to 0$. This can bee seen from (4.13) or (4.17) by realizing that $\kappa_1 \to 0$ and $a \to 0$, when we increase the number of grid points. However, when $\Delta x, \Delta y \to 0$ the form of the solution (4.7) approaches the form of the continuous solution, i.e.,

$$
\begin{aligned}
\hat{u}_j &= (a_\omega + b_\omega x_j)(1-\Delta x\omega)^j + (c_\omega + d_\omega(x_j - 1))(1+\Delta x\omega)^{j-N} \\
&\quad + \mathcal{O}(\Delta x + \Delta y) \\
&= (a_\omega + b_\omega x_j)e^{-\omega x_j} + (c_\omega + d_\omega(x_j-1))e^{\omega(x_j-1)} \\
&\quad + \mathcal{O}(\Delta x + \Delta y) ,
\end{aligned} \tag{4.18}
$$

$$
\begin{aligned}
\hat{v}_j &= \tfrac{i}{\omega\Delta x}(-\omega\Delta x(a_\omega + b_\omega x_j) + \Delta x b_w)(1-\Delta x\omega)^j + \\
&\quad \tfrac{i}{\omega\Delta x}(\omega\Delta x c_\omega + d_\omega(x_j-1) + \Delta x d_\omega)(1+\Delta x\omega)^{j-N} \\
&\quad + \mathcal{O}(\Delta x + \Delta y) \\
&= i(-a_\omega + \tfrac{1}{\omega}b_\omega - b_\omega x_j)e^{-\omega x_j} + \\
&\quad i(c_\omega + \tfrac{1}{\omega}d_\omega + d_\omega(x_j-1))e^{\omega(x_j-1)} \\
&\quad + \mathcal{O}(\Delta x + \Delta y) ,
\end{aligned} \tag{4.19}
$$

$$
\begin{aligned}
\hat{p}_j &= \nu(2-\omega\Delta x)b_\omega(1-\Delta x\omega)^j + \nu(2+\omega\Delta x)d_\omega(1+\Delta x\omega)^{j-N} \\
&\quad + \mathcal{O}(\Delta x + \Delta y) \\
&= 2\nu(b_\omega e^{-\omega x_j} + d_\omega e^{\omega(x_j-1)}) + \mathcal{O}(\Delta x + \Delta y) .
\end{aligned} \tag{4.20}
$$

In the proof on Theorem 2.1 we showed that this system is well conditioned in the limit $\Delta x = \Delta y = 0$, and since all components are continuous with respect to Δx and Δy we conclude that this system is well conditioned also in a neighborhood of $\Delta x = \Delta y = 0$.

Indeed, this shows that $|\mathcal{D}| \geq \delta > 0$, δ independent of $\Delta x, \Delta y$ and ω, $\omega > 0$. Since all elements of the matrix are bounded, we conclude that the condition number is also bounded. The same result holds also for $\omega < 0$, by symmetry. For $\omega = 0$, the determination of the constants in (4.9) is well conditioned.

Therefore, the constants, $a_\omega, \tilde{b}_\omega, c_\omega, \tilde{d}_\omega$ are bounded in terms of the boundary data $\hat{w}_0, \hat{u}_1, \hat{v}_0, \hat{v}_1$. Hence, the corresponding estimate for the original constants for the discrete case is for $\omega \neq 0$

$$|a_\omega|^2 + |c_\omega|^2 \leq \text{const}\,(|\hat{w}_0(\omega)|^2 + |\hat{u}_1(\omega)|^2 + |\hat{v}_0(\omega)|^2 + |\hat{v}_1(\omega)|^2)\,, \qquad (4.21)$$

$$|b_\omega|^2 + |d_\omega|^2 \leq \text{const}\,|\omega|^2(|\hat{w}_0(\omega)|^2 + |\hat{u}_1(\omega)|^2 + |\hat{v}_0(\omega)|^2 + |\hat{v}_1(\omega)|^2)\,, \quad (4.22)$$

and for $\omega = 0$

$$|a_0|^2 + |b_0|^2 + |c_0|^2 + |d_0|^2 \leq \text{const}\,(|\hat{u}_1(0)|^2 + |\hat{v}_0(0)|^2 + |\hat{v}_1(0)|^2 + |\hat{q}_0|^2)\,. \qquad (4.23)$$

