A Complete Axiomatisation for Timed Automata

Huimin Lin
Laboratory for Computer Science
Institute of Software, CAS
Email: lhm@ios.ac.cn

Wang Yi
Department of Computer Systems
Uppsala University
Email: yi@csd.uu.se

Abstract

A proof system of timed bisimulation equivalence for timed automata is presented, based on a CCS-style regular language for describing timed automata. It consists of the standard monoid laws for bisimulation and a set of inference rules. The judgments of the proof system are *conditional equations* of the form $\phi \rhd t = u$ where ϕ is a clock constraint and t, u are terms denoting timed automata. The proof of the completeness result relies on the notion of *symbolic timed bisimulation*, adapted from the work on value–passing processes.

1 Introduction

The last decade has seen a growing interest in extending various concurrency theories with timing constructs so that real-time aspects of concurrent systems can be modeled. Among them timed automata [AD94] has stood out as a fundamental model for real-timed systems.

A timed automaton is a finite automaton extended with a finite set of real-valued *clock variables*. A node of a timed automata is associated with an *invariant* constraint on the clock variables, while an edge is decorated with a clock constraint, an action label, and a subset of clocks to be reset after the transition. At each node a timed automaton may perform two kinds of transitions: it may let time pass for any amount (a delay transition), as long as the invariant is satisfied, or choose an edge whose constraint is met, make the move, reset the relevant clocks to zero, and arrive at the target node (an action transition). Although a timed automaton has only finite number of nodes, due to (real-valued) delay transitions it typically exhibits infinite-state behaviour. Two timed automata are *timed bisimilar* if they can match each other's action transitions as well as delay transitions, and their residuals remain timed bisimilar. The explicit presence of clock variables and resetting, features that mainly associated with the so-called "imperative languages", distinguishes timed automata from process calculi such as CCS, CSP and their timed extensions which are "applicative" in nature and therefore more amenable to axiomatisation. By now most theoretical aspects of timed automata have been well studied, but they still lack a satisfactory algebraic theory.

In this paper we shall develop a complete axiomatisation for timed automata, in the form of an inference system, in which the equalities between pairs of timed automata that are timed bisimilar can be derived. To this end we first propose a language, in CCS style, equipping it with a symbolic transitional semantics in such a way that each term in the language denotes a timed automaton. The language has a conditional construct $\phi{\rightarrow}t$, read "if ϕ then t", an action prefixing $a(\mathbf{x}).t$, meaning "perform the action a, reset the clocks in $\mathbf{x}$ to zero, then behave like t", and a recursion $\mathbf{fix}Xt$ which allows infinite behaviours to be described. The proof system consists of a set of inference rules and the standard monoid laws for bisimulation. Roughly speaking the monoid laws characterize bisimulation, while the inference rules deal with specific constructs in the language. The judgments of the inference system are of the form

$$\phi \rhd t = u$$

where ϕ is a time constraint and t, u are terms. Intuitively it means: t and u are timed bisimilar over clock evaluations satisfying ϕ. A typical inference rule takes the form:

$$\text{GUARD}\quad \frac{\phi \wedge \psi \rhd t = u \qquad \phi \wedge \neg\psi \rhd \mathbf{0} = u}{\phi \rhd (\psi{\rightarrow}t) = u}$$

It performs a case analysis on the constraint ψ: $\psi{\rightarrow}t$ behaves like t when ψ is true, and like the inactive process $\mathbf{0}$ otherwise. Note that the guarding constraint ψ of $\psi{\rightarrow}t$ in the conclusion is *part of the object language* describing timed automata, while in the premise it is shifted to the condition part of the judgment in our *meta language* for reasoning about timed automata.

A crucial rule, as might be expected, is the one for action prefixing:

$$\text{ACTION}\quad \frac{\phi{\downarrow_{\mathbf{x}}}{\Uparrow} \rhd t = u}{\phi \rhd a(\mathbf{x}).t = a(\mathbf{x}).u}$$

Here $\downarrow_{\mathbf{x}}$ and $\Uparrow$ are postfixing operations on clock constraints. $\phi{\downarrow_{\mathbf{x}}}{\Uparrow}$ is a clock constraint obtained from ϕ by first setting the clocks in $\mathbf{x}$ to zero (operator $\downarrow_{\mathbf{x}}$), then removing up-bounds on all clocks of ϕ (operator $\Uparrow$). Readers familiar with Hoare Logic may notice some similarity between this rule and the rule dealing with assignment there:

$$\{P[e/x]\}\ x := e\ \{P\}$$

But here the operator $\downarrow_{\mathbf{x}}$ is slightly more complicated than substitution with zero, because clocks are required to increase uniformly. Also we need $\Uparrow$ to allow time to pass indefinitely.

A standard way to reasoning with recursion is to use, apart from the usual rule for folding/unfolding recursions, the following *unique fixpoint induction*:

$$\text{UFI}\quad \frac{t = u[t/X]}{t = \mathbf{fix}Xu}\quad X \text{ guarded in } u$$

This rule was adopted in [Mil84] for a complete axiomatisation of bisimulation equivalence for regular pure-CCS. Here we use it in a quite different context: terms in our setting normally contain clock variables, namely they are *open terms*. In spit of this, it turns out that this rule is still sound and sufficient for a complete axiomatisation of regular behaviour, though the proof is slightly more complicated than in the pure calculi.

The completeness proof relies on the introduction of the notion of *symbolic timed bisimulation*, $t \sim^{\phi} u$, which captures timed bisimulation in the following sense: $t \sim^{\phi} u$ if and only if $t\rho$ and $u\rho$ are timed bisimilar for any clock evaluation ρ satisfying ϕ. Following [Mil84], to show that the inference system is complete, that is $t \sim^{\phi} u$ implies $\vdash \phi \rhd t = u$, we first transform t and u into *standard equation sets* which are the syntactical representations of timed automata. We then construct a product equation set out of the two and prove that t and u both satisfy this new equation set, by exploiting the assumption that t and u are symbolically timed bisimilar. Due to the presence of clock variables the notion of satisfiability is parameterised on a set of clock constraints. Finally we show that, using UFI, if two terms satisfy the same set of standard equations then they are provably equal.

The result of this paper fills a gap in the theory of timed automata. It demonstrates that bisimulation equivalence of timed automata are as mathematically tractable as those of standard process algebras.

The rest of the paper is organised as follows: In the next section we first recall the definition of timed automata, then propose a language to describe them. Section 3 introduces symbolic timed bisimulation. The inference system is presented and its soundness discusses in Section 4. Section 5 is devoted to proving the completeness of the proof system. The paper concludes with Section 6 where related work is also briefly discussed.

2 A Language for Timed Automata

2.1 Timed Automata

A timed automaton is a standard finite-state automaton extended with a finite collection of real-valued clocks. In a timed automaton, each node is associated with an *invariant*, while a transition is labelled with a *guard* (a constraint on clocks), a *synchronisation action*, and a *clock reset* (a subset of clocks to be reset). Intuitively, a timed automaton starts execution with all clocks initialized to zero. The automaton can stay at a node while the invariant of the node is satisfied, with all clocks increasing at the same rate. A transition can be taken if the clocks fulfill the guard. By taking the transition, all clocks in the clock reset are set to zero, while the others keep their values. Semantically, a state of an automaton is a pair of a control node and a *clock valuation*, i.e. the current setting of the clocks. Transitions in the semantic interpretation are either labelled with a synchronisation action (if it is an instantaneous switch from the current node to another) or with a positive real number i.e. a time delay (if the automaton stays within a node letting time pass).

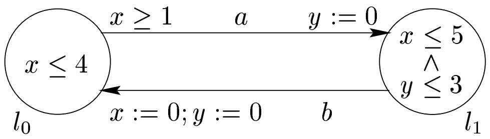


Figure 1: A Timed Automaton.

Consider the timed automaton of Figure 1. It has two control nodes l_0 and l_1 and two clocks x and y. A *state* of the automaton is of the form $(l, < s, t >)$, where l is a control node and s and t are non–negative reals giving the values of x and y. Assuming that the automaton starts to operate in the state $(l_0, < 0, 0 >)$, it may stay in node l_0 for any amount of time, as long as the invariant $x \leq 4$ of l_0 is satisfied. During this time the values of x and y increase uniformly, at the same rate. Thus from the initial state, all states of the form $(l_0, < t, t >)$ with $0 \leq t \leq 4$ are reachable, but only at the states $(l_0, < t, t >)$, where $t \geq 1$, the edge from l_0 to l_1 is enabled. When following the edge from l_0 to l_1 the action a is performed to synchronize with the environment and the clock y is reset to 0 leading to states of the form $(l_1, < t, 0 >)$ where $t \geq 1$.

For the formal definition, we assume a finite set $\mathcal{A}$ for synchronization actions and a finite set $\mathcal{C}$ for real-valued clock variables. We use a, b etc. to range over $\mathcal{A}$ and x, y etc. to range over $\mathcal{C}$. We use $\mathcal{B}(C)$, ranged over by ϕ, ψ etc., to denote the set of conjunctive formulas of atomic constraints in the form: $x_i \bowtie m$ or $x_i - x_j \bowtie n$, where $x_i, x_j \in \mathcal{C}$, $\bowtie \in \{\leq, <, \geq, >\}$ and m, n are natural numbers. The elements of $\mathcal{B}(C)$ are called *clock constraints*.

Definition 2.1 *A timed automaton over actions $\mathcal{A}$ and clocks $\mathcal{C}$ is a tuple $\langle N, l_0, E \rangle$ where*

- *N is a finite set of nodes,*
- *$l_0 \in N$ is the initial node,*
- *$E \subseteq N \times \mathcal{B}(C) \times \mathcal{A} \times 2^C \times N$ is the set of edges.*

When $\langle l, g, a, r, l' \rangle \in E$, we write $l \xrightarrow{g,a,r} l'$. □

We shall present the operational semantics for timed automata in terms of a process algebraic language in which each term denotes an automaton.

