Boundary conditions and estimates for the linearized Navier-Stokes equations on staggered grids

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Abstract

In this paper we consider the linearized Navier-Stokes equations in two dimensions under specified boundary conditions. We study both the continuous case and a discretization using a second order finite difference method on a staggered grid and derive estimates for both the analytic solution and the approximation on staggered grids. We present numerical experiments to verify our results.

1 Introduction

We consider the linearized incompressible Navier-Stokes equations in two dimensions. Let $\mathbf{U} = (u, v, p)^T$ be the solution vector to the following system:

$$\begin{aligned} u_t + \bar{u}u_x + \bar{v}u_y + p_x &= \nu(u_{xx} + u_{yy})\,, \\ v_t + \bar{u}v_x + \bar{v}v_y + p_y &= \nu(v_{xx} + v_{yy})\,, \\ u_x + v_y &= 0\,. \end{aligned} \tag{1}$$

Here u, v, p are the velocity components in the x- and y-direction and the pressure, respectively and $\bar{u}$, $\bar{v}$ are assumed to be constants $\in \mathbb{R}$ and $\nu > 0$.

The incompressible Navier-Stokes equations have been analyzed in the book by Kreiss and Lorenz [8, Chapter 10]. That analysis is based on the energy method, and uses an artificial pressure term ϵp_t in the continuity equation. By deriving estimates with bounds that are independent of ϵ, estimates are obtained for the limit $\epsilon = 0$.

In this paper, we develop a detailed direct analysis of the original equations (1) by assuming periodic solutions in the y-direction and by using a Laplace transform in time, and use the same type of boundary conditions that were used for the steady state Stokes equations considered in [7], with the special modification of the velocity prescription. In this way we can use the same type of technique for deriving the same type of estimates also for the discrete case with a staggered grid method.

For the discretization in space, we use a second order finite difference method on staggered grids. With this, we will avoid spurious solutions. For the time discretization we use a second order implicit-explicit BDF2-scheme, which is mainly motivated by the generaliztion to the fully nonlinear Navier-Stokes equations, which are treated in [2] and [6] using a fourth order staggered grid formulation in space. See also [9]. A stability analysis for the fourth order accurate approximation for periodic solutions is done in [5].

In section 2 of this paper we analyze the well-posedness of the system in a bounded domain, with the boundary conditions stated below. In section 3 we consider a staggered grid method and give an estimate for the discrete approximation to (1). Numerical experiments in the last section demonstrate that the estimates hold also for the non-periodic case.

2 The Continuous Problem

We consider (1) in the domain $\Omega = \{(t,x,y)|(t,x,y) \in \mathbb{R}^+ \times [0,1] \times [0,2\pi]\}$ and for the analysis we assume periodicity in the y-direction:

$$\mathbf{U}(t,x,y) = \mathbf{U}(t,x,y+2\pi)\,.$$

We formulate the boundary conditions in the following way:

$$\begin{aligned} u(t,0,y) - \frac{1}{2\pi}\int_0^{2\pi} u(t,0,y)dy = w_0(t,y)\,, &\qquad u(t,1,y) = u_1(t,y)\,, \\ v(t,0,y) = v_0(t,y)\,, &\qquad v(t,1,y) = v_1(t,y)\,, \\ \int_0^{2\pi} p(t,0,y)dy = q_0(t)\,, & \end{aligned} \tag{2}$$

with $\int_0^{2\pi} w_0(t,y)dy = 0$. This is a direct generalization of the boundary conditions formulated for the steady state Stokes equations in [7]. In choosing the boundary conditions in this way, we avoid difficulties in the discretization procedure, as is explained in detail in [7]. Furthermore, we can assume initial conditions $u(0,x,y) = v(0,x,y) = p(0,x,y) = 0$, see [4], and we also assume that the boundary conditions are compatible with zero initial data.

We use the notation $\mathbf{v} = (u,v)^T$, $\mathbf{v}_B = (w_0,u_1,v_0,v_1)^T$, and define the norms

$$\|\mathbf{v}(\cdot,\cdot,\cdot)\| = \left(\int_0^\infty \int_0^{2\pi} \int_0^1 |\mathbf{v}(t,x,y)|^2\,dx\,dy\,dt\right)^{1/2}, \qquad |\mathbf{v}|^2 = u^2+v^2\,,$$

$$\|\mathbf{v}_B(\cdot,\cdot)\| = \left(\int_0^\infty \int_0^{2\pi} |\mathbf{v}_B(t,y)|^2\,dy\,dt\right)^{1/2},$$

$$|\mathbf{v}_B|^2 = |w_0|^2 + |u_1|^2 + |v_0|^2 + |v_1|^2 \,,$$

$$\|p(\cdot,\cdot,\cdot)\| = \left(\int_0^\infty \int_0^{2\pi} \int_0^1 |p(t,x,y)|^2 \, dx \, dy \, dt \right)^{1/2} .$$