Note, that a factor ω occurs in the estimate of the coefficients b_ω and d_ω because of the rescaling (4.12). However, we can estimate the effect of this. For $\hat{p}$ the extra factor $2\sin(\eta/2)\omega/\eta$ corresponds to the introduction of the difference operator $D_{+,y}$ in physical space, since

$$D_{+,y}e^{i\omega y_k} = \frac{e^{i\omega\Delta y} - 1}{\Delta y}e^{i\omega y_k} = \frac{2i\sin(\eta/2)}{\eta}\omega e^{i\omega(k+1/2)\Delta y}\,. \qquad (4.24)$$

Furthermore, one can show that the estimate for the velocity components u and v does not deteriorate because of the extra factor. Consider the case $\omega > 0$ and the expressions (4.7) for $\hat{u}$ and $\hat{v}$ near $x_j = 0$. In this case, the constants b_ω, d_ω are multiplied either by

$$\frac{i}{2\lambda\sin(\eta/2)}\frac{1+\kappa_1}{\sqrt{\kappa_1}}\frac{\Delta x}{2}\kappa_1^j$$

or by

$$x_j\kappa_1^j\,.$$

The first factor can be rewritten as

$$\frac{1+\kappa_1}{\sqrt{\kappa_1}}\frac{i\eta}{4\omega\sin(\eta/2)}\kappa_1^j$$

and obviously the extra factor $2\sin(\eta/2)\omega/\eta$ is canceled. For the second term, we note that $\omega x_j \kappa_1^j$ has a bounded maximum

$$\max_{\substack{\omega\Delta x>0 \\ j=0,1,\dots,N}} (\omega x_j \kappa_1^j) = \frac{\sqrt{3}}{9} \frac{\eta}{\sin(\eta/2)}$$

for $j = 1$ and $\omega\Delta x = \frac{\eta}{\sqrt{3}\sin(\eta/2)}$. By symmetry, the corresponding results hold near $x = 1$ and all these arguments also apply for negative ω as well. From Parseval's identity the estimate for u, v, does not deteriorate because of the extra factor $2\sin(\eta/2)\omega/\eta$. Hence, the estimates (4.3) and (4.4) hold, and the theorem is proven.

■

Consider now the case where the pressure is prescribed on the boundary instead of one of the velocity components. For this case, there is not possible to leave out one of the velocity components completely. The conditions are

$$\begin{array}{ll}
\frac{1}{2}(u_{-\frac{1}{2},k} + u_{\frac{1}{2},k}) - & \\
\frac{1}{2\pi}\sum_{k=0}^{M-1} \frac{1}{2}(u_{-\frac{1}{2},k} + u_{\frac{1}{2},k})\Delta y = w_0(y_k)\,, & \frac{1}{2}(u_{N-\frac{1}{2},k} + u_{N+\frac{1}{2},k}) = u_1(y_k)\,, \\
p_{0,k} - \frac{1}{2\pi}\sum_{k=0}^{M-1} p_{0,k}\Delta y = q_0(y_k)\,, & p_{N,k} = p_1(y_k)\,, \\
\sum_{k=0}^{M-1} v_{0,k-\frac{1}{2}}\Delta y = z_0\,, & \sum_{k=0}^{M-1} v_{N,k-\frac{1}{2}}\Delta y = z_1\,.
\end{array} \tag{4.25}$$

With this set of boundary conditions, we have

Theorem 4.2 *Assume that the boundary data w_0, u_1, q_0, p_1 are 2π-periodic in y and $\sum_{k=0}^{M-1} w_0(y_k)\Delta y = \sum_{k=0}^{M-1} q_0(y_k)\Delta y = 0$, but otherwise arbitrary. Then the second order scheme (4.1) with boundary conditions (4.25) has a unique solution, and there is an estimate*

$$\begin{aligned}
||\mathbf{U}||_h^2 \leq const \Big(& |z_0|^2 + |z_1|^2 + \\
& \sum_{k=0}^{M-1} \left(|w_0(y_k)|^2 + |u_1(y_k)|^2 + |q_0(y_k)|^2 + |p_1(y_k)|^2\right)\Delta y \Big)\,.
\end{aligned} \tag{4.26}$$

where the constants are independent of $\Delta x, \Delta y$.

Proof: As in Theorem 4.1 we write the solutions as a discrete Fourier series and get (4.6) with the general form of the solution given by (4.7), (4.8) and (4.9).