2.2 The Language

We preassume a set of process variables, ranged over by $X,\ Y,\ Z,\ \ldots$. The language for timed automata over $\mathcal{C}$ can be given by the following BNF grammar:

$$
\begin{array}{rcl}
s & ::= & \{\phi\}t \\
t & ::= & \mathbf{0} \quad | \quad \phi \rightarrow t \quad | \quad a(\mathbf{x}).s \quad | \quad t + t \quad | \quad X \quad | \quad \mathbf{fix} X t
\end{array}
$$

$$\text{DELAY}\quad \frac{}{t\rho \xrightarrow{d} t(\rho+d)}\ \rho+d \models Inv(t) \qquad \text{CHOICE}\quad \frac{t\rho \xrightarrow{a} t'\rho'}{(t+u)\rho \xrightarrow{a} t'\rho'}$$

$$\text{ACTION}\quad \frac{}{(a(\mathbf{x}).t)\rho \xrightarrow{a} t\rho\{\mathbf{x}:=0\}} \qquad \text{GUARD}\quad \frac{t\rho \xrightarrow{a} t'\rho'}{(\phi \to t)\rho \xrightarrow{a} t'\rho'}\ \rho \models \phi$$

$$\text{REC}\quad \frac{(t[\mathbf{fix}Xt/X])\rho \xrightarrow{a} t'\rho'}{(\mathbf{fix}Xt)\rho \xrightarrow{a} t'\rho'} \qquad \text{INV}\quad \frac{t\rho \xrightarrow{a} t'\rho'}{(\{\phi\}t)\rho \xrightarrow{a} t'\rho'}\ \rho \models \phi$$

Figure 2: Standard Transitional Semantics

$\mathbf{0}$ is the inactive process which can do nothing, except for allowing time to pass. $\phi \to t$, read "if ϕ then t", is the usual (one-armed) conditional construct. $a(\mathbf{x}).t$ is action prefixing. $+$ is nondeterministic choice. The $\{\phi\}t$ construct introduces an invariant. Note that invariants can only occur at places which correspond to locations in timed automata. For instance, strings having the forms $\phi \to \{\psi\}t$, $\{\phi\}t + \{\psi\}u$ or $\{\phi\}\{\psi\}t$ are *not* terms of the language, while $\{\phi\}(t+u)$ and $\phi \to a(\mathbf{x}).\{\psi\}t$ are allowed.

A recursion $\mathbf{fix}Xt$ binds X in t. This is the only binding operator in this language. It induces the notions of bound and free process variables as usual. Terms not containing free process variables are *closed*. A recursion $\mathbf{fix}Xt$ is *guarded* if every occurrence of X in t is within the scope of an action prefixing.

The set of clock variables used in a term t is denoted $\mathcal{C}(t)$.

A *clock valuation* is a function from $\mathcal{C}$ to $\mathbf{R}^{\geq 0}$ (non-negative real numbers), and we use ρ to range over clock valuations. The notations $\rho\{\mathbf{x}:=0\}$ and $\rho+d$ are defined thus

$$\rho\{\mathbf{x}:=0\}(y) = \begin{cases} 0 & \text{if } y \in \mathbf{x} \\ \rho(y) & \text{otherwise} \end{cases}$$
$$(\rho+d)(x) = \rho(x) + d \quad \text{for all } x$$

To give a transitional semantics to our language, we first assign each term t an invariant constraint $Inv(t)$ by letting

$$Inv(t) = \begin{cases} \phi & \text{if } t \text{ has the form } \{\phi\}s \\ \mathtt{tt} & \text{otherwise} \end{cases}$$

We shall require that all invariants are downward closed:

$$\text{For all } d \in \mathbf{R}^{\geq 0},\ \rho+d \models \phi \text{ implies } \rho \models \phi$$

Given a clock valuation $\rho : \mathcal{C} \to \mathbf{R}^{\geq 0}$, a term can be interpreted according to the rules in Figure 2, where the symmetric rule for $+$ has been omitted. We call $t\rho$ a *process*, where t is a term and ρ a valuation; we use $p,\ q,\ldots$ to range over the set of processes. We also write μ for either an action or a delay (a real number). The transitional semantics uses two types of transition relations: action transition $\xrightarrow{a}$ and delay transition $\xrightarrow{d}$. Note that a process can have only a finite number of

$$\text{ACTION}\quad \frac{}{a(\mathbf{x}).t \xrightarrow{\mathtt{tt},a,\mathbf{x}} t} \qquad \text{CHOICE}\quad \frac{t \xrightarrow{b,a,\mathbf{x}} t'}{t+u \xrightarrow{b,a,\mathbf{x}} t'}$$

$$\text{GUARD}\quad \frac{t \xrightarrow{\psi,a,\mathbf{x}} t'}{\phi{\to}t \xrightarrow{\phi\wedge\psi,a,\mathbf{x}} t'} \qquad \text{REC}\quad \frac{t[\mathbf{fix}Xt/X] \xrightarrow{b,a,\mathbf{x}} t'}{\mathbf{fix}Xt \xrightarrow{b,a,\mathbf{x}} t'}$$

$$\text{INV}\quad \frac{t \xrightarrow{\psi,a,\mathbf{x}} t'}{\{\phi\}t \xrightarrow{\psi,a,\mathbf{x}} t'}$$

Figure 3: Symbolic Transitional Semantics

action transitions, but it may have infinite many delay transitions. It is the later that makes timed processes infinite branching (and infinite states).

Definition 2.2 A symmetric relation R over processes is a timed bisimulation if $(p,q) \in R$ implies

whenever $p \xrightarrow{\mu} p'$ then $q \xrightarrow{\mu} q'$ for some q' with $(p',q') \in R$.

We write $p \sim q$ if $(p,q) \in R$ for some timed bisimulation R. $\square$

The symbolic transitional semantics of this language is listed in Figure 3. Again the symmetric rule for $+$ has been omitted. Note that invariants are simply forgotten in the symbolic transitional semantics. This reflects our intention that symbolic transitions correspond to edges in timed automata, while invariants reside in nodes. Note also that there is no rule to deduce "delay" transitions in the symbolic semantics. As noted above, delay transitions are the source of infinity in the semantics of timed automata. The purpose of the symbolic transitional semantics, and the symbolic timed bisimulation built on it, is to avoid such infinity. Thus "delays" do not appear explicitely in the symbolic semantics. Instead they will be implicitly encoded in the notion of "upward-closeness" used to define symbolic bisimulation in the next section.

According to the symbolic semantics, each guarded closed term of the language gives rise to a timed automaton; On the other hand, it is not difficult to see that every timed automaton can be generated from a guarded closed term in the language. In the sequel we will use the phrases "timed automata" and "terms" interchangeably. The two versions of transitional semantics can be related as follows (note again only action transitions are related):

Lemma 2.3 *1. If $t \xrightarrow{\phi,a,\mathbf{x}} t'$ then $t\rho \xrightarrow{a} t'\rho\{\mathbf{x}:=0\}$ for any $\rho \models \phi \wedge Inv(t)$.*

2. If $t\rho \xrightarrow{a} t'\rho'$ then there exist ϕ, $\mathbf{x}$ such that $\rho \models \phi \wedge Inv(t)$, $\rho' = \rho\{\mathbf{x}:=0\}$ and $t \xrightarrow{\phi,a,\mathbf{x}} t'$.

Proof: Both are proved by transition induction.
1. Assuming $t \xrightarrow{\phi,a,\mathbf{x}} t'$ and $\rho \models \phi \wedge Inv(t)$, we show $t\rho \xrightarrow{a} t'\rho\{\mathbf{x}:=0\}$.

- $\phi \equiv \mathtt{tt}$ and $t \equiv a(\mathbf{x}).t' \xrightarrow{\mathtt{tt},a,\mathbf{x}} t'$. Then $(a(\mathbf{x}).t')\rho \xrightarrow{a} t'\rho\{\mathbf{x} := 0\}$ by ACTION and $\rho \models \phi$.

- $\phi \equiv \phi' \wedge \psi$ and $t \equiv \phi' \rightarrow t'' \xrightarrow{\phi' \wedge \psi,a,\mathbf{x}} t'$ is because $t'' \xrightarrow{\psi,a,\mathbf{x}} t'$. Then $Inv(t) = \mathtt{tt}$. Since $\rho \models \psi \wedge Inv(t)$, by induction we get $t''\rho \xrightarrow{a} t'\rho\{\mathbf{x} := 0\}$. Since $\rho \models \phi'$, by GUARD, $t\rho \xrightarrow{a} t'\rho\{\mathbf{x} := 0\}$.

- $Inv(t) = \phi'$ and $t \equiv \{\phi'\}t'' \xrightarrow{\psi,a,\mathbf{x}} t'$ is because $t'' \xrightarrow{\psi,a,\mathbf{x}} t'$. Then $Inv(t'') = \mathtt{tt}$, and $\rho \models \psi \wedge Inv(t'')$. By induction, we get $t''\rho \xrightarrow{a} t'\rho\{\mathbf{x} := 0\}$. Since $\rho \models \phi'$, by INV, $t\rho \xrightarrow{a} t'\rho\{\mathbf{x} := 0\}$.

- The other cases are similar.

2. Assuming $t\rho \xrightarrow{a} t'\rho'$, we show $t \xrightarrow{\phi,a,\mathbf{x}} t'$ for some ϕ, $\mathbf{x}$ such that $\rho \models \phi \wedge Inv(t)$ and $\rho' = \rho\{\mathbf{x} := 0\}$.

- $t \equiv a(\mathbf{x}).t'$ and $t\rho \xrightarrow{a} t'\rho\{\mathbf{x} := 0\}$. Then $Inv(t) = \mathtt{tt}$. By ACTION we have $t \xrightarrow{\mathtt{tt},a,\mathbf{x}} t'$ and $\rho \models \mathtt{tt} \wedge Inv(t)$.

- $t \equiv \psi \rightarrow t''$ and $t\rho \xrightarrow{a} t'\rho'$ is because $\rho \models \psi$ and $t''\rho \xrightarrow{a} t'\rho'$. Then $Inv(t) = \mathtt{tt}$. By induction we get $t'' \xrightarrow{\phi,a,\mathbf{x}} t'$ for some ϕ, $\mathbf{x}$ such that $\rho \models \phi \wedge Inv(t'')$ and $\rho' = \rho\{\mathbf{x} := 0\}$. By GUARD, $t \xrightarrow{\phi \wedge \psi,a,\mathbf{x}} t'$ and $\rho \models \phi \wedge \psi \wedge Inv(t)$.

- $t \equiv \{\phi'\}t''$ and $t\rho \xrightarrow{a} t'\rho'$ is because $\rho \models \phi'$ and $t''\rho \xrightarrow{a} t'\rho'$. Then $Inv(t) = \phi'$ and $Inv(t'') = \mathtt{tt}$. By induction we get $t'' \xrightarrow{\phi,a,\mathbf{x}} t'$ for some ϕ, $\mathbf{x}$ such that $\rho \models \phi$ and $\rho' = \rho\{\mathbf{x} := 0\}$. By INV, $t \xrightarrow{\phi,a,\mathbf{x}} t'$. Also $\rho \models \phi \wedge Inv(t)$.

- The other cases are similar.