Because of the periodicity in the y-direction we can write the solution as a Fourier series representation:

$$\mathbf{U}(t,x,y) = \frac{1}{2\pi} \sum_{\omega=-\infty}^{\infty} \hat{\mathbf{U}}(t,x,\omega) e^{i\omega y} \,, \tag{3}$$

where $\hat{\mathbf{U}} = (\hat{u}, \hat{v}, \hat{p})^T$. Furthermore, we perform a Laplace transformation in the t-direction

$$\tilde{\mathbf{U}}(s,x,\omega) = \int_0^\infty \hat{\mathbf{U}}(t,x,\omega) e^{-st} \, dt \,, \tag{4}$$

where $\tilde{\mathbf{U}} = (\tilde{u}, \tilde{v}, \tilde{p})^T$ and $s = i\xi + \eta$, $\eta > 0$. When inserting (3) and (4) into the differential equations (1), we obtain the Fourier- and Laplace-transformed system

$$\begin{aligned} s\tilde{u} + \bar{u}\tilde{u}_x + \bar{v}i\omega\tilde{u} + \tilde{p}_x &= \nu(\tilde{u}_{xx} - \omega^2 \tilde{u}) \,, \\ s\tilde{v} + \bar{u}\tilde{v}_x + \bar{v}i\omega\tilde{v} + i\omega\tilde{p} &= \nu(\tilde{v}_{xx} - \omega^2 \tilde{v}) \,, \\ \tilde{u}_x + i\omega\tilde{v} &= 0 \,. \end{aligned} \tag{5}$$

In a similar way, we obtain the Fourier- and Laplace-transformation of the boundary conditions (2):

$\underline{\omega = 0}$

$$\begin{aligned} & & \tilde{u}(s,1,0) &= \tilde{u}_1(s,0) \,, \\ \tilde{v}(s,0,0) &= \tilde{v}_0(s,0) \,, & \tilde{v}(s,1,0) &= \tilde{v}_1(s,0) \,, \\ \tilde{p}(s,0,0) &= \tilde{q}_0(s) \,, & & \end{aligned} \tag{6}$$

$\underline{\omega \neq 0}$

$$\begin{aligned} \tilde{u}(s,0,\omega) &= \tilde{w}_0(s,\omega) \,, & \tilde{u}(s,1,\omega) &= \tilde{u}_1(s,\omega) \,, \\ \tilde{v}(s,0,\omega) &= \tilde{v}_0(s,\omega) \,, & \tilde{v}(s,1,\omega) &= \tilde{v}_1(s,\omega) \,. \end{aligned} \tag{7}$$

In the Fourier-Laplace-space, we use the notation $\tilde{\mathbf{v}} = (\tilde{u}, \tilde{v})^T$ for the velocity components and $\tilde{\mathbf{v}}_B = (\tilde{w}_0, \tilde{u}_1, \tilde{v}_0, \tilde{v}_1)^T$ for the boundary data, and define the norms by

$$\begin{aligned} \|\tilde{\mathbf{v}}(s,\cdot,\omega)\| &= \left(\int_0^1 |\tilde{\mathbf{v}}(s,x,\omega)|^2 \, dx \right)^{1/2} , \quad |\tilde{\mathbf{v}}|^2 = \tilde{u}^2 + \tilde{v}^2 \,, \\ \|\tilde{p}(s,\cdot,\omega)\| &= \left(\int_0^1 |\tilde{p}(s,x,\omega)|^2 \, dx \right)^{1/2} , \\ |\tilde{\mathbf{v}}_B|^2 &= |\tilde{w}_0|^2 + |\tilde{u}_1|^2 + |\tilde{v}_0|^2 + |\tilde{v}_1|^2 \,. \end{aligned}$$

We have the following lemma

Lemma 2.1 *The system (5) with the boundary conditions (6) and (7) has a unique solution, and there is an estimate*

$$\begin{aligned} \|\tilde{\mathbf{v}}(s,\cdot,\omega)\|^2 &\leq const\big(|\tilde{q}_0|^2 + |\tilde{\mathbf{v}}_B|^2\big)\,, & (8)\\ \|\tilde{p}(s,\cdot,\omega)\|^2 &\leq const\big(|\tilde{q}_0|^2 + |\tilde{\mathbf{v}}_B|^2 + |\omega\tilde{\mathbf{v}}_B|^2 + |s\tilde{\mathbf{v}}_B|^2\big)\,. & (9)\end{aligned}$$

Proof: We distinguish between the three different cases: $\omega > 0$, $\omega < 0$ and $\omega = 0$. For $\omega > 0$, we obtain the general solution