For $\omega > 0$, the boundary conditions in transformed space are

$$\begin{aligned} \tfrac{\kappa_1+1}{2\sqrt{\kappa_1}}a_\omega + \tfrac{\Delta x}{4}\tfrac{\kappa_1-1}{\sqrt{\kappa_1}}b_\omega + \tfrac{\kappa_2+1}{2\sqrt{\kappa_2}}\kappa_2^{-N}c_\omega + \left(\tfrac{\Delta x}{4}\tfrac{\kappa_2-1}{\sqrt{\kappa_2}} - \tfrac{\kappa_2+1}{2\sqrt{\kappa_2}}\right)\kappa_2^{-N}d_\omega &= \hat{w}_0(\omega)\,,\\ \tfrac{\kappa_1+1}{2\sqrt{\kappa_1}}\kappa_1^N a_\omega + \left(\tfrac{\kappa_1+1}{2\sqrt{\kappa_1}} + \tfrac{\Delta x}{4}\tfrac{\kappa_1-1}{\sqrt{\kappa_1}}\right)\kappa_1^N b_\omega + \tfrac{\kappa_2+1}{2\sqrt{\kappa_2}}c_\omega + \tfrac{\Delta x}{4}\tfrac{\kappa_2-1}{\sqrt{\kappa_2}}d_\omega &= \hat{u}_1(\omega)\,,\\ \tfrac{\nu(\kappa_1+1)}{\sqrt{\kappa_1}}b_\omega + \tfrac{\nu(\kappa_2+1)}{\sqrt{\kappa_2}}\kappa_2^{-N}d_\omega &= \hat{p}_0(\omega)\,,\\ \tfrac{\nu(\kappa_1+1)}{\sqrt{\kappa_1}}\kappa_1^N b_\omega + \tfrac{\nu(\kappa_2+1)}{\sqrt{\kappa_2}}d_\omega &= \hat{p}_1(\omega)\,, \end{aligned} \tag{4.27}$$

and for $\omega = 0$

$$\begin{aligned} a_0 &= \hat{u}_1(0)\,,\\ b_0 &= z_0\,,\\ b_0 + c_0 &= z_1\,,\\ d_0 &= \hat{p}_1(0)\,. \end{aligned} \tag{4.28}$$

This yields for $\omega > 0$ the determinant of the coefficient matrix

$$\mathcal{D} = -\frac{\nu}{4}(\kappa_1+1)^2(\kappa_2+1)^2\left(\kappa_1^{2N}-1\right)^2\,, \tag{4.29}$$

which is independent of ω for a fixed $\lambda, \eta > 0$, and one can show that the condition number is bounded independent of ω, even as $\omega \to \infty$. A similar analysis as in Theorem 4.1 shows that the system is also well conditioned when $\Delta x, \Delta y \to 0$, and for the case $\omega = 0$.

This shows that all the constants $a_\omega, b_\omega, c_\omega, d_\omega$ can be estimated in terms of the given data independent of ω. Since the j-dependent functions occuring in (4.7) and (4.9) are all bounded, the estimate (4.26) follows.

■

Finally, the third set of boundary conditions for the discrete case is

$$\begin{aligned} &v_{0,k-\frac{1}{2}} = v_0(y_{k-\frac{1}{2}})\,, && v_{N,k-\frac{1}{2}} = v_1(y_{k-\frac{1}{2}})\,,\\ &p_{0,k} - \tfrac{1}{2\pi}\textstyle\sum_{k=0}^{M-1} p_{0,k}\Delta y = q_0(y_k)\,, && p_{N,k} = p_1(y_k)\,,\\ &\textstyle\sum_{k=0}^{M-1}\tfrac{1}{2}(u_{-\frac{1}{2},k} + u_{\frac{1}{2},k})\Delta y = w_0\,. \end{aligned} \tag{4.30}$$

In analogy with the previous cases, we have

Theorem 4.3 *Assume that the boundary data v_0, v_1, q_0, p_1 are 2π-periodic in y and $\sum_{k=0}^{M-1} q_0(y_k)\Delta y = 0$, but otherwise arbitrary. Then the second order scheme (4.1) with boundary conditions (4.30) has a unique solution, and there is an estimate*

$$\begin{aligned}
&||\mathbf{U}||_h^2 \leq const \Bigg(|w_0|^2 + \\
&\quad \sum_{k=0}^{M-1} \left(|v_0(y_{k-\frac{1}{2}})|^2 + |v_1(y_{k-\frac{1}{2}})|^2 + |q_0(y_k)|^2 + |p_1(y_k)|^2 \right) \Delta y \Bigg) .
\end{aligned} \tag{4.31}$$

where the constants are independent of $\Delta x, \Delta y$.

Proof: As above we write the solutions as a discrete Fourier series and get (4.6) with the general form of the solution given by (4.7), (4.8) and (4.9).