$\square$

3 Symbolic Timed Bisimulation

In this section we shall define a symbolic version of timed bisimulation. To simplify the presentation we fix two timed automata. To avoid clock variables of one automaton being reset by the other, we assume the sets of clocks of the two timed automata under consideration are disjoint, and write C for the union of the two clock sets [1]. Let N be the largest natural number occurring in the constraints of the two automata. An atomic constraint over C with ceiling N has one of the three forms: $x > N$, $x \bowtie m$ or $x - y \bowtie n$ where $x, y \in C, \bowtie \in \{\leq, <, \geq, >\}$ and $m, n \leq N$ are natural numbers.

[1] This does not impose any restriction on our results, because we can always rename clock variables of an automaton without affecting its behaviour.

In the following, "atomic constraint" always means "atomic constraint over C with ceiling N". Note that given two timed automata there are only finite number of such atomic constraints. We shall use c to range over atomic constraints.

A constraint, or *zone*, is a boolean combination of atomic constraints. A constraint ϕ is consistent if there is some ρ such that $\rho \models \phi$. Let ϕ and ψ be two constraints. We write $\phi \models \psi$ to mean $\rho \models \phi$ implies $\rho \models \psi$ for any ρ. Note that the relation $\models$ is decidable.

A *region constraint*, or *region* for short, over n clock variables $x_1, \ldots, x_n$ is a consistent constraint containing the following atomic conjuncts:

- For each $i \in \{1, \ldots, n\}$ either $x_i = m_i$ or $m_i < x_i < m_i + 1$ or $x_i > N$;
- For each pair of $i, j \in \{1, \ldots, n\}$, $i \neq j$, such that both x_i and x_j are not greater than N, either $x_i - m_i = x_j - m_j$ or $x_i - m_i < x_j - m_j$ or $x_j - m_j < x_i - m_i$.

where the m_i in $x_i - m_i$ of the second clause refers to the m_i related to x_i in the first clause. In words, m_i is the integral part of x_i and $x_i - m_i$ its fractional part.

Given a finite set of clock variables C and a ceiling N, the set of region constraints over C is finite and is denoted $\mathcal{RC}_N^C$. In the sequel, we will omit the sub- and superscripts when they can be supplied by the context.

Fact 1 *Let ϕ be a region constraint. If $\rho \models \phi$ and $\rho' \models \phi$ then*

- *For all $i \in \{1, \ldots, n\}$, if $\rho(x_i) \leq N$ then $\lfloor \rho(x_i) \rfloor = \lfloor \rho'(x_i) \rfloor$.*
- *For any $i, j \in \{1, \ldots, n\}$, $i \neq j$,*
  - *$\{\rho(x_i)\} = \{\rho(x_j)\}$ iff $\{\rho'(x_i)\} = \{\rho'(x_j)\}$ and*
  - *$\{\rho(x_i)\} < \{\rho(x_j)\}$ iff $\{\rho'(x_i)\} < \{\rho'(x_j)\}$.*

where $\lfloor x \rfloor$ and $\{x\}$ are the integral and fractional parts of x, respectively.

That is, two valuations satisfying the same region constraint must agree on their integral parts as well as on the ordering of their fractional parts.

Lemma 3.1 *Suppose that ϕ is a region constraint and ψ a zone. Then either $\phi \Rightarrow \psi$ or $\phi \Rightarrow \neg\psi$.*

Proof: We first transform ψ into disjunctive normal form: $\psi = \bigvee_i \bigwedge_j e_{ij}$ where each e_{ij} is an atomic constraint. Now $\psi \wedge \phi = \bigvee_i \bigwedge_j (e_{ij} \wedge \phi)$. It is easy to see, by examining the possible forms of e_{ij}, that each $e_{ij} \wedge \phi$ is either equal to ϕ or false. Hence $\psi \wedge \phi$ is either equal to ϕ or false. In the former case we have $\phi \Rightarrow \psi$, and in the later case we get $\phi \Rightarrow \neg\psi$. $\square$

According to this lemma, a region is either entirely contained in a zone, or is completely outside a zone. In other words, regions are the finest polyhedra that can be described by our constraint language.

The notion of a region constraint enjoy an important property: processes in the same region behave uniformly with respect to timed bisimulation ([Cer92]):

Fact 2 *Let t, u be two terms with disjoint sets of clock variables and ϕ a region constraint over the union of the two clock sets. Suppose that both ρ and ρ' satisfy ϕ. Then $t\rho \sim u\rho$ iff $t\rho' \sim u\rho'$.*

A *canonical* constraint is a disjunction of regions. Given a constraint we can first transform it into disjunctive normal form, then decompose each disjunct into a disjoint set of regions. Both steps can be effectively implemented. As a corollary to Lemma 3.1, if we write $\mathcal{RC}(\phi)$ for the set of regions contained in the zone ϕ, then $\bigvee \mathcal{RC}(\phi) = \phi$, i.e. $\bigvee \mathcal{RC}(\phi)$ is the canonical form of ϕ.

We will need two (postfixing) operators to deal with resetting. The first one is $\downarrow_{\mathbf{x}}$ where $\mathbf{x} \subseteq C \subseteq \mathcal{C}$. We first define it on regions, then generalise it to zones. With abuse of notation, we will write $c \in \phi$ to mean c is a conjunct of ϕ.

For a region ϕ,

$$\begin{aligned}\phi{\downarrow_{\mathbf{x}}} = \phi \downarrow'_{\mathbf{x}} \;\; & \wedge \;\; \bigwedge\{\, x_i = 0 \mid x_i \in \mathbf{x}\,\} \wedge \bigwedge\{\, x_i = x_j \mid x_i, x_j \in \mathbf{x}\,\} \\ & \wedge \;\; \bigwedge\{\, x_i = x_j - m \mid x_i \in \mathbf{x}, x_j \notin \mathbf{x}, x_j = m \in \phi\,\} \\ & \wedge \;\; \bigwedge\{\, x_i < x_j - m \mid x_i \in \mathbf{x}, x_j \notin \mathbf{x}, x_j > m \in \phi\,\}\end{aligned}$$

and $\downarrow'_{\mathbf{x}}$ is defined by

$$\begin{array}{ll} \mathsf{tt} \downarrow'_{\mathbf{x}} = \mathsf{tt} & \\ (c \wedge \phi) \downarrow'_{\mathbf{x}} = \phi \downarrow'_{\mathbf{x}} & \text{if } \mathbf{x} \cap \mathit{fv}(c) \neq \emptyset \\ (c \wedge \phi) \downarrow'_{\mathbf{x}} = c \wedge \phi \downarrow'_{\mathbf{x}} & \text{if } \mathbf{x} \cap \mathit{fv}(c) = \emptyset \end{array}$$

where $\mathit{fv}(c)$ is the set of clock variables appearing in (atomic constraint) c.

For a canonical constraint $\bigvee_i \phi_i$ with each ϕ_i a region, $(\bigvee_i \phi_i){\downarrow_{\mathbf{x}}} = \bigvee_i(\phi_i{\downarrow_{\mathbf{x}}})$. For an arbitrary constraint ϕ, $\phi{\downarrow_{\mathbf{x}}}$ is understood as the result of applying $\downarrow_{\mathbf{x}}$ to the canonical form of ϕ.

Lemma 3.2 *1. $\rho \models \phi$ implies $\rho\{\mathbf{x} := 0\} \models \phi{\downarrow_{\mathbf{x}}}$.*

2. If ϕ is a region constraint then so is $\phi{\downarrow_{\mathbf{x}}}$.

Proof:

1. Let $\bigvee_i \phi_i$ be the canonical form of ϕ. Since $\rho \models \phi$, $\rho \models \phi_i$ for some i. Now ϕ_i is a region constraint, so $\rho\{\mathbf{x} := 0\} \models \phi_i{\downarrow_{\mathbf{x}}}$ follows immediately from the definition of $\downarrow_{\mathbf{x}}$.

2. Immediately from the definition of $\downarrow_{\mathbf{x}}$.

$\square$

The second operator $\Uparrow$ is defined similarly. We first define it on regions:

$$\phi\Uparrow = \phi\Uparrow' \wedge \bigwedge_{i \leq j} e_{ij}(\phi)$$

where $\Uparrow'$ is defined by

$$
\begin{array}{l}
(x < m \wedge \phi)\Uparrow' = \phi\Uparrow' \\
(x = m \wedge \phi)\Uparrow' = m \leq x \wedge \phi\Uparrow' \\
(x_i - m_i < x_j - m_j \wedge \phi)\Uparrow' = x_i - m_i < x_j - m_j \wedge x_j - m_j < x_i - m_i + 1 \wedge \phi\Uparrow' \\
(c \wedge \phi)\Uparrow' = c \wedge \phi\Uparrow' \quad \text{for other atomic constraint c} \\
\mathsf{tt}\Uparrow' = \mathsf{tt}
\end{array}
$$

and

$$e_{ij}(\phi) = \begin{cases} x_i - m_i = x_j - m_j & \textit{if } x_i = m_i, x_j = m_j \in \phi \\ \mathsf{tt} & \text{otherwise} \end{cases}$$

For an arbitrary constraint ϕ, $\phi\Uparrow$ is understood as the result of applying $\Uparrow$ to each disjunct of the canonical form of ϕ. ϕ is $\Uparrow$-closed if and only if $\phi\Uparrow = \phi$.

Lemma 3.3 *1. $\rho \models \phi$ implies $\rho \models \phi\Uparrow$.*

2. $\phi\Uparrow$ is $\Uparrow$-closed.

3. If ϕ is $\Uparrow$-closed then $\rho \models \phi$ implies $\rho + d \models \phi$ for all $d \in \mathbf{R}^{\geq 0}$.

Proof:

1. Immediately from the definition of $\Uparrow$.

2. It is sufficient to consider the case when ϕ is a region constraint. We check if each conjunct introduced by the $\Uparrow$ operator is preserved by the a further application of it. The only interesting case is the third clause in the definition of $\Uparrow'$. Direct calculation gives:

$$
\begin{array}{rl}
& (x_i - m_i < x_j - m_j \wedge x_j - m_j < x_i - m_i + 1)\Uparrow' \\
= & (x_i - m_i < x_j - m_j \wedge x_j - m_j < x_i - m_i + 1) \wedge \\
& (x_j - m_j < x_i - m_i + 1 \wedge x_i - m_i + 1 < x_j - m_j + 1) \\
= & x_i - m_i < x_j - m_j \wedge x_j - m_j < x_i - m_i + 1
\end{array}
$$

3. Let $\bigvee_i \phi_i$ be the canonical form of ϕ. We have $\phi\Uparrow = \bigvee_i \phi_i\Uparrow = \phi$. Since $\rho \models \phi$, $\rho \models \phi_i\Uparrow$ for some i. It is straightforward to check that $\rho + d \models \psi$ for each conjunct ψ of $\phi_i\Uparrow$.