$$\begin{aligned}\tilde{u}(s,x,\omega) &= a^s_\omega e^{-\omega x} + b^s_\omega e^{\omega(x-1)} + c^s_\omega \frac{(\sigma_3+\sigma_4)(e^{-\omega x} - e^{-\omega\sigma_3 x})}{(\sigma_3-1)(\sigma_4+1)}\\ &\quad + d^s_\omega \frac{(\sigma_3+\sigma_4)(e^{\omega(x-1)} - e^{\omega\sigma_4(x-1)})}{(\sigma_4-1)(\sigma_3+1)}\,, \qquad (10)\\ \tilde{v}(s,x,\omega) &= -a^s_\omega i e^{-\omega x} + b^s_\omega i e^{\omega(x-1)} + c^s_\omega i \frac{(\sigma_3+\sigma_4)(-e^{-\omega x} + \sigma_3 e^{-\omega\sigma_3 x})}{(\sigma_3-1)(\sigma_4+1)}\\ &\quad + d^s_\omega \frac{(\sigma_3+\sigma_4)(e^{\omega(x-1)} - \sigma_4 e^{\omega\sigma_4(x-1)})}{(\sigma_4-1)(\sigma_3+1)}\,,\\ \tilde{p}(s,x,\omega) &= \nu\omega\Big(-a^s_\omega(\sigma_3-1)(\sigma_4+1)e^{-\omega x} + b^s_\omega(\sigma_4-1)(\sigma_3+1)e^{\omega(x-1)}\\ &\quad + (\sigma_3+\sigma_4)\Big(c^s_\omega e^{-\omega x} + d^s_\omega e^{\omega(x-1)}\Big)\Big)\,,\end{aligned}$$

where

$$\begin{aligned}\tau &= \frac{s}{\nu\omega^2}\,,\\ \sigma_3 &= -\frac{1}{2}\frac{\bar{u}}{\nu\omega} + \sqrt{\left(\frac{1}{2}\frac{\bar{u}}{\nu\omega}\right)^2 + (1+\tau+\frac{i}{\nu\omega}\bar{v})}\,,\\ \sigma_4 &= \frac{1}{2}\frac{\bar{u}}{\nu\omega} + \sqrt{\left(\frac{1}{2}\frac{\bar{u}}{\nu\omega}\right)^2 + (1+\tau+\frac{i}{\nu\omega}\bar{v})}\,.\end{aligned}$$

The four undetermined coefficients a^s_ω, b^s_ω, c^s_ω, d^s_ω are determined by the boundary conditions (7), where $\tilde{w}_0$, $\tilde{u}_1$, $\tilde{v}_0$, $\tilde{v}_1$ are the Fourier- and Laplace-transforms of w_0, u_1, v_0, v_1. This gives rise to a linear system for a^s_ω, b^s_ω, c^s_ω and d^s_ω with the following coefficient matrix:

$$\begin{pmatrix} 1 & e^{-\omega} & 0 & \frac{(\sigma_3+\sigma_4)(e^{-\omega}-e^{-\omega\sigma_4})}{(\sigma_4-1)(\sigma_3+1)}\\ e^{-\omega} & 1 & \frac{(\sigma_3+\sigma_4)(e^{-\omega}-e^{-\omega\sigma_3})}{(\sigma_3-1)(\sigma_4+1)} & 0\\ -i & ie^{-\omega} & i\frac{(\sigma_3+\sigma_4)(-1+\sigma_3)}{(\sigma_3-1)(\sigma_4+1)} & \frac{(\sigma_3+\sigma_4)(e^{-\omega}-\sigma_4 e^{-\omega\sigma_4})}{(\sigma_4-1)(\sigma_3+1)}\\ -ie^{-\omega} & i & i\frac{(\sigma_3+\sigma_4)(-e^{-\omega}+\sigma_3 e^{-\omega\sigma_3})}{(\sigma_3-1)(\sigma_4+1)} & i\frac{(\sigma_3+\sigma_4)(1-\sigma_4)}{(\sigma_4-1)(\sigma_3+1)} \end{pmatrix}$$

All the elements of the matrix are bounded, thus if the determinant is bounded, the constants are bounded in terms of the boundary data. The determinant for this system is

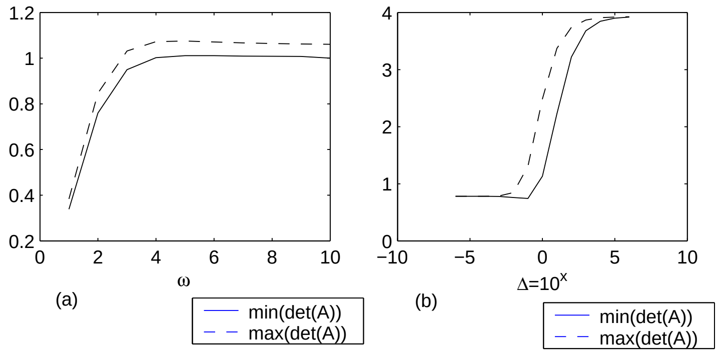

Figure 1: Numerical investigation of the determinant for different values of ω and τ. (a) $\omega = 1, \ldots, 10$; $Re(\tau) = 0.01 \cdot [1 : 10]$; $Im(\tau) = 0.01 \cdot [-10 : 10]$; $\bar{u} = \bar{v} = 1$ (b) $\omega = 2$; $\Delta = 10^{-6} : 10^{6}$; $Re(\tau) = \Delta \cdot [1 : 10]$; $Im(\tau) = \Delta \cdot [-10 : 10]$; $\bar{u} = \bar{v} = 1$

$$\begin{aligned} \mathcal{D} \quad &= \quad \frac{(\sigma_3 + \sigma_4)^2 (1 - e^{-\omega(\sigma_3+\sigma_4)})(1 - e^{-2\omega})}{(\sigma_4 + 1)(\sigma_3 + 1)} \\ &\quad -2\frac{(\sigma_3 + \sigma_4)^2 (\sigma_3 + \sigma_4)(e^{-\omega} - e^{-\omega\sigma_3})(e^{-\omega} - e^{-\omega\sigma_4})}{(1 - \sigma_3)(1 - \sigma_4)(\sigma_4 + 1)(\sigma_3 + 1)} \end{aligned}$$

The analysis of the determinant has been done numerically. For simplicity we assume $\nu = 1$, which can be achieved by transformation, transferring changes in ν into the constants $\bar{u}$ and $\bar{v}$.