For $\omega > 0$ we get

$$\begin{aligned}
\hat{V}(\omega) \left(\tfrac{\kappa_1 - 1}{\sqrt{\kappa_1}} a_\omega + \tfrac{\kappa_1+1}{\sqrt{\kappa_1}} \tfrac{\Delta x}{2} b_\omega + \tfrac{\kappa_2 - 1}{\sqrt{\kappa_2}} \kappa_2^{-N} c_\omega + \left(\tfrac{\kappa_2+1}{\sqrt{\kappa_2}} \tfrac{\Delta x}{2} - \tfrac{\kappa_2-1}{\sqrt{\kappa_2}}\right) \kappa_2^{-N} d_\omega \right) &= \hat{v}_0(\omega) , \\
\hat{V}(\omega) \left(\tfrac{\kappa_1 - 1}{\sqrt{\kappa_1}} \kappa_1^N a_\omega + \left(\tfrac{\kappa_1-1}{\sqrt{\kappa_1}} + \tfrac{\kappa_1+1}{\sqrt{\kappa_1}} \tfrac{\Delta x}{2}\right) \kappa_1^N b_\omega + \tfrac{\kappa_2 - 1}{\sqrt{\kappa_2}} c_\omega + \tfrac{\kappa_2+1}{\sqrt{\kappa_2}} \tfrac{\Delta x}{2} d_\omega \right) &= \hat{v}_1(\omega) , \\
\tfrac{\nu(\kappa_1+1)}{\sqrt{\kappa_1}} b_\omega + \tfrac{\nu(\kappa_2+1)}{\sqrt{\kappa_2}} \kappa_2^{-N} d_\omega &= \hat{q}_0(\omega) , \\
\tfrac{\nu(\kappa_1+1)}{\sqrt{\kappa_1}} \kappa_1^N b_\omega + \tfrac{\nu(\kappa_2+1)}{\sqrt{\kappa_2}} d_\omega &= \hat{p}_1(\omega) ,
\end{aligned} \tag{4.32}$$

and for $\omega = 0$

$$\begin{aligned}
a_0 &= w_0 , \\
b_0 &= \hat{v}_0(0) , \\
b_0 + c_0 &= \hat{v}_1(0) , \\
d_0 &= \hat{p}_1(0) .
\end{aligned} \tag{4.33}$$

This yields for $\omega > 0$ the determinant of the coefficient matrix

$$\mathcal{D} = -\hat{V}(\omega)^2 (\kappa_1^2 - 1)(\kappa_2^2 - 1) \left(\kappa_1^{2N} - 1\right)^2 \tag{4.34}$$

which is independent of ω for a fixed $\lambda, \eta > 0$ and a similar analysis as above proves the theorem.

■

5 The Staggered Grid Method for the Pressure Form

Consider now the discrete formulation of the pressure form of the Stokes equations. For the inner points we discretize the equations on a staggered grid with the second order scheme

$$\begin{aligned} D_{-,x}p_{j,k} &= \nu(D_{+,x}D_{-,x} + D_{+,y}D_{-,y})u_{j-\frac{1}{2},k}\,, \\ D_{+,y}p_{j,k} &= \nu(D_{+,x}D_{-,x} + D_{+,y}D_{-,y})v_{j,k+\frac{1}{2}}\,, \\ 0 &= (D_{+,x}D_{-,x} + D_{+,y}D_{-,y})p_{j,k}\,. \end{aligned} \tag{5.1}$$

As in the previous section we assume that the solutions are 2π-periodic in the y-direction. This yields the Fourier-transformed system

$$\begin{aligned} \Delta x(\hat{p}_j - \hat{p}_{j-1}) &= \nu(\hat{u}_{j+\frac{1}{2}} - 2\hat{u}_{j-\frac{1}{2}} + \hat{u}_{j-\frac{3}{2}} + 2\lambda^2(\cos\eta - 1)\hat{u}_{j-\frac{1}{2}})\,, \\ 2i\lambda\sin(\eta/2)\Delta x\hat{p}_j &= \nu(\hat{v}_{j+1} - 2\hat{v}_j + \hat{v}_{j-1} + 2\lambda^2(\cos\eta - 1)\hat{v}_j)\,, \\ 0 &= \hat{p}_{j+1} - 2\hat{p}_j + \hat{p}_{j-1} + 2\lambda^2(\cos\eta - 1)\hat{p}_j\,. \end{aligned} \tag{5.2}$$