□

Symbolic bisimulation will be defined as a family of binary relations indexed by clock constraints. Following [Cer92] we use constraints over the union of the (disjoint) clock sets of two timed automata as indices. The reason for this is as follows: the definition of timed bisimulation requires two processes to match action transitions as well as delay transitions, which amounts to requiring them to match action transitions *while their clocks progress at the same rate.* In the definition of symbolic bisimulation indexing constraints are subject to the $\Uparrow$ operation which introduces equalities between clock variables (the e_{ij} component in the above definition), which guarantees the "same rate" requirement when such constraints are over the union of the two clock sets.

Given a constraint ϕ, a finite set of constraints Φ is called a *ϕ-partition* if $\bigvee \Phi = \phi$. A ϕ-partition Φ is called *finer* than another such partition Ψ if Φ can be obtained from Ψ by decomposing some of its elements. By the corollary to Lemma 3.1, $\mathcal{RC}(\phi)$ is a ϕ-partition, and is the finest such partition. In particular, if ϕ is a region constraint then $\{\phi\}$ is the only partition of ϕ.

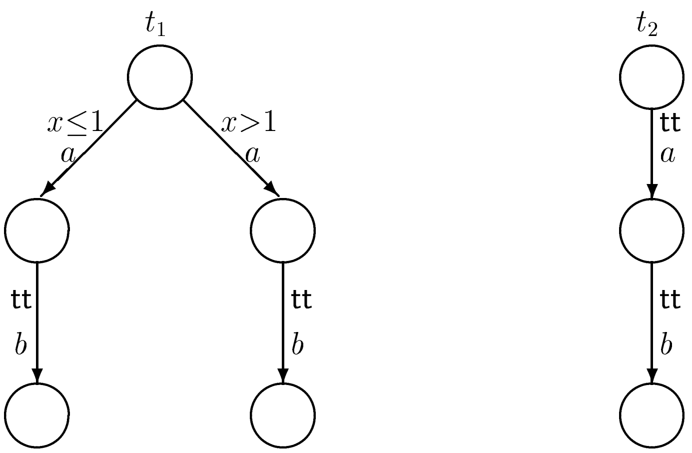
Definition 3.4 A constraint indexed family of symmetric relations over terms $\mathbf{S} = \{ S^\phi \mid \phi \text{ is } \Uparrow\!-\text{closed} \}$ is a symbolic timed bisimulation if $(t, u) \in S^\phi$ implies

1. $\phi \models Inv(t) \Leftrightarrow Inv(u)$ and
2. whenever $t \xrightarrow{\psi,a,\mathbf{x}} t'$ then there is a $Inv(t) \wedge \phi \wedge \psi$-partition Φ such that for each $\phi' \in \Phi$ there is $u \xrightarrow{\psi',a,\mathbf{y}} u'$ for some ψ', $\mathbf{y}$ and u' such that $\phi' \Rightarrow \psi'$ and $(t', u') \in S^{\phi' \downarrow_{\mathbf{xy}} \Uparrow}$.

We write $t \sim^\phi u$ if $(t, u) \in S^\phi \in \mathbf{S}$ for some symbolic bisimulation $\mathbf{S}$. □

Note that there is no clause for delay transitions in the definition, because delays are encoded in the $\Uparrow$-closeness property of the indexing constraints.

The use of a partition when matching a symbolic transition is essential. Without it we will not be able to characterise timed bisimulation using symbolic transitions. For example, consider the two timed automata t_1 and t_2 below (we have omitted the empty resets). They are apparently timed bisimilar. But the symbolic transition $t_2 \xrightarrow{\mathtt{tt},a,\{\}}$ can not be entirely matched by either of the two symbolic transitions from t_1. We must use a partition, say $\{x \leq 1,\ x > 1\}$: t_1 can match the symbolic transition from t_2 using its left branch over the constraint $x \leq 1$, and the right branch over $x > 1$.



Symbolic timed bisimulation captures $\sim$ in the following sense:

Theorem 3.5 *For $\Uparrow\!-$closed ϕ, $t \sim^\phi u$ iff $t\rho \sim u\rho$ for any $\rho \models \phi \wedge Inv(t) \wedge Inv(u)$.*

Proof: ($\Longrightarrow$) Assume $(t, u) \in S^\phi \in \mathbf{S}$ for some symbolic bisimulation $\mathbf{S}$. Define

$$R = \{ (t\rho, u\rho) \mid \text{there exists some } \phi \text{ such that } \rho \models \phi \text{ and } (t, u) \in S^\phi \in \mathbf{S} \}$$

We show R is a timed bisimulation. Suppose $(t\rho, u\rho) \in R$, i.e. there is some ϕ such that $\rho \models \phi$ and $(t, u) \in S^\phi$. By the first claus in Definition 3.4, we have $\rho \models \mathit{Inv}(t)$ if and only if $\rho \models \mathit{Inv}(u)$.

- $t\rho \xrightarrow{a} t'\rho'$. By Lemma 2.3 there are ψ, $\mathbf{x}$ such that $\rho \models \psi \wedge \mathit{Inv}(t)$, $\rho' = \rho\{\mathbf{x} := 0\}$ and $t \xrightarrow{\psi,a,\mathbf{x}} t'$. So there is a $\phi \wedge \psi$-partition Φ with the properties specified in Definition 3.4. Since $\rho \models \phi \wedge \psi$, $\rho \models \phi'$ for some $\phi' \in \Phi$. Let $u \xrightarrow{\psi',a,\mathbf{y}} u'$ be the symbolic transition associated with this ϕ', as guaranteed by Definition 3.4. Then $\phi' \Rightarrow \psi'$ and $(t', u') \in S^{\phi'\downarrow_{\mathbf{xy}}\Uparrow}$. Since $\rho \models \psi' \wedge \mathit{Inv}(u)$, $u\rho \xrightarrow{a} u'\rho\{\mathbf{y} := 0\}$. By Lemma 3.2, $\rho\{\mathbf{xy} := 0\} \models \phi'\downarrow_{\mathbf{xy}}$. By Lemma 3.3, $\rho\{\mathbf{xy} := 0\} \models \phi'\downarrow_{\mathbf{xy}}\Uparrow$. Therefore $(t'\rho\{\mathbf{xy} := 0\}, u'\rho\{\mathbf{xy} := 0\}) \in R$. Since $t'\rho\{\mathbf{xy} := 0\} \equiv t'\rho\{\mathbf{x} := 0\}$ and $u'\rho\{\mathbf{xy} := 0\} \equiv u'\rho\{\mathbf{y} := 0\}$, this is the same as $(t'\rho\{\mathbf{x} := 0\}, u'\rho\{\mathbf{y} := 0\}) \in R$.

- $t\rho \xrightarrow{d} t(\rho + d)$. Since ϕ is $\Uparrow$-closed, $\rho + d \models \phi$. Then $\rho + d \models \mathit{Inv}(u)$ and hence $u\rho \xrightarrow{d} u(\rho + d)$. Therefore $(t(\rho + d), u(\rho + d)) \in R$.

($\Longleftarrow$) Assume $t\rho \sim u\rho$ for any $\rho \models \phi_0 \wedge \mathit{Inv}(t) \wedge \mathit{Inv}(u)$, *i.e.* $(t\rho, u\rho) \in R$ for some timed bisimulation R, we show $t \sim^{\phi_0} u$ as follows. For each $\Uparrow-$closed ϕ, define

$$S^\phi = \{ (t, u) \mid \forall \phi' \in \mathcal{RC}(\phi),\ (t\rho, u\rho) \in R \text{ for any } \rho \models \phi' \wedge \mathit{Inv}(t) \wedge \mathit{Inv}(u) \}$$

and let $\mathbf{S} = \{ S^\phi \mid \phi \text{ is } \Uparrow - \text{closed} \}$. Then $(t, u) \in S^{\phi_0}$. $\mathbf{S}$ is well-defined because of Fact 2. We show $\mathbf{S}$ is a symbolic bisimulation. Suppose $(t, u) \in S^\phi$. Consider any $\phi' \in \mathcal{RC}(\phi)$. There exists $\rho \models \phi' \wedge \mathit{Inv}(t) \wedge \mathit{Inv}(u)$ such that $(t\rho, u\rho) \in R$. Since ϕ' is a region it must be entirely contained in $\mathit{Inv}(t) \wedge \mathit{Inv}(u)$, i.e. $\phi' \models \mathit{Inv}(t) \Leftrightarrow \mathit{Inv}(u)$. Therefore $\phi \models \mathit{Inv}(t) \Leftrightarrow \mathit{Inv}(u)$. Now let $t \xrightarrow{\psi,a,\mathbf{x}} t'$. Define $\Phi' = \{ \phi' \mid \phi' \in \mathcal{RC}(\phi) \text{ and } \phi' \Rightarrow \psi \}$. Then Φ' is a $\phi \wedge \psi$-partition. For each $\phi' \in \Phi'$, there exists ρ s.t. $\rho \models \phi'$ with $(t\rho, u\rho) \in R$. By the definition of Φ', $\rho \models \psi$. By Lemma 2.3, $t\rho \xrightarrow{a} t'\rho\{\mathbf{x} := 0\}$. Since $(t\rho, u\rho) \in R$, $u\rho \xrightarrow{a} u'\rho'$ for some u' and ρ' with $(t'\rho\{\mathbf{x} := 0\}, u'\rho') \in R$. By Lemma 2.3 again, $u \xrightarrow{\psi',a',\mathbf{y}} u'$ for some ψ' and $\mathbf{y}$ with $\rho \models \psi'$ and $\rho' = \rho\{\mathbf{y} := 0\}$. Hence $(t\rho\{\mathbf{x} := 0\}, u\rho\{\mathbf{y} := 0\}) \in R$, which is the same as $(t\rho\{\mathbf{xy} := 0\}, u\rho\{\mathbf{xy} := 0\}) \in R$. Since $\rho \models \phi'$, by Lemma 3.2 we have $\rho\{\mathbf{xy} := 0\} \models \phi'\downarrow_{\mathbf{xy}}$. Since ϕ' is a region constraint, so is $\phi'\downarrow_{\mathbf{xy}}$ which is the only element of $\mathcal{RC}(\phi'\downarrow_{\mathbf{xy}})$. Therefore $(t', u') \in S^{\phi'\downarrow_{\mathbf{xy}}\Uparrow}$. □