We investigate the behavior of the determinant for various values of ω and τ. In Figure 1 (a) the maximum and minimum of the determinant over the considered τ for each ω is depicted. Here $Re(\tau)$ ranges from 0.01 to 0.1 with steps of 0.01. $Im(\tau)$ ranges from -0.1 to 0.1 with the same step size. When changing the range of τ to higher values, this behavior will not change, and the determinant stays in the same region. In Figure 1 (b) we consider the case $\omega = 2$, with different step sizes Δ and $Re(\tau)$ ranges from Δ to $10 \cdot \Delta$ and $Im(\tau)$ ranges from $-10 \cdot \Delta$ to $10 \cdot \Delta$. We see that the range of the determinant stays the same for $\omega > 5$ and for extremely large and small values of τ. Also when changing to the time dependent Stokes equations by setting $\bar{u} = \bar{v} = 0$, we observe that no problems occur. We have also done experiments for negative $\bar{u}$, $\bar{v}$ and no different behaviors occur.

Thus, we find that the determinant of this linear system of equations is uniformly bounded from above and below away from zero. Therefore, we conclude that the estimate

$$\begin{aligned} &|a^s_\omega|^2 + |b^s_\omega|^2 + |c^s_\omega|^2 + |d^s_\omega|^2 \leq \\ &\qquad \text{const}(|\tilde{w}_0(s,\omega)|^2 + |\tilde{u}_1(s,\omega)|^2 + |\tilde{v}_0(s,\omega)|^2 + |\tilde{v}_1(s,\omega)|^2)\,, \end{aligned}$$

holds. By symmetry, the corresponding estimate holds for $\omega < 0$.

For $\omega = 0$ the Fourier- and Laplace-transformed system (5) reduces to

$$\begin{aligned} \tilde{p}_x + s\tilde{u} + \bar{u}\tilde{u}_x &= \nu\tilde{u}_{xx}\,, \\ s\tilde{v} + \bar{u}\tilde{v}_x &= \nu\tilde{v}_{xx}\,, \\ \tilde{u}_x &= 0\,, \end{aligned} \tag{11}$$

with the general solution of the form

$$\begin{aligned} \tilde{u}(s,x,0) &= a_0^s\,, \\ \tilde{v}(s,x,0) &= \frac{e^{\sigma_5(x-1)} + e^{-\sigma_6 x}}{2} b_0^s + \frac{e^{\sigma_5(x-1)} - e^{-\sigma_6 x}}{1 - e^{-\sigma_6}e^{-\sigma_5}} c_0^s\,, \\ \tilde{p}(s,x,0) &= -s\,x\,a_0^s + d_0^s\,, \end{aligned} \tag{12}$$

where

$$\begin{aligned} \sigma_5 &= \frac{\bar{u}}{s\nu} + \sqrt{\left(\frac{\bar{u}}{s\nu}\right)^2 + \frac{s}{\nu}}\,, \\ \sigma_6 &= -\frac{\bar{u}}{s\nu} + \sqrt{\left(\frac{\bar{u}}{s\nu}\right)^2 + \frac{s}{\nu}}\,. \end{aligned}$$

Here, the structure of the solution only allows $\tilde{u}$ to be prescribed by one condition. The form of the boundary condition (2) implies that one of the conditions for $\tilde{u}$ is substituted by a condition on $\tilde{p}$, as we have for the Fourier- and Laplace-transformed boundary conditions (6).

For the undetermined coefficients a_0^s, b_0^s, c_0^s and d_0^s we obtain the following system:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{e^{-\sigma_5}+1}{2} & \frac{e^{-\sigma_5}-1}{1-e^{-\sigma_6}e^{-\sigma_5}} & 0 \\ 0 & \frac{1+e^{-\sigma_6}}{2} & \frac{1-e^{-\sigma_6}}{1-e^{-\sigma_6}e^{-\sigma_5}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_0^s \\ b_0^s \\ c_0^s \\ d_0^s \end{pmatrix} = \begin{pmatrix} \tilde{u}_1(s,0) \\ \tilde{v}_0(s,0) \\ \tilde{v}_1(s,0) \\ \tilde{q}_0(s,0) \end{pmatrix}.$$

The determinant of this coefficient matrix is one. Therefore, we have

$$\begin{aligned} &|a_0^s|^2 + |b_0^s|^2 + |c_0^s|^2 + |d_0^s|^2 \leq \\ &\qquad \text{const}(|\tilde{u}_1(s,\omega)|^2 + |\tilde{v}_0(s,\omega)|^2 + |\tilde{v}_1(s,\omega)|^2 + |\tilde{q}_0(s,\omega)|^2)\,. \end{aligned}$$

Thus, the coefficients a_ω^s, b_ω^s, c_ω^s and d_ω^s are bounded by the boundary data for all ω. Considering the representation (11) and (12) of the solution, we have that $\tilde{u}$ and $\tilde{v}$ are bounded. However, for $\tilde{p}$ we have that $|\tilde{p}(s,x,\omega)| \sim |\omega| + |s|$. Hence, the estimates (8) and (9) hold, and the lemma is proven.