We first exclude negative values of ω, and obtain the general form of the solution for $\omega > 0$ as

$$\begin{aligned} \hat{u}_j(\omega) = &\ (a_\omega + b_\omega x_j)\kappa_1^j + (c_\omega + d_\omega(x_j - 1))\kappa_2^{j-N}\,, \\ \hat{v}_j(\omega) = &\ \left(\alpha_\omega + \tfrac{i\Delta y}{2\Delta x\sin(\omega\Delta y/2)}\tfrac{\kappa_1 - 1}{\sqrt{\kappa_1}}b_\omega x_j\right)\kappa_1^j + \\ &\ \left(\beta_\omega + \tfrac{i\Delta y}{2\Delta x\sin(\omega\Delta y/2)}\tfrac{\kappa_2 - 1}{\sqrt{\kappa_2}}d_\omega(x_j - 1)\right)\kappa_2^{j-N}\,, \\ \hat{p}_j(\omega) = &\ \tfrac{\nu(\kappa_1+1)}{\sqrt{\kappa_1}}b_\omega\kappa_1^j + \tfrac{\nu(\kappa_2+1)}{\sqrt{\kappa_2}}d_\omega\kappa_2^{j-N}\,, \end{aligned} \tag{5.3}$$

where κ_1, κ_2 are given by (4.8). For $\omega = 0$ the general solution has the form

$$\begin{aligned} \hat{u}_j(0) &= a_0 + \alpha_0 x_j + \tfrac{1}{2\nu}\beta_0 x_j^2\,, \\ \hat{v}_j(0) &= b_0 + c_0 x_j\,, \\ \hat{p}_j(0) &= d_0 + \beta_0 x_j\,. \end{aligned} \tag{5.4}$$

In accordance with the continuous solution for Stokes equations on the pressure form, we have two extra constants $\alpha_\omega, \beta_\omega$. Therefore, we add the divergence condition as extra boundary conditions at $x = 0, 1$. The proposed set of boundary conditions are

$$
\begin{array}{ll}
\frac{1}{2}(u_{-\frac{1}{2},k}+u_{\frac{1}{2},k})- & \\
\frac{1}{2\pi}\sum_{k=0}^{M-1}\frac{1}{2}(u_{-\frac{1}{2},k}+u_{\frac{1}{2},k})\Delta y = w_0(y_k)\,, & \frac{1}{2}(u_{N-\frac{1}{2},k}+u_{N+\frac{1}{2},k}) = u_1(y_k)\,,\\
v_{0,k-\frac{1}{2}} = v_0(y_{k-\frac{1}{2}})\,, & v_{N,k-\frac{1}{2}} = v_1(y_{k-\frac{1}{2}})\,,\\
\sum_{k=0}^{M-1} p_{0,k}\Delta y = q_0\,, & \\
(u_{-\frac{1}{2},k}+u_{\frac{1}{2},k})/\Delta x + & (u_{N-\frac{1}{2},k}+u_{N+\frac{1}{2},k})/\Delta x+\\
(v_{0,k-\frac{1}{2}}+v_{0,k+\frac{1}{2}})/\Delta y = 0\,, & (v_{N,k-\frac{1}{2}}+v_{N,k+\frac{1}{2}})/\Delta y = 0\,,
\end{array}
\tag{5.5}
$$

where it is assumed that $\sum_{k=0}^{M-1} w_0(y_k)\Delta y = 0$. By applying the discrete Fourier transformed divergence condition at the boundaries $j = 0, N$, the extra constants can be determined as

$$
\begin{aligned}
\alpha_\omega &= \frac{i\Delta y}{2\Delta x\sin(\frac{\omega\Delta y}{2})}\left(\frac{\kappa_1-1}{\sqrt{\kappa_1}}a_\omega + \frac{\kappa_1+1}{\sqrt{\kappa_1}}\frac{\Delta x}{2}b_\omega\right)\,,\\
\beta_\omega &= \frac{i\Delta y}{2\Delta x\sin(\frac{\omega\Delta y}{2})}\left(\frac{\kappa_2-1}{\sqrt{\kappa_2}}c_\omega + \frac{\kappa_2+1}{\sqrt{\kappa_2}}\frac{\Delta x}{2}d_\omega\right)\,,
\end{aligned}
\tag{5.6}
$$

for $\omega \neq 0$, and

$$\alpha_0 = \beta_0 = 0 \tag{5.7}$$

for $\omega = 0$. We then have exactly the form (4.7)-(4.9) of the solution, and we can apply the same arguments as in Theorem 4.1 to prove

Theorem 5.1 *Assume that the boundary data w_0, u_1, v_0, v_1 are 2π-periodic in y and $\sum_{k=0}^{M-1} w_0(y_k)\Delta y = 0$, but otherwise arbitrary. Then the second order scheme (5.3) for the pressure form of the Stokes equations with boundary conditions (5.5) has a unique solution, and there is an estimate*

$$||\mathbf{v}||_h^2 \le const\,(|q_0|^2 + ||\mathbf{v}_B||_h^2)\,, \tag{5.8}$$

$$||p||_h^2 \le const\,(|q_0|^2 + ||\mathbf{v}_B||_h^2 + ||D_{+,y}\mathbf{v}_B||_h^2)\,. \tag{5.9}$$

where the constants are independent of $\Delta x, \Delta y$.