4 The Proof System

The proposed proof system consists of a set of inference rules in Figure 4 and a set of equational axioms in Figure 5. The judgments of the inference system are *conditional equations* of the form

$$\phi \rhd t = u$$

$$\text{EQUIV} \quad \frac{}{t = t} \quad \frac{\phi \rhd t = u}{\phi \rhd u = t} \quad \frac{\phi \rhd t = u \quad \phi \rhd u = v}{\phi \rhd t = v}$$

$$\text{AXIOM} \quad \frac{}{t = u} \quad t = u \text{ an axiom instance}$$

$$\text{CHOICE} \quad \frac{\phi \rhd t = t'}{\phi \rhd t + u = t' + u}$$

$$\text{GUARD} \quad \frac{\phi \wedge \psi \rhd t = u \quad \phi \wedge \neg\psi \rhd \mathbf{0} = u}{\phi \rhd \psi \rightarrow t = u}$$

$$\text{INV} \quad \frac{\phi \wedge \psi \rhd t = u \quad \phi \wedge \neg\psi \rhd \{\mathsf{ff}\}\mathbf{0} = u}{\phi \rhd \{\psi\}t = u}$$

$$\text{ACTION} \quad \frac{\phi{\downarrow_{\mathbf{x}}}\Uparrow \rhd t = u}{\phi \rhd a(\mathbf{x}).t = a(\mathbf{x}).u}$$

$$\text{THINNING} \quad \frac{}{a(\mathbf{xy}).t = a(\mathbf{x}).t} \quad \mathbf{y} \cap \mathcal{C}(t) = \emptyset$$

$$\text{REC} \quad \frac{}{\mathbf{fix}Xt = t[\mathbf{fix}Xt/X]}$$

$$\text{UFI} \quad \frac{t = u[t/X]}{t = \mathbf{fix}Xu} \quad X \text{ guarded in } u$$

$$\text{UNG} \quad \frac{}{\mathbf{fix}X(X + t) = \mathbf{fix}Xt}$$

$$\text{PARTITION} \quad \frac{\phi_1 \rhd t = u \quad \phi_2 \rhd t = u}{\phi \rhd t = u} \quad \phi \models \phi_1 \vee \phi_2$$

$$\text{ABSURD} \quad \frac{}{\mathsf{ff} \rhd t = u}$$

Figure 4: The Inference Rules

where ϕ is a constraint and t, u are terms. Its intended meaning is "$t \sim^{\phi} u$", or "$t\rho \sim u\rho$ for any $\rho \models \phi \wedge \mathit{Inv}(t) \wedge \mathit{Inv}(u)$". $\mathsf{tt} \rhd t = u$ will be abbreviated as $t = u$.

The axioms are the standard monoid laws for bisimulation in process algebras. More interesting are the inference rules. For each construct in the language there is a corresponding introduction rule. CHOICE expresses the fact that timed bisimulation is preserved by $+$. The rule GUARD permits a case analysis on conditional. The rule INV deals with invariants. It also does a case analysis and appears very similar to GUARD. However, there is a crucial difference: When the guard ψ is false $\psi \rightarrow t$ behaves like $\mathbf{0}$, the process which is inactive but can allow time to pass; On the other hand, when the invariant ψ is false $\{\psi\}t$ behaves like $\{\mathsf{ff}\}\mathbf{0}$, the process usually referred to as *time-stop*, which is not only inactive but also "still", can not even let time elapse. ACTION is the introduction rule for action prefixing (with clock resetting). The THINNING rule allows to introduce/remove redundant clocks. REC is the usual rule for folding/unfolding recursions, while UFI says if X is guarded in u then $\mathbf{fix}Xu$ is the unique solution of the equation $X = u$. UNG can be used to transform unguarded terms into guarded ones. Finally the two rules PARTITION and ABSURD do not handle any specific constructs in the language.

They are so-called "structural rules" used to "glue" pieces of derivation together.

Taking $\phi_1 = \phi_2$ PARTITION specialises to a useful rule

$$\text{CONSEQUENCE}\quad \frac{\phi_1 \rhd t = u}{\phi \rhd t = u} \quad \phi \models \phi_1$$

Let us write $\vdash \phi \rhd t = u$ to mean $\phi \rhd t = u$ can be derived from this proof system.

Some useful properties of the proof system are summarised in the following proposition:

Proposition 4.1 *1.* $\vdash \phi{\rightarrow}(\psi{\rightarrow}t) = \phi \wedge \psi{\rightarrow}t$

2. $\vdash t = t + \phi{\rightarrow}t$

3. If $\phi \models \psi$ *then* $\vdash \phi \rhd t = \psi{\rightarrow}t$

4. $\vdash \phi \wedge \psi \rhd t = u$ *implies* $\vdash \phi \rhd \psi{\rightarrow}t = \psi{\rightarrow}u$

5. $\vdash \phi{\rightarrow}(t + u) = \phi{\rightarrow}t + \phi{\rightarrow}u$

6. $\vdash \phi{\rightarrow}t + \psi{\rightarrow}t = \phi \vee \psi{\rightarrow}t$

7. For any t *and* u, $\vdash \{\mathsf{ff}\}t = \{\mathsf{ff}\}u$

Proof: We only give proofs for *1*, *4* and *7*, leaving the others to the readers.

We first prove a lemma:

$$\text{If } \phi \wedge \psi = \mathsf{ff} \text{ then } \vdash \phi \rhd \psi{\rightarrow}t = \mathbf{0} \tag{1}$$

By GUARD we need to show

$$\phi \wedge \psi \rhd t = \mathbf{0} \quad \text{and} \quad \phi \wedge \neg\psi \rhd \mathbf{0} = \mathbf{0}$$

The first follows from the hypothesis and ABSURD, while the second from EQUIV.
1. An application of GUARD gives

$$\phi \rhd \psi{\rightarrow}t = \phi \wedge \psi{\rightarrow}t \tag{2}$$

and

$$\neg\phi \rhd \mathbf{0} = \phi \wedge \psi{\rightarrow}t \tag{3}$$

(3) is an instance of 1. To show (2), we apply GUARD again and obtain two subgoals:

$$\phi \wedge \psi \rhd t = \phi \wedge \psi{\rightarrow}t \quad \text{and} \quad \phi \wedge \neg\psi \rhd \mathbf{0} = \phi \wedge \psi{\rightarrow}t$$

The second subgoal can be settled by (1). Apply symmetricity (the middle rule in EQUIV) followed by GUARD, the first subgoal is reduced to

$$(\phi \wedge \psi) \wedge (\phi \wedge \psi) \rhd t = t \quad \text{and} \quad (\phi \wedge \psi) \wedge \neg(\phi \wedge \psi) \rhd \mathbf{0} = t$$

S1	$X + \mathbf{0} = X$	S2	$X + X = X$
S3	$X + Y = Y + X$	S4	$(X + Y) + Z = X + (Y + Z)$

Figure 5: The Equational Axioms

which can be settled by EQUIV (plus CONSEQUENCE) and ABSURD, respectively.
4. By GUARD, $\vdash \phi \rhd \psi{\rightarrow}t = \psi{\rightarrow}u$ can be reduced to

$$\phi \wedge \psi \rhd t = \phi{\rightarrow}u \quad \text{and} \quad \phi \wedge \neg\psi \rhd \mathbf{0} = \phi{\rightarrow}u$$

The second subgoal is an instance of (1). For the first one we apply GUARD again obtaining

$$(\phi \wedge \psi) \wedge \psi \rhd t = u \quad \text{and} \quad (\phi \wedge \psi) \wedge \neg\psi \rhd t = \mathbf{0}$$

Now the first subgoal follows from the hypothesis and the second from ABSURD.
7. It is sufficient to prove $\vdash \{\mathsf{ff}\}t = \{\mathsf{ff}\}\mathbf{0}$ for any t. By INV this can be reduced to $\vdash \mathsf{ff} \rhd t = \{\mathsf{ff}\}\mathbf{0}$ and $\vdash \neg\mathsf{ff} \rhd \{\mathsf{ff}\}\mathbf{0} = \{\mathsf{ff}\}\mathbf{0}$. The first subgoal is settled by ABSURD while the second by EQUIV. □

The following lemma shows how to "push" a condition through an action prefix:

Lemma 4.2 $\vdash \phi \rhd a(\mathbf{x}).\{\psi\}t = a(\mathbf{x}).\{\psi\}\phi{\downarrow_{\mathbf{x}}}{\Uparrow}{\rightarrow}t$.

Proof: By ACTION this can be reduced to

$$\phi{\downarrow_{\mathbf{x}}}{\Uparrow} \rhd \{\psi\}t = \{\psi\}\phi{\downarrow_{\mathbf{x}}}{\Uparrow}{\rightarrow}t$$

An applications of INV gives two subgoals:

$$\phi{\downarrow_{\mathbf{x}}}{\Uparrow} \wedge \psi \quad \rhd \quad t = \{\psi\}\phi{\downarrow_{\mathbf{x}}}{\Uparrow}{\rightarrow}t \tag{4}$$

$$\phi{\downarrow_{\mathbf{x}}}{\Uparrow} \wedge \neg\psi \quad \rhd \quad \{\mathsf{ff}\}\mathbf{0} = \{\psi\}\phi{\downarrow_{\mathbf{x}}}{\Uparrow}{\rightarrow}t \tag{5}$$

Apply INV again to (4) we get

$$\phi{\downarrow_{\mathbf{x}}}{\Uparrow} \wedge \psi \wedge \psi \rhd t = \phi{\downarrow_{\mathbf{x}}}{\Uparrow}{\rightarrow}t \quad \text{and} \quad \phi{\downarrow_{\mathbf{x}}}{\Uparrow} \wedge \psi \wedge \neg\psi \rhd t = \{\mathsf{ff}\}\mathbf{0}$$

the first follows from Proposition 4.1.*3*, while the second from ABSURD.