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From this lemma we obtain

Theorem 2.1 *Assume that the boundary data w_0, u_1, v_0, v_1 are 2π-periodic in y and $\int_0^{2\pi} w_0(t,y)dy = 0$, but otherwise arbitrary. Then the system (1) with*

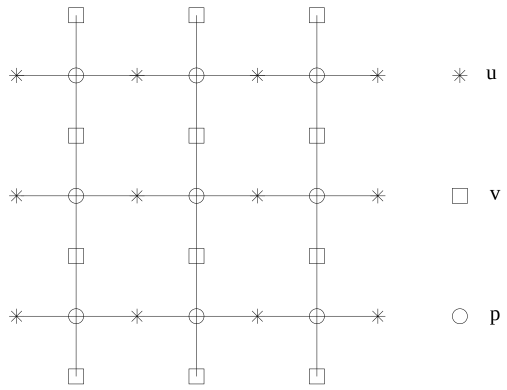

Figure 2: A staggered grid.

boundary conditions (2) and zero initial conditions has a unique solution, and there is an estimate

$$\begin{aligned}
\int_0^\infty e^{-2\eta t}\|\mathbf{v}(t,\cdot,\cdot)\|^2\,dt &\le const\left(\int_0^\infty e^{-2\eta t}\left(\|q_0(t)\|^2+\|\mathbf{v}_B(t,\cdot)\|^2\right)\,dt\right),\\
\int_0^\infty e^{-2\eta t}\|p(t,\cdot,\cdot)\|^2\,dt &\le const\left(\int_0^\infty e^{-2\eta t}\left(\|q_0(t)\|^2+\|\mathbf{v}_B(t,\cdot)\|^2\right.\right.\\
&\quad\left.\left.+\|\frac{\partial \mathbf{v}_B}{\partial y}(t,\cdot)\|^2+\|\frac{\partial \mathbf{v}_B}{\partial t}(t,\cdot)\|^2\right)\,dt\right),\\
&\quad\forall\eta>0\,.
\end{aligned}$$

Proof: The theorem follows by Parseval's identity, and by observing that multiplication with ω and s respectively in transformed space corresponds to taking derivatives in the y- and t-direction, respectively, in physical space.

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3 The Staggered Grid Method

In this section, we will investigate the stability for a second order discretization of the linearized Navier-Stokes equations. The discretization in space is done on a staggered grid for a uniform Cartesian mesh. The velocity components in the x-direction and in the y-direction are taken at the points $(t_n, x_j - \frac{1}{2}\Delta x, y_k)$ and $(t_n, x_j, y_k - \frac{1}{2}\Delta y)$ respectively, and the pressure values are taken at the points (t_n, x_j, y_k). We use the notation

$$\begin{aligned}
u^n_{j-\frac{1}{2},k} &= u(t_n, x_j - \frac{1}{2}\Delta x, y_k)\,,\\
v^n_{j,k-\frac{1}{2}} &= u(t_n, x_j, y_k - \frac{1}{2}\Delta y)\,,\\
p^n_{j,k} &= p(t_n, x_j, y_k)\,,
\end{aligned}$$

where $x_j = j\Delta x$, $y_k = k\Delta y$, $t_n = n\Delta t$, $\Delta x = 1/N$ and $\Delta y = 2\pi/M$, see Figure 2.

We use the notation $\mathbf{U}^n_{j,k} = (u^n_{j-\frac{1}{2},k}, v^n_{j,k-\frac{1}{2}}, p^n_{j,k})^T$, and consider periodic solutions in the y-direction, i.e. $\mathbf{U}^n_{j,k} = \mathbf{U}^n_{j,k+M}$. In time we use a second order semi-implicit BDF2 scheme, see [3], [1]. This is mainly done to easily generalize the scheme to the fully incompressible Navier-Stokes equations, where the nonlinear terms are treated explicitly. We obtain the following system:

$$\begin{aligned}
\left(\tfrac{3}{2}D_{+,t} - \tfrac{1}{2}D_{-,t}\right) u^n_{j-\frac{1}{2},k} &= f^{n+1}_{impl} + 2f^n_{expl} - f^{n-1}_{expl}\,, \\
\left(\tfrac{3}{2}D_{+,t} - \tfrac{1}{2}D_{-,t}\right) v^n_{j,k-\frac{1}{2}} &= g^{n+1}_{impl} + 2g^n_{expl} - g^{n-1}_{expl}\,, \\
D_{+,x}u^n_{j-\frac{1}{2},k} + D_{+,y}v^n_{j,k-\frac{1}{2}} &= 0\,,
\end{aligned} \tag{13}$$