In a similar way, we obtain the Theorem 4.2 and the Theorem 4.3 without further modification for the type of boundary conditions (4.25) and (4.30), where the pressure is prescribed instead of one of the velocity components, if we add the divergence condition as a boundary condition on both sides.

6 Numerical Examples

The first test problem is the one that has been analyzed above, i.e. the Stokes equations in the divergence form with periodic solutions in the domain $\Omega = \{0 \leq x \leq 1, 0 \leq y \leq 2\pi\}$. In order to obtain a simple analytic solution, we add a forcing function in the second equation. The problem we solved was

$$\begin{aligned} \nu(u_{xx} + u_{yy}) - p_x &= 0\,, \\ \nu(v_{xx} + v_{yy}) - p_y &= 4\nu \cos(x)\sin(y)\,, \qquad (x, y) \in \Omega, \\ u_x + v_y &= 0\,, \end{aligned} \tag{6.1}$$

where $\nu = 1$. In this case the exact solutions are

$$\begin{aligned} u^* &= \sin(x)\cos(y)\,, \\ v^* &= -\cos(x)\sin(y)\,, \\ p^* &= 2\nu\cos(x)\cos(y)\,. \end{aligned} \tag{6.2}$$

The test problem is discretized in space according to the second order scheme (4.1). A sparse direct solver was used to solve the algebraic system of equations.

In Table 1 we display the errors of the calculations with the boundary conditions (4.2). The error is measured in l^2-norm, for u i.e.,

$$||u - u^*||_h = \left(\sum_{k=0}^{M-1} \sum_{j=0}^{N+1} |u_{j-\frac{1}{2},k} - u^*(x_{j-\frac{1}{2}}, y_k)|^2 \Delta x \Delta y \right)^{1/2} . \tag{6.3}$$

In Table 2 and in Table 3 the calculations with the boundary conditions (4.25) and (4.30) respectively are shown. We can see that we achieved second-order accuracy both for the velocity components and the pressure in all three cases. This verifies the theory. However, note that while the accuracy of the velocity components is almost of the same order of magnitude in all three cases, the results for the pressure are more accurate for the cases when the pressure is specified at the boundary by more than three order of magnitude. To this can be attributed that the error in the pressure is almost independent of the error in the divergence equations.

The same results hold, with non-periodic solutions. As a test problem we take the same equations as in the previous example but on the domain $\Omega = \{0 \leq x \leq 1, 0 \leq y \leq 1\}$. In Table 4 - Table 6 we display the results

$N \times M$	$\|\|u-u^*\|\|_h$	$\|\|v-v^*\|\|_h$	$\|\|p-p^*\|\|_h$
10×10	2.75e-3	2.95e-2	3.03e-1
20×20	5.93e-4	7.13e-3	6.71e-2
40×40	1.40e-4	1.77e-3	1.59e-2
80×80	3.41e-5	4.41e-4	3.89e-3

Table 1: Numerical results for the periodic case, where u and v are specified at the boundary.

$N \times M$	$\|\|u-u^*\|\|_h$	$\|\|v-v^*\|\|_h$	$\|\|p-p^*\|\|_h$
10×10	3.77e-3	3.04e-2	2.11e-4
20×20	8.63e-4	7.24e-3	5.66e-5
40×40	2.07e-4	1.78e-3	1.39e-5
80×80	5.05e-5	4.42e-4	3.39e-6

Table 2: Numerical results for the periodic case, where u and p are specified at the boundary.

$N \times M$	$\|\|u-u^*\|\|_h$	$\|\|v-v^*\|\|_h$	$\|\|p-p^*\|\|_h$
10×10	5.01e-2	8.43e-3	2.11e-4
20×20	1.12e-2	2.10e-3	5.66e-5
40×40	2.64e-3	5.24e-4	1.39e-5
80×80	6.43e-4	1.31e-4	3.39e-6

Table 3: Numerical results for the periodic case, where v and p are specified at the boundary.

of the calculations with the boundary conditions (4.2), (4.25) and (4.30) respectively on the boundary $x = 0$ and $x = 1$. For the inner points we use the same discretization scheme as for the previous example and in all three cases we use the following boundary conditions at $y = 0$ and $y = 1$

$$\begin{aligned} u_{j-\frac{1}{2},0} &= u^*(x_{j-\frac{1}{2}}, 0)\,, & u_{j-\frac{1}{2},M} &= u^*(x_{j-\frac{1}{2}}, 1)\,, \\ \tfrac{1}{2}(v_{j,-\frac{1}{2}} + v_{j,\frac{1}{2}}) &= v^*(x_j, 0)\,, & \tfrac{1}{2}(v_{j,M-\frac{1}{2}} + v_{j,M+\frac{1}{2}}) &= v^*(x_j, 1)\,. \end{aligned} \tag{6.4}$$

Also for this problem we achieve second-order accuracy both for the velocity and the pressure, which illustrates that the results of the analysis of the periodic case are also valid for the non-periodic case.