(5) can be settled similarly by an application of INV followed by EQUIV and ABSURD. □

The UFI rule, as presented in Figure 4, is unconditional. However, a conditional version can be derived:

Proposition 4.3 *Suppose X is guarded in u. Then from $\vdash \phi \rhd t = u[\phi{\rightarrow}t/X]$ infer $\vdash \phi \rhd t = \mathbf{fix}X\phi{\rightarrow}u$.*

Proof: Assume $\vdash \phi \rhd t = u[\phi{\rightarrow}t/X]$. By Proposition 4.1.4 we have $\vdash \phi{\rightarrow}t = \phi{\rightarrow}u[\phi{\rightarrow}t/X]$, i.e.

$$\vdash \phi{\rightarrow}t = (\phi{\rightarrow}u)[\phi{\rightarrow}t/X]$$

Since X is guarded in u, it is also guarded in $\phi{\rightarrow}u$. By UFI, $\vdash \phi{\rightarrow}t = \mathbf{fix}X\phi{\rightarrow}u$. Hence

$$\begin{array}{rcl}\vdash \phi{\rightarrow}t & \overset{\text{REC}}{=} & (\phi{\rightarrow}u)[\mathbf{fix}X\phi{\rightarrow}u/X] \\ & = & \phi{\rightarrow}u[\mathbf{fix}X\phi{\rightarrow}u/X] \\ & = & \phi{\rightarrow}(\phi{\rightarrow}u)[\mathbf{fix}X\phi{\rightarrow}u/X] \\ & \overset{\text{REC}}{=} & \phi{\rightarrow}\mathbf{fix}X\phi{\rightarrow}u\end{array}$$

Therefore, by Proposition 4.1.4 again, $\vdash \phi \rhd t = \mathbf{fix}X\phi{\rightarrow}u$. □

The rule PARTITION has a more general form:

Proposition 4.4 *Suppose Ψ is a ϕ-partition and $\vdash \psi \rhd t = u$ for each $\psi \in \Psi$, then $\vdash \phi \rhd t = u$.*

Proof: By induction on the size of Ψ. The base case when Ψ contains only one element is trivial. For the induction step, assume the statement of the proposition holds for ϕ-partitions of size k and let $\Psi = \{\, \psi_i \mid 1 \leq i \leq k+1 \,\}$. Set $\Psi' = \{\, \neg\psi_{k+1} \wedge \psi_i \mid 1 \leq i \leq k \,\}$. Since $\vdash \psi_i \rhd t = u$, by CONSEQUENCE $\vdash \neg\psi_{k+1} \wedge \psi_i \rhd t = u$. Therefore by the induction hypothesis,

$$\vdash \bigvee \Psi' \rhd t = u$$

From this and the assumption $\vdash \psi_{k+1} \rhd t = u$, by PARTITION we obtain

$$\vdash \psi_{k+1} \vee \bigvee \Psi' \rhd t = u$$

Since $\psi_{k+1} \vee \bigvee \Psi' = \psi_{k+1} \vee (\neg\psi_{k+1} \wedge \bigvee_{1 \leq i \leq k} \psi_i) = \bigvee_{1 \leq i \leq k+1} \psi_i = \bigvee \Psi = \phi$, this completes the induction. □

In the rest of this section we discuss the soundness of the proof system. First we show that the rule UFI is sound with respect to $\sim$. Following [Mil89] we use the technique of *bisimulation up to*.

Definition 4.5 A symmetric relation R is a *timed bisimulation up to* $\sim$ if $(p, q) \in R$ implies

- whenever $p \xrightarrow{d} p'$ then $q \xrightarrow{d} q'$ for some q' and $(p', q') \in R$.
- whenever $p \xrightarrow{a} p'$ then $q \xrightarrow{a} q'$ for some q' and $(p', q') \in \sim R \sim$.

□

Note that the derivatives of delay transitions are required to be in the same relation, while those of action transitions are allowed to be related modular $\sim$.

Lemma 4.6 *If R is a timed bisimulation up to $\sim$ then $R \subseteq \sim$.*

Proof: Let $(p, q) \in R$ and $p \xrightarrow{\mu} p'$. We need to show that there is some q' such that $q \xrightarrow{\mu} q'$ and $(p', q') \in R$. The case when μ is an action is settled in the same way as in the proof of Proposition 6, Section 4.3, [Mil89]. The case when μ is a delay follows directly from Definition 4.5. $\square$

Lemma 4.7 *If X is guarded in v and $v[t/X] \xrightarrow{a} t'$, then t' has the form $v'[t/X]$, and moreover, for any u, $v[u/X] \xrightarrow{a} v'[u/X]$.*

This lemma concerns only action transitions and its proof is the same as that of Lemma 13, Section 4.5, [Mil89].

Proposition 4.8 *Suppose $fv(v) \subseteq \{X\}$ and X is guarded in v. If $t\rho \sim v[t/X]\rho$ and $u\rho \sim v[u/X]\rho$ then $t\rho \sim u\rho$.*

Proof: We show the relation

$$R = \{\, (v[t/X]\rho, v[u/X]\rho) \mid fv(v) \subseteq \{X\}\,\}$$

is a timed bisimulation upto $\sim$. Assume $(v[t/X]\rho, v[u/X]\rho) \in R$ and consider the following two cases:

- $v[t/X]\rho \xrightarrow{d} v[t/X](\rho + d)$. Then also $v[u/X]\rho) \xrightarrow{d} v[u/X](\rho + d)$ and $(v[t/X](\rho + d), v[u/X](\rho + d)) \in R$.
- $v[t/X]\rho \xrightarrow{a} t'\rho'$. In this case we can find a matching transition $v[u/X]\rho \xrightarrow{a} u'\rho'$ such that $(t'\rho', u'\rho') \in \sim R \sim$, as in the proof of Proposition 14, Section 4.5, [Mil89].

$\square$

Soundness of the proof system is stated below:

Theorem 4.9 *If $\vdash \phi \rhd t = u$ and ϕ is $\Uparrow$-closed then $t\rho \sim u\rho$ for any $\rho \models \phi \wedge Inv(t) \wedge Inv(u)$.*

The standard approach to the soundness proof is by induction on the length of derivations, and perform a case analysis on the last rule/axiom used. However, this does not quite work here. The reason is that the definition of timed bisimulation requires two processes to simulate each other after any time delays. To reflect this in the proof system, we apply the $\Uparrow$ operator, after $\downarrow_{\mathbf{x}}$ for clock resetting, in the premise of the ACTION rule. But not all the inference rules preserve the $\Uparrow$-closeness property. An example is GUARD. In order to derive $\phi \rhd \psi \rightarrow t = u$, we need to establish $\phi \wedge \psi \rhd t = u$ and $\phi \wedge \neg\psi \rhd \mathbf{0} = u$. Even if ϕ is $\Uparrow$-closed, $\phi \wedge \psi$ may not be so.

To overcome this difficulty, we introduce the notion of "timed bisimulation up to a time bound", formulated as follows

Definition 4.10 Two processes p and q are timed bisimular up to $d_0 \in \mathbf{R}^{\geq 0}$, written $p \sim^{d_0} q$, if for any d such that $0 \leq d \leq d_0$

- whenever $p \xrightarrow{d} p'$ then $q \xrightarrow{d} q'$ for some q' and $p' \dot{\sim} q'$,
- whenever $q \xrightarrow{d} q'$ then $p \xrightarrow{d} p'$ for some p' and $p' \dot{\sim} q'$.

where $p \dot{\sim} q$ is defined thus

- whenever $p \xrightarrow{a} p'$ then $q \xrightarrow{a} q'$ for some q' and $p' \sim q'$,
- whenever $q \xrightarrow{a} q'$ then $p \xrightarrow{a} p'$ for some p' and $p' \sim q'$.

□

The difference between timed bisimulation up to d and the standard notion of timed bisimulation only concerns initial delay transitions: in timed bisimulation up to d two processes are required to match only those initial delay transitions with duration no greater that d. Note that $\dot{\sim}$ is the same as $\sim^0$, and $\sim^{d_0} \subseteq \dot{\sim}$ in general.

Lemma 4.11 *1. If $p \sim^{d_0} q$ for any $d_0 \in \mathbf{R}^{\geq 0}$ then $p \sim q$.*

2. Let ρ_i and d_i, $0 \leq i \leq n$, be such that $\rho_{i+1} = \rho_i + d_i$, $0 \leq i < n$. If $t\rho_i \sim^{d_i} u\rho_i$ for all i such that $0 \leq i \leq n$, then $t\rho_0 \sim^{d} u\rho_0$ where $d = d_0 + \ldots + d_n$.

Proof: Both follow directly from the definition of $\sim^{d_0}$. □

Now the following proposition, of which Theorem 4.9 is a special case when ϕ is $\Uparrow$-closed, can be proved by standard induction on the length of derivations :

Proposition 4.12 *If $\vdash \phi \rhd t = u$ then $t\rho \sim^{d_0} u\rho$ for any ρ and d_0 such that $\rho + d \models \phi \wedge Inv(t) \wedge Inv(u)$ for all $0 \leq d \leq d_0$.*

Proof: By induction on the length of inference. The base case when the length is 0 is straightforward. For the induction step we do case ananysis on the last rule applied.

- ACTION. Assume ρ, d_0 are such that $\rho + d \models \phi$ for any $0 \leq d \leq d_0$. To show $(a(\mathbf{x}).t)\rho \sim^{d_0} (a(\mathbf{x}).u)\rho$, let $(a(\mathbf{x}).t)\rho \xrightarrow{d} (a(\mathbf{x}).t)(\rho + d)$ with $0 \leq d \leq d_0$. This can be matched by $(a(\mathbf{x}).u)\rho \xrightarrow{d} (a(\mathbf{x}).u)(\rho + d)$. To see $(a(\mathbf{x}).t)(\rho + d) \dot{\sim} (a(\mathbf{x}).u)(\rho + d)$, observe that the only possible action transition from both sides is an a-transition:

$$\begin{array}{l} (a(\mathbf{x}).t)(\rho + d) \xrightarrow{a} t(\rho + d)\{\mathbf{x} := 0\} \\ (a(\mathbf{x}).u)(\rho + d) \xrightarrow{a} u(\rho + d)\{\mathbf{x} := 0\} \end{array}$$

  Write ρ' for $(\rho + d)\{\mathbf{x} := 0\}$. For any d_0' and any $0 \leq d' \leq d_0'$, $\rho' + d' \models \phi{\downarrow_{\mathbf{x}}}\Uparrow$. By induction, $t\rho' \sim^{d_0'} u\rho'$. Therefore $t\rho' \sim u\rho'$ by Lemma 4.11. This establishes $(a(\mathbf{x}).t)(\rho + d) \dot{\sim} (a(\mathbf{x}).u)(\rho + d)$. Hence $(a(\mathbf{x}).t)\rho \sim^{d_0} (a(\mathbf{x}).u)\rho$.

- GUARD. Assume ρ, d_0 are such that $\rho + d \models \phi$ for any $0 \leq d \leq d_0$. The line $[\rho,\ \rho + d_0]$ is divided by regions into finite many segments $[\rho, \rho_1)$, $[\rho_1, \rho_1]$, (ρ_1, ρ_2), $[\rho_2, \rho_2]$, ..., $[\rho_n, \rho_n]$ and $(\rho_n, \rho + d_0]$, where $\rho_{i+1} = \rho_i + d'_i$, such that each segment is entirely contained in some region. By Lemma 4.11 we only need to show $(\psi \rightarrow t)\rho_i \sim^{d'_i} u\rho_i$ for each $1 \leq i \leq n$.