where

$$\begin{aligned}
f^n_{expl} &= (-\bar{u}D_{0,x} - \bar{v}D_{0,y})\, u^n_{j-\frac{1}{2},k}\,, \\
f^n_{impl} &= \nu(D_{+,x}D_{-,x} + D_{+,y}D_{-,y})u^n_{j-\frac{1}{2},k} - D_{-,x}p^n_{j,k}\,, \\
g^n_{expl} &= (-\bar{u}D_{0,x} - \bar{v}D_{0,y})\, v^n_{j,k-\frac{1}{2}}\,, \\
g^n_{impl} &= \nu(D_{+,x}D_{-,x} + D_{+,y}D_{-,y})v^n_{j,k-\frac{1}{2}} - D_{-,y}p^n_{j,k}\,,
\end{aligned}$$

with the difference operators defined as

$$\begin{aligned}
D_{+,x}u^n_{j,k} &= \frac{u^n_{j+1,k} - u^n_{j,k}}{\Delta x}\,, \\
D_{-,x}u^n_{j,k} &= \frac{u^n_{j,k} - u^n_{j-1,k}}{\Delta x}\,, \\
D_{0,x}u^n_{j,k} &= \frac{u^n_{j+1,k} - u^n_{j-1,k}}{2\Delta x}\,,
\end{aligned}$$

and similarly for the operators in the y- and t-direction.

We propose the following boundary conditions

$$\begin{aligned}
\tfrac{1}{2}(u^n_{-\frac{1}{2},k} + u^n_{\frac{1}{2},k}) - \tfrac{1}{2\pi}\textstyle\sum_{k=0}^{M-1} \tfrac{1}{2}(u^n_{-\frac{1}{2},k} + u^n_{\frac{1}{2},k})\Delta y &= w_0(y_k, t_n)\,, \\
\tfrac{1}{2}(u^n_{N-\frac{1}{2},k} + u^n_{N+\frac{1}{2},k}) &= u_1(y_k, t_n)\,, \\
v^n_{0,k-\frac{1}{2}} &= v_0(y_{k-\frac{1}{2}}, t_n)\,, \\
v^n_{N,k-\frac{1}{2}} &= v_1(y_{k-\frac{1}{2}}, t_n)\,, \\
\textstyle\sum_{k=0}^{M-1} p^n_{0,k}\Delta y &= q_0(t_n)\,,
\end{aligned} \tag{14}$$

where $\sum_{k=0}^{M-1} w_0(y_k, t_n)\Delta y = 0$.

By periodicity we expand the solution in a discrete Fourier series representation in the y-direction, and use a z-transform in the t-direction, i.e.

$$\mathbf{U}^n_{j,k} = \frac{1}{\sqrt{2\pi}} \sum_{\omega=-M/2}^{M/2-1} \hat{\mathbf{U}}^n_j(\omega) e^{i\omega y_k}\,, \tag{15}$$

where $\hat{\mathbf{U}}_j^n = (\hat{u}^n_{j-\frac{1}{2}}, \hat{v}^n_j, \hat{p}^n_j)^T$ and

$$\tilde{\mathbf{U}}_j(z,\omega) = \sum_{n=0}^{\infty} \hat{\mathbf{U}}_j^n(\omega) z^{-n}\,, \quad |z| > 1\,, \tag{16}$$

where $\tilde{\mathbf{U}}_j = (\tilde{u}_{j-\frac{1}{2}}, \tilde{v}_j, \tilde{p}_j)^T$. Note that $z = e^{\Delta t s}$, where s comes from the continuous Laplace transformation.

By inserting (15) and (16) into the system (13) we obtain the transformed system

$$\begin{aligned}
&\frac{3}{2}\frac{\Delta x^2}{\Delta t z^2}(z-1)(z-\frac{1}{3})\tilde{u}_{j-\frac{1}{2}} = \nu(\tilde{u}_{j+\frac{1}{2}} - 2\tilde{u}_{j-\frac{1}{2}} + \tilde{u}_{j-\frac{3}{2}} - (2\lambda\sin\frac{\eta}{2})^2\tilde{u}_{j-\frac{1}{2}})\\
&-(\tilde{p}_j - \tilde{p}_{j-1})\Delta x - \frac{(2-\frac{1}{z})}{z}(\frac{\bar{u}\Delta x}{2}(\tilde{u}_{j+\frac{1}{2}} - \tilde{u}_{j-\frac{3}{2}}) + i\bar{v}\sin\eta\tilde{u}_{j-\frac{1}{2}}\lambda\Delta x)\,,\\
&\frac{3}{2}\frac{\Delta x^2}{\Delta t z^2}(z-1)(z-\frac{1}{3})\tilde{v}_j = \nu(\tilde{v}_{j+1} - 2\tilde{v}_j + \tilde{v}_{j-1} - (2\lambda\sin\frac{\eta}{2})^2\tilde{v}_j)\\
&-\tilde{p}_j 2i\lambda\sin\frac{\eta}{2}\Delta x - \frac{(2-\frac{1}{z})}{z}(\frac{\bar{u}\Delta x}{2}(\tilde{v}_{j+1} - \tilde{v}_{j-1}) + i\bar{v}\sin\eta\tilde{v}_j\lambda\Delta x)\,,\\
&\quad 0 = \tilde{u}_{j+\frac{1}{2}} - \tilde{u}_{j-\frac{1}{2}} + \tilde{v}_j 2i\lambda\sin\frac{\eta}{2}\,,
\end{aligned} \tag{17}$$

where $\lambda = \frac{\Delta x}{\Delta y}$, $\eta = \omega\Delta y$.