The second test problem was the Stokes equations on the pressure form

$N \times M$	$\|\|u-u^*\|\|_h$	$\|\|v-v^*\|\|_h$	$\|\|p-p^*\|\|_h$
10×10	6.54e-4	6.54e-4	4.91e-3
20×20	1.52e-4	1.52e-4	1.52e-3
40×40	3.71e-5	3.71e-5	4.49e-4
80×80	9.21e-6	9.21e-6	1.29e-4

Table 4: Numerical results for the non-periodic case, where u and v are specified at the boundary.

$N \times M$	$\|\|u-u^*\|\|_h$	$\|\|v-v^*\|\|_h$	$\|\|p-p^*\|\|_h$
10×10	6.64e-4	6.65e-4	6.37e-3
20×20	1.52e-4	1.54e-4	1.57e-3
40×40	3.70e-5	3.70e-5	3.99e-4
80×80	9.17e-6	9.08e-6	1.06e-4

Table 5: Numerical results for the non-periodic case, where u and p are specified at the boundary.

$N \times M$	$\|\|u-u^*\|\|_h$	$\|\|v-v^*\|\|_h$	$\|\|p-p^*\|\|_h$
10×10	7.18e-4	6.04e-4	6.44e-3
20×20	1.63e-4	1.34e-4	1.58e-3
40×40	3.89e-5	3.17e-5	3.93e-4
80×80	9.55e-6	7.74e-6	9.89e-5

Table 6: Numerical results for the non-periodic case, where v and p are specified at the boundary.

with periodic solutions in the y-direction,

$$\begin{aligned} \nu(u_{xx}+u_{yy})-p_x &= 0\,, \\ \nu(v_{xx}+v_{yy})-p_y &= 4\nu\cos(x)\sin(y)\,, \qquad (x,y)\in\Omega, \\ p_{xx}+p_{yy} &= -4\nu\cos(x)\cos(y)\,, \end{aligned} \tag{6.5}$$

where $\Omega = \{0 \le x \le 1, 0 \le y \le 2\pi\}$ and $\nu = 1$. This formulation has the same solution (6.2) as in the first problem. The problem is discretized according to the second order scheme (5.1) on the same staggered grids as in the previous problem and with boundary conditions (5.5).

We can see in Table 7, where we have displayed the computational errors, that we achieved second-order accuracy also in this case. The same results hold for the Stokes equation on the pressure form together with one of the other types of boundary conditions (4.25) and (4.30), where the pressure is prescribed instead of one of the velocity components, see Table 8 and Table 9.

$N\times M$	$\|\|u-u^*\|\|_h$	$\|\|v-v^*\|\|_h$	$\|\|p-p^*\|\|_h$
10×10	2.76e-03	2.10e-02	1.85e-01
20×20	5.93e-04	5.06e-03	4.01e-02
40×40	1.40e-04	1.25e-03	9.47e-03
80×80	3.40e-05	3.12e-04	2.31e-03

Table 7: Numerical results for the periodic case, where u and v are specified at the boundary.

$N\times M$	$\|\|u-u^*\|\|_h$	$\|\|v-v^*\|\|_h$	$\|\|p-p^*\|\|_h$
10×10	4.02e-03	2.33e-02	9.27e-03
20×20	9.22e-04	5.53e-03	2.23e-03
40×40	2.21e-04	1.35e-03	5.45e-04
80×80	5.41e-05	3.36e-04	1.35e-04

Table 8: Numerical results for the periodic case, where u and p are specified at the boundary.

$N\times M$	$\|\|u-u^*\|\|_h$	$\|\|v-v^*\|\|_h$	$\|\|p-p^*\|\|_h$
10×10	4.97e-02	7.67e-03	9.27e-03
20×20	1.11e-02	1.91e-03	2.23e-03
40×40	2.63e-03	4.76e-04	5.45e-04
80×80	6.40e-04	1.19e-04	1.35e-04

Table 9: Numerical results for the periodic case, where v and p are specified at the boundary.