  By Lemma 3.1, each $(\rho_i, \rho_i + d'_i)$ is either entirely contained in $\phi \wedge \psi$ or entirely contained in $\phi \wedge \neg\psi$. By induction, in the former case we have $(\psi \rightarrow t)\rho_i \sim^{d'_i} u\rho_i$. In the later case we have $\mathbf{0}\rho_i \sim^{d'_i} u\rho_i$. Since $\rho_i + d' \not\models \psi$ for any $0 \leq d' \leq d'_i$ in this case, it follows $\mathbf{0}(\rho_i + d') \sim (\psi \rightarrow t)(\rho_i + d')$. Hence $(\psi \rightarrow t)\rho_i \sim^{d'_i} u\rho_i$. Therefore we have $(\psi \rightarrow t)\rho_i \sim^{d'_i} u\rho_i$ in both cases.

- UFI. See Proposition 4.8.

- The other cases are similar (and easier).

$\square$

5 Completeness

This section is devoted to proving the completeness of the proof system which is stated thus: if $t \sim^{\phi} u$ then $\vdash \phi \rhd t = u$. The structure of the proof follows from that of [Mil84]. The intuition behind the proof is as follows: A timed automaton is presented as a set of standard equations in which the left hand-side of each equation is a formal process variable corresponding to a node of the automaton, while the right hand-side encodes the outgoing edges from the node. We first transform, within the proof system, both t and u into such equation sets (Proposition 5.1). We then construct a "product" of the two equation sets, representing the product of the two underlying timed automata. Because t and u are timed bisimilar over ϕ, each should also bisimilar to the product over ϕ. Using this as a guide we show that such bisimilarity is derivable within the proof system, i.e. both t and u provably ϕ-satisfy the product equation set (Proposition 5.2). Finally we demonstrate that a standard set of equations has only one solution, therefore the required equality between t and u can be derived. The unique fixpoint induction is only employed in the last step of the proof, namely Proposition 5.3.

Let $\mathbf{X} = \{ X_i \mid i \in I \}$ and $\mathbf{W}$ be two disjoint sets of process variables and $\mathbf{x}$ a set of clock variables. Let also u_i, $i \in I$, be terms with free process variables in $\mathbf{X} \cup \mathbf{W}$ and clock variables in $\mathbf{x}$. Then

$$E: \quad \{ X_i = u_i \mid i \in I \}$$

is an equation set with formal process variables $\mathbf{X}$ and free process variables in $\mathbf{W}$. E is closed if $\mathbf{W} = \emptyset$. E is a *standard* equation set if each u_i has the form

$$\{\psi_i\}(\sum_{k \in K_i} \phi_{ik} \rightarrow a_{ik}(\mathbf{x}_{ik}).X_{f(i,k)} + \sum_{k' \in K'_i} \psi_{ik'} \rightarrow W_{f'(i,k')})$$

A term t provably ϕ-satisfies an equation set E if there exist a vector of terms $\{\, t_i \mid i \in I \,\}$, each t_i being of the form $\{\psi_i'\}t_i'$, and a vector of conditions $\{\, \phi_i \mid i \in I \,\}$ such that $\phi_1 = \phi$, $\vdash \phi \rhd t_1 = t$, $\phi_i \models Inv(u_i) \Leftrightarrow \psi_i'$, and

$$\vdash \phi_i \rhd t_i = u_i[\{\psi_i'\}(\phi_i \rightarrow t_i')/X_i|i \in I]$$

for each $i \in I$. We will simply say "t provably satisfies E" when $\phi_i = \mathtt{tt}$ for all $i \in I$.

Proposition 5.1 *For any guarded term t with free process variables $\mathbf{W}$ there exists a standard equation set E, with free process variables in $\mathbf{W}$, which is provably satisfied by t. In particular, if t is closed then E is also closed.*

Proof: By induction on the structure of t. The only non-trivial case is recursion when $t \equiv \mathbf{fix} X t'$ with X guarded in t'. By induction there is a standard equation set $E' : \ \{\, X_i = u_i \mid i \in I \,\}$ with free process variables in $FV(t) \cup \{X\}$ and $t_i' : s$ such that $\vdash t' = t_1'$ and

$$\vdash t_i' = u_i[t_i'/X_i|i \in I]$$

We may assume that X is different from any X_i. Let $v_i = u_i[u_1/X]$ for each i. Note that since X is under an action prefixing in t', it does not occur free in u_1. Hence $v_1 = u_1$. Consider the equation set

$$E : \ \{\, X_i = v_i \mid i \in I \,\}$$

To show t satisfies E, set $t_i = t_i'[t/X]$. Then

$$\begin{array}{rcl} \vdash t & = & \mathbf{fix} X t' \\ & = & \mathbf{fix} X t_1' \\ & \stackrel{REC}{=} & t_1'[\mathbf{fix} X t_1'/X] \\ & = & t_1'[t/X] \\ & = & t_1 \end{array}$$

Now

$$\begin{array}{rcl} \vdash t & = & t_1'[t/X] \\ & = & u_1[t_i'/X_i|i \in I][t/X] \\ & = & u_1[t_i'[t/X]/X_i|i \in I] \\ & = & u_1[t_i/X_i|i \in I] \end{array}$$

and

$$\begin{array}{rcl} \vdash t_i & = & t_i'[t/X] \\ & = & u_i[t_i'/X_i|i \in I][t/X] \\ & = & u_i[t, t_i'[t/X]/X, X_i|i \in I] \\ & = & u_i[t, t_i/X, X_i|i \in I] \\ & = & u_i[u_1[t_i/X_i|i \in I], t_i/X, X_i|i \in I] \\ & = & u_i[u_1/X][t_i/X_i|i \in I] \\ & = & v_i[t_i/X_i|i \in I] \end{array}$$

$\square$

Proposition 5.2 *For closed terms t and u, if $t \sim^{\phi} u$ then there exist a ϕ' such that $\phi \Rightarrow \phi'$ and a standard, closed equation set E which is provably ϕ'-satisfied by both t and u.*

Proof: It easy to see that, using rule UNG, any unguarded term can be transformed into a guarded one, so we may assume both t and u are guarded.

Let the sets of clock variables of t, u be $\mathbf{x}$, $\mathbf{y}$, respectively, with $\mathbf{x} \cap \mathbf{y} = \emptyset$. Let also E_1 and E_2 be the standard equation sets for t and u, respectively:

$$\begin{aligned} E_1 : &\quad \{\, X_i = \{\phi_i\} \sum_{k \in K_i} \phi_{ik} {\rightarrow} a_{ik}(\mathbf{x}_{ik}).X_{f(i,k)} \mid i \in I \,\} \\ E_2 : &\quad \{\, Y_j = \{\psi_j\} \sum_{l \in L_j} \psi_{jl} {\rightarrow} b_{jl}(\mathbf{y}_{jl}).Y_{g(j,l)} \mid j \in J \,\} \end{aligned}$$

So there are $t_i \equiv \{\phi_i'\}t_i'$, $u_j \equiv \{\psi_j'\}u_j'$ with $\vdash t_1 = t$, $\vdash u_1 = u$ such that $\models \phi_i \Leftrightarrow \phi_i'$, $\models \psi_i \Leftrightarrow \psi_i'$, and

$$\vdash t_i = \{\phi_i\} \sum_{k \in K_i} \phi_{ik} {\rightarrow} a_{ik}(\mathbf{x}_{ik}).t_{f(i,k)} \qquad \vdash u_j = \{\psi_j\} \sum_{l \in L_j} \psi_{jl} {\rightarrow} b_{jl}(\mathbf{y}_{jl}).u_{g(j,l)}$$

Without loss of generality, we may assume $a_{ik} = b_{jl} = a$ for all i, k, j, l.

For each pair of i, j, let

$$\Phi_{ij} = \{\, \Delta \in \mathcal{RC}(\mathbf{xy}) \mid t_i \sim^{\Delta\Uparrow} u_j \,\}$$

Set $\phi_{ij} = \bigvee \Phi_{ij}$. By the definition of Φ_{ij}, ϕ_{ij} is the weakest condition over which t_i and u_j are symbolically bisimilar, that is, $\psi \Rightarrow \phi_{ij}$ for any ψ such that $t_i \sim^{\psi} u_j$. In particular, $\phi \Rightarrow \phi_{11}$. Also for each $\Delta \in \Phi_{ij}$, $\Delta \models \mathit{Inv}(t_i) \Leftrightarrow \mathit{Inv}(u_j)$, *i.e.*, $\Delta \models \phi_i' \Leftrightarrow \psi_j'$, hence $\Delta \models \phi_i \Leftrightarrow \psi_j$.

For each $\Delta \in \Phi_{ij}$ let

$$I_{ij}^{\Delta} = \{\, (k,l) \mid t_{f(i,k)} \sim^{\Delta\downarrow_{\mathbf{x}_{ik}\mathbf{y}_{jl}}\Uparrow} u_{g(j,l)} \,\}$$

Define

$$E : \quad Z_{ij} = \{\phi_i\} \sum_{\Delta \in \Phi_{ij}} \Delta {\rightarrow} \sum_{(k,l) \in I_{ij}^{\Delta}} a(\mathbf{x}_{ik}\mathbf{y}_{jl}).Z_{f(i,k)g(j,l)}$$

We claim that E is provably ϕ_{11}-satisfied by t when each Z_{ij} is instantiated with t_i over ϕ_{ij}. We need to show, for each i,

$$\vdash \phi_{ij} \rhd t_i = \{\phi_i\} \sum_{\Delta \in \Phi_{ij}} \Delta {\rightarrow} \Big(\sum_{(k,l) \in I_{ij}^{\Delta}} a(\mathbf{x}_{ik}\mathbf{y}_{jl}).\{\phi'_{f(i,k)}\}\phi_{f(i,k)g(j,l)} {\rightarrow} t'_{f(i,k)} \Big)$$

Since the elements of Φ_{ij} are mutually disjoint, by Propositions 4.4 and 4.1, it is sufficient to show that, for each $\Delta \in \Phi_{ij}$,

$$\vdash \Delta \rhd t_i = \{\phi_i\} \sum_{(k,l) \in I_{ij}^{\Delta}} a(\mathbf{x}_{ik}\mathbf{y}_{jl}).\{\phi'_{f(i,k)}\}\phi_{f(i,k)g(j,l)} {\rightarrow} t'_{f(i,k)}$$

By the definition of I^{Δ}_{ij}, we have $t_{f(i,k)} \sim^{\Delta\downarrow_{\mathbf{x}_{ik}\mathbf{y}_{jl}}\Uparrow} u_{g(j,l)}$. Hence, from the definition of Φ_{ij},

$$\Delta\downarrow_{\mathbf{x}_{ik}\mathbf{y}_{jl}}\Uparrow \Rightarrow \phi_{f(i,k)g(j,l)}$$