The corresponding transformed boundary conditions yield for $\omega = 0$

$$\begin{array}{rclrcl}
 & & & \frac{1}{2}(\tilde{u}_{N-\frac{1}{2}} + \tilde{u}_{N+\frac{1}{2}}) &=& \tilde{u}_1(z,0)\,,\\
\tilde{v}_0 &=& \tilde{v}_0(z,0)\,, & \tilde{v}_N &=& \tilde{v}_1(z,0)\,,\\
\tilde{p}_0 &=& \tilde{q}_0(z)\,, & & &
\end{array} \tag{18}$$

and for $\omega \neq 0$

$$\begin{array}{rclrcl}
\frac{1}{2}(\tilde{u}_{-\frac{1}{2}} + \tilde{u}_{\frac{1}{2}}) &=& \tilde{w}_0(z,\omega)\,, & \frac{1}{2}(\tilde{u}_{N-\frac{1}{2}} + \tilde{u}_{N-\frac{1}{2}}) &=& \tilde{u}_1(z,\omega)\,,\\
\tilde{v}_0 &=& \tilde{v}_0(z,\omega)\,, & \tilde{v}_N &=& \tilde{v}_1(z,\omega)\,.
\end{array} \tag{19}$$

We use the notation $\tilde{\mathbf{v}}_j = (\tilde{u}_{j-\frac{1}{2}}, \tilde{v}_j)^T$, $\tilde{\mathbf{v}}_B = (\tilde{w}_0, \tilde{u}_1, \tilde{v}_0, \tilde{v}_1)^T$, and define the norms by

$$\begin{aligned}
\|\tilde{\mathbf{v}}(z,\omega)\|_h^2 &= \left(\sum_{j=0}^{N+1} |\tilde{u}_{j-\frac{1}{2}}|^2 + \sum_{j=0}^{N} |\tilde{v}_j|^2\right)\Delta x\\
\|\tilde{p}(z,\omega)\|_h^2 &= \sum_{j=0}^{N+1} |\tilde{p}_j|^2 \Delta x\\
|\tilde{\mathbf{v}}_B|^2 &= |\tilde{w}_0|^2 + |\tilde{u}_1|^2 + |\tilde{v}_0|^2 + |\tilde{v}_1|^2\,.
\end{aligned}$$

We have in analogy to Lemma 2.1 for the continuous case the following lemma.

Lemma 3.1 *The Fourier-Laplace-transformed system (17) with the boundary conditions (18) and (19) has a unique solution for Δx sufficiently small, and there is an estimate*

$$\|\tilde{\mathbf{v}}(z,\omega)\|_h^2 \leq const\left(|\tilde{q}_0(z)|^2 + |\tilde{\mathbf{v}}_B(z,\omega)|^2\right), \tag{20}$$

$$\|\tilde{p}(z,\omega)\|_h^2 \leq const\Bigg(|\tilde{q}_0(z)|^2 + |\tilde{\mathbf{v}}_B(z,\omega)|^2 + \left|\frac{2\sin\frac{\eta}{2}}{\eta}\omega\tilde{\mathbf{v}}_B(z,\omega)\right|^2 + |\beta(z)\tilde{\mathbf{v}}_B(z,\omega)|^2\Bigg), \tag{21}$$

where $\beta(z) = \frac{3}{2}\frac{1}{\Delta t z^2 \nu}(z-1)(z-\frac{1}{3})$.

Proof: For solving the system (17) we assume that $\frac{1}{2}(2-\frac{1}{z})\frac{1}{z}\bar{u}\Delta x - \nu \neq 0$ by choosing Δx small enough. This is necessary to avoid that the coefficients in front of the outer terms of the approximation become zero.

As for the continuous case, we distinguish between the three cases: $\omega > 0$, $\omega < 0$, $\omega = 0$. Consider first $\omega > 0$, we obtain the general solution to (17) in the form