The last two test problems are designed to demonstrate that the estimates in Theorem 2.1 and Theorem 4.1 are sharp. The estimate (4.4) for the solution of the pressure demands that we can bound the term $||D_{+,y}\mathbf{v}_b||_h$. If we cannot limit this norm, we cannot restrict $||p||_h$.

To illustrate this, we choose the test problem to be the Stokes equations in the divergence form on a rectangle with periodic solutions in the y-direction

$$\begin{aligned} u_{xx}+u_{yy}-p_x &= 0\,, \\ v_{xx}+v_{yy}-p_y &= 0\,, \qquad (x,y)\in\Omega, \\ u_x+v_y &= 0\,, \end{aligned} \tag{6.6}$$

where $\Omega=\{0\le x\le 1, 0\le y\le 2\pi\}$. The discretization is done as in the first test problem and the boundary conditions are implemented according to (4.2), where $w_0(y_k)=(-1)^k$ and $u_1(y_k)=v_0(y_k)=v_1(y_k)=0$. For this choice of boundary data the norm of $D_{+,y}\mathbf{v}_B$ is unbounded when $\Delta y \to 0$, since

$$||D_{+,y} w_0(y_k)||_h^2 = \frac{4}{\Delta y}\,. \tag{6.7}$$

Table 10 shows the norm of the solutions both for velocity components u, v and the pressure, for different mesh sizes. We can see that the solution for the velocity is bounded for increased number of grid points, but the pressure is not.

$N \times M$	$\|\|u\|\|_h$	$\|\|v\|\|_h$	$\|\|p\|\|_h$
20×10	1.62	0.896	8.75
20×20	1.29	0.515	10.9
20×30	1.11	0.418	14.5
20×40	1.02	0.363	18.3
20×60	0.920	0.300	26.8
20×80	0.874	0.261	37.0
20×120	0.837	0.210	63.5
20×160	0.825	0.175	98.9
20×240	0.817	0.129	198.0
20×320	0.815	0.101	335.9
20×480	0.814	0.069	728.9
20×640	0.814	0.053	1278.8

Table 10: Numerical results for the periodic case, where v and p are specified at the boundary.

Finally, the proof of the Theorem 2.1 indicates that boundary data with discontinuities in its derivatives, may cause a jump in the solution for the pressure. This is based on the fact that it appears an extra factor ω in the pressure solution, and we can therefore expect the regularity to decrease one step. To demonstrate this, we solve the same problem as in the previous example, but with the boundary data

$$w_0(y) = v_0(y) = \begin{cases} (y-\pi)\left(y-\frac{\pi}{2}\right)\left(y-\frac{3\pi}{2}\right)\,, & y \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)\,, \\ 0\,, & y < \frac{\pi}{2} \text{ or } \frac{3\pi}{2} < y\,, \end{cases} \tag{6.8}$$

and $u_1(y) = v_1(y) = 0$.

In Figure 2 we have plotted the solution for the pressure at $x = 0$, for $M = 160$. The computations are done with the boundary conditions (4.2), and we can see that there are discontinuities in the solution for the pressure at $y = \pi/2$ and $y = 3\pi/2$.

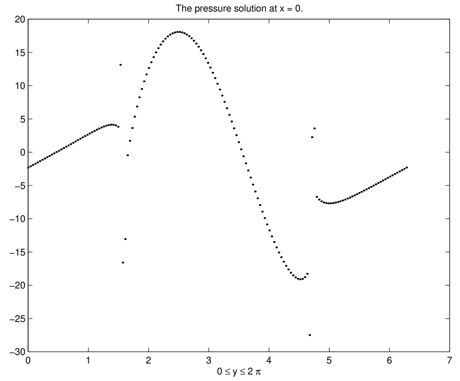

Figure 2: The pressure solution at the boundary $x = 0$, for grid sizes $N \times M = 10 \times 160$, with the boundary conditions (4.2).

7 Conclusions

We have showed how to formulate boundary conditions for the steady state Stokes equations with periodic solutions, so that the boundary value problem becomes non-singular. An analogous analysis for a second order finite difference approximation on a staggered grid have also been performed, and by applying the proposed boundary conditions we were able to derive the same type of the estimates of the solution for the discrete case as to the continuous one. The estimates of the solution are derived for the case where both the velocity components are prescribed at the boundary, and moreover for the case where the pressure is specified instead of one of the velocity components. We have also showed that the same estimates hold when the continuity condition is substituted by the Poisson equation for the pressure if we add the divergence condition as a boundary condition. Numerical experiments have been performed to demonstrate that the estimates hold also for the non-periodic case.

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