Therefore

$$\begin{array}{lcl}
\vdash \Delta & \rhd & \{\phi_i\} \sum\limits_{(k,l)\in I^{\Delta}_{ij}} a(\mathbf{x}_{ik}\mathbf{y}_{jl}).\{\phi'_{f(i,k)}\}\phi_{f(i,k)g(j,l)}\!\rightarrow\! t'_{f(i,k)} \\
& \overset{Lemma\ 4.2}{=} & \{\phi_i\} \sum\limits_{(k,l)\in I^{\Delta}_{ij}} a(\mathbf{x}_{ik}\mathbf{y}_{jl}).\{\phi'_{f(i,k)}\}\Delta\downarrow_{\mathbf{x}_{ik}\mathbf{y}_{jl}}\Uparrow\!\rightarrow\!\phi_{f(i,k)g(j,l)}\!\rightarrow\! t'_{f(i,k)} \\
& \overset{Prop.\ 4.1}{=} & \{\phi_i\} \sum\limits_{(k,l)\in I^{\Delta}_{ij}} a(\mathbf{x}_{ik}\mathbf{y}_{jl}).\{\phi'_{f(i,k)}\}\Delta\downarrow_{\mathbf{x}_{ik}\mathbf{y}_{jl}}\Uparrow\!\rightarrow\! t'_{f(i,k)} \\
& \overset{Lemma\ 4.2}{=} & \{\phi_i\} \sum\limits_{(k,l)\in I^{\Delta}_{ij}} a(\mathbf{x}_{ik}\mathbf{y}_{jl}).\{\phi'_{f(i,k)}\}t'_{f(i,k)} \\
& \overset{\text{THINNING}}{=} & \{\phi_i\} \sum\limits_{(k,l)\in I^{\Delta}_{ij}} a(\mathbf{x}_{ik}).t_{f(i,k)} \\
& \overset{\text{S1-S4}}{=} & \{\phi_i\} \sum\limits_{k\in K_i} a(\mathbf{x}_{ik}).t_{f(i,k)} \\
& = & t_i
\end{array}$$

Symmetrically we can show E is provably ϕ_{11}-satisfied by u when Z_{ij} is instantiated with u_j over ϕ_{ij}. □

Proposition 5.3 *If both t and u provably ϕ-satisfy an equation set E then $\vdash \phi \rhd t = u$.*

Proof: By induction on the size of E. For the base case when E contains only one equation $X_1 = v_1$, we have $\vdash \phi \rhd t = v_1[\phi\!\rightarrow\! t/X_1]$. Since E is standard, X_1 is guarded in v_1. Therefore by Proposition 4.3, $\vdash \phi \rhd t = \mathbf{fix}X_1\phi\!\rightarrow\! v_1$. Similarly $\vdash \phi \rhd u = \mathbf{fix}X_1\phi\!\rightarrow\! v_1$. Hence $\vdash \phi \rhd t = u$.

Assume the result for m and let E contain $m+1$ equations:

$$X_i = v_i \quad 1 \le i \le m+1$$

Since t provably ϕ-satisfies E, there are t_i and ϕ_i, $1 \le i \le m+1$, such that $\vdash t_1 = t$, $\phi_1 = \phi$, and

$$\vdash \phi_i \rhd t_i = v_i[\phi_i\!\rightarrow\! t_i/X_i | 1 \le i \le m+1]$$

for each $1 \le i \le m+1$. In particular

$$\begin{array}{rcl}
\vdash \phi_{m+1} \rhd t_{m+1} & = & v_{m+1}[\phi_i\!\rightarrow\! t_i/X_i | 1 \le i \le m+1] \\
& = & (v_{m+1}[\phi_i\!\rightarrow\! t_i/X_i | 1 \le i \le m])[\phi_{m+1}\!\rightarrow\! t_{m+1}/X_{m+1}]
\end{array}$$

By Proposition 4.3, noting that X_{m+1} is guarded in $v_{m+1}[\phi_i\!\rightarrow\! t_i/X_i | 1 \le i \le m]$,

$$\vdash \phi_{m+1} \rhd t_{m+1} = \mathbf{fix}X_{m+1}\phi_{m+1}\!\rightarrow\! v_{m+1}[\phi_i\!\rightarrow\! t_i/X_i | 1 \le i \le m]$$

Let w_{m+1} be $\mathbf{fix}X_{m+1}\phi_{m+1}{\rightarrow}v_{m+1}$. We have

$$\vdash \phi_{m+1} \rhd t_{m+1} = w_{m+1}[\phi_i{\rightarrow}t_i/X_i|1 \leq i \leq m]$$

By Proposition 4.1,

$$\vdash \phi_{m+1}{\rightarrow}t_{m+1} = \phi_{m+1}{\rightarrow}w_{m+1}[\phi_i{\rightarrow}t_i/X_i|1 \leq i \leq m]$$

Now, writing w_i for $v_i[\phi_{m+1}{\rightarrow}w_{m+1}/X_{m+1}]$, we have

$$\begin{aligned}
\vdash \phi_i \rhd t_i &= v_i[\phi_i{\rightarrow}t_i/X_i|1 \leq i \leq m+1] \\
&= v_i[\phi_i{\rightarrow}t_i/X_i|1 \leq i \leq m][\phi_{m+1}{\rightarrow}t_{m+1}/X_{m+1}] \\
&= v_i[\phi_i{\rightarrow}t_i/X_i|1 \leq i \leq m][\phi_{m+1}{\rightarrow}w_{m+1}[\phi_i{\rightarrow}t_i/X_i|1 \leq i \leq m]/X_{m+1}] \\
&= v_i[\phi_{m+1}{\rightarrow}w_{m+1}/X_{m+1}][\phi_i{\rightarrow}t_i/X_i|1 \leq i \leq m] \\
&= w_i[\phi_i{\rightarrow}t_i/X_i|1 \leq i \leq m]
\end{aligned}$$

This shows t provably ϕ-satisfies the equation set

$$E': \quad X_i = w_i \quad 1 \leq i \leq m$$

Symmetrically we can show u provably ϕ-satisfies E'. By induction we conclude $\vdash \phi \rhd t = u$.

□

Putting together Propositions 5.2 and 5.3 we obtain the main theorem:

Theorem 5.4 *For closed terms t and u, if $t \sim^{\phi} u$ then $\vdash \phi \rhd t = u$.*

Proof: By Proposition 5.2, there is a standard equation set E which are ϕ'-satisfied by both t and u for some ϕ' such that $\phi' \Rightarrow \phi$. By Proposition 5.3, $\vdash \phi' \rhd t = u$. Finally, by CONSEQUENCE, $\vdash \phi \rhd t = u$. □

6 Conclusion And Related Work

We have presented an axiomatisation, in the form of an inference system, of timed bisimulation for timed automata, and proved its completeness. To the best of our knowledge, this is the first complete axiomatisation for the full set of timed automata. There are two key rules in this axiomatisation: ACTION for action prefixing and UFI for recursion. The former caters for clock reseting and progressing. The form of the later rule is syntactically the same as that used for parameterless processes [Mil84], but here it is implicitly parameterised on clock variables, since the terms involved may contain free clock variables.

We have shown that by generalising pure equational reasoning to a set of inference rules dealing with specific language constructs needed for timed automata, the standard monoid laws for bisimulation are sufficient for characterizing bisimulation in the timed world. This result agrees with the previous works on proof systems for value-passing processes [HL96] and for π-calculus [Lin94], providing a further evidence that the four monoid laws capture the essence of bisimulation.

The most interesting development so far in algebraic characterizations for timed automata are presented in [ACM97, BP99]. As the main result, they established that each timed automaton is equivalent to an algebraic expression out of the standard operators in formal languages, such as union, intersection, concatenation and variants of Kleene's star operator, in the sense that the automaton recognize the same timed language as denoted by the expression. However, the issue of axiomatisation was not considered there. In [DAB96] a set of equational axioms was proposed for timed automata, but no completeness result was reported. [HS98] presents an algebraic framework for real-time systems which is similar to timed automata where "invariants" are replaced by "deadlines" (to express "urgency"), together with some equational laws. Apart from these, we are not aware of any other published work on axiomatising timed automata. On the other hand, most timed extensions of process algebras came with equational axiomatisations. Of particular relevance are [Bor96] and [AJ94]. The former developed a symbolic theory for a timed process algebra, while the later used the unique fixpoint induction to achieve a complete axiomatisation for the regular subset of the timed-CCS proposed in [Wan91].

References

[ACM97] E. Asarin, P. Caspi and O. Maler. A Kleene theorem for timed automata. In proceedings of LICS'97, 1997.

[AD94] R. Alur and D.L. Dill. A theory of timed automata. *Theoretical Computer Science*, 126:183–235. 1994.

[AJ94] L. Aceto and A. Jeffrey. A complete axiomatization of timed bisimulation for a class of timed regular behaviours. *Theoretical Computer Science*, 152(2):251–268. 1995.

[Bor96] M. Boreale. Symbolic Bisimulation for Timed Processes. In *AMAST'96*, LNCS 1101 pp.321-335. Springer–Verlag. 1996.

[BP99] P. Bouyer and A. Petit. Decomposition and Composition of Timed Automata. In *ICALP'99*, LNCS 1644, pp. 210-219. Springer–Verlag. 1999.

[HS98] S. Bornot and J. Sifakis. An Algebraic Framework for Urgency. In *Calculational System Design*, NATO Science Series, Computer and Systems Science 173, Marktoberdorf, July 1998.

[Cer92] K. Čeräns. Decidability of Bisimulation Equivalences for Parallel Timer Processes. In *CAV'92*, LNCS 663, pp.302-315. Springer–Verlag. 1992.

[DAB96] P.R. D'Argenio and Ed Brinksma. A Calculus for Timed Automata (Extended Abstract). In *FTRTFTS'96*, LNCS 1135, pp.110-129. Springer–Verlag. 1996.

[HL95] M. Hennessy and H. Lin. Symbolic bisimulations. *Theoretical Computer Science*, 138:353–389, 1995.

[HL96] M. Hennessy and H. Lin. Proof systems for message-passing process algebras. *Formal Aspects of Computing*, 8:408–427, 1996.

[Lin94] H. Lin. Symbolic bisimulations and proof systems for the π-calculus. Report 7/94, Computer Science, University of Sussex, 1994.

[LW00] H. Lin and Y. Wang. A proof system for timed automata. Fossacs'2000, LNCS 1784. March 2000.

[Mil84] R. Milner. A complete inference system for a class of regular behaviours. *J. Computer and System Science*, 28:439–466, 1984.

[Mil89] R. Milner. *Communication and Concurrency.* Prentice-Hall, 1989.

[Wan91] Wang Yi. *A Calculus of Real Time Systems.* Ph.D. thesis, Chalmers University, 1991.

[WPD94] Wang Yi, Paul Pettersson, and Mats Daniels. Automatic Verification of Real-Time Communicating Systems By Constraint-Solving. In *Proc. of the 7*th *International Conference on Formal Description Techniques*, 1994.