$$\begin{aligned}
\tilde{u}_j(z,\omega) &= a_\omega^z \kappa_1^j + b_\omega^z \kappa_2^{j-N} \\
&+ c_\omega^z \frac{\kappa_4-\kappa_3}{\rho_3(z,\omega)}(\kappa_1^j - \kappa_3^j) + d_\omega^z \frac{\kappa_4-\kappa_3}{\rho_4(z,\omega)}(\kappa_2^{j-N} - \kappa_4^{j-N}), \\
\tilde{v}_j(z,\omega) &= -a_\omega^z i\kappa_1^j + b_\omega^z i\kappa_2^{j-N} \\
&- c_\omega^z \frac{(\kappa_4-\kappa_3)}{\rho_3(z,\omega)} i(\kappa_1^j - \frac{1}{2\lambda\sin\eta/2}\frac{\kappa_3-1}{\sqrt{\kappa_3}}\kappa_3^j) \\
&+ d_\omega^z \frac{(\kappa_4-\kappa_3)}{\rho_4(z,\omega))} i(\kappa_2^{j-N} - \frac{1}{2\lambda\sin\eta/2}\frac{\kappa_4-1}{\sqrt{\kappa_4}}\kappa_4^{j-N}), \\
\tilde{p}_j(z,\omega) &= \nu(a_\omega^z \rho_3(z,\omega)\kappa_1^j + b_\omega^z \rho_4(z,\omega)\kappa_2^{j-N} \\
&+ c_\omega^z(\kappa_4-\kappa_3)\kappa_1^j + d_\omega^z(\kappa_4-\kappa_3)\kappa_2^{j-N},
\end{aligned} \tag{22}$$

where

$$
\begin{aligned}
\kappa_1 &= \alpha(\omega) - \sqrt{\alpha(\omega)^2 - 1}\,, \\
\kappa_2 &= \alpha(\omega) + \sqrt{\alpha(\omega)^2 - 1}\,, \\
\kappa_3 &= \frac{\alpha(\omega) + \beta(z)\Delta x^2/2 + \delta(z)\gamma(\omega)}{1 - \delta(z)\bar{u}\Delta x} \\
&- \frac{\sqrt{(\alpha(\omega) + \beta(z)\Delta x^2/2 + \delta(z)\gamma(\omega))^2 - 1 + (\delta(z)\bar{u}\Delta x)^2}}{1 - \delta(z)\bar{u}\Delta x}\,, \\
\kappa_4 &= \frac{\alpha(\omega) + \beta(z)\Delta x^2/2 + \delta(z)\gamma(\omega)}{1 - \delta(z)\bar{u}\Delta x} \\
&+ \frac{\sqrt{(\alpha(\omega) + \beta(z)\Delta x^2/2 + \delta(z)\gamma(\omega))^2 - 1 + (\delta(z)\bar{u}\Delta x)^2}}{1 - \delta(z)\bar{u}\Delta x}\,, \\
\rho_3 &= \frac{\beta(z)\Delta x^2/2 + \delta(z)\gamma(\omega) - \delta(z)\bar{u}\Delta x\sqrt{\alpha(\omega)^2 - 1}}{2\lambda\sin(\frac{\eta}{2})}\,, \\
\rho_4 &= \frac{\beta(z)\Delta x^2/2 + \delta(z)\gamma(\omega) + \delta(z)\bar{u}\Delta x\sqrt{\alpha(\omega)^2 - 1}}{2\lambda\sin(\frac{\eta}{2})}\,,
\end{aligned}
$$

and

$$
\begin{aligned}
\alpha(\omega) &= 1 + 2\lambda^2\sin^2(\frac{\eta}{2})\,, \\
\beta(z) &= \frac{3}{2}\frac{1}{\Delta t z^2\nu}(z-1)(z-\frac{1}{3})\,, \\
\gamma(\omega) &= i\bar{v}\lambda\Delta x\sin\eta\,, \\
\delta(z) &= \frac{1}{2\nu}(2 - \frac{1}{z})\frac{1}{z}\,.
\end{aligned}
$$

As for the continuous case we obtain a linear system determining the unknown coefficients a_ω^z, b_ω^z, c_ω^z, d_ω^z from the boundary conditions (19). We again perform a numerical investigation of the determinant for this coefficient matrix for different values of ω, τ, Δx, Δy and Δt. The determinant behaves in a similar way as in the continuous case. Hence, we conclude that the system is well conditioned, i.e.

$$
\begin{aligned}
&|a_\omega^z|^2 + |b_\omega^z|^2 + |c_\omega^z|^2 + |d_\omega^z|^2 \leq \\
&\qquad \text{const}(|\tilde{w}_0(z,\omega)| + |\tilde{u}_1(z,\omega)| + |\tilde{v}_0(z,\omega)| + |\tilde{v}_1(z,\omega)|)\,.
\end{aligned} \tag{23}
$$

As in the continuous case, $\omega < 0$ follows by symmetry.

It now remains to investigate the case $\omega = 0$. For $\omega = 0$ we have the following system:

$$
\begin{aligned}
\beta\Delta x^2\tilde{u}_{j-\frac{1}{2}} &= \nu(\tilde{u}_{j+\frac{1}{2}} - 2\tilde{u}_{j-\frac{1}{2}} + \tilde{u}_{j-\frac{3}{2}}) - (\tilde{p}_j - \tilde{p}_{j-1})\Delta x \\
&- \nu\delta\bar{u}\Delta x(\tilde{u}_{j+\frac{1}{2}} - \tilde{u}_{j-\frac{3}{2}})\,, \\
\beta\Delta x^2\tilde{v}_j &= \nu(\tilde{v}_{j+1} - 2\tilde{v}_j + \tilde{v}_{j-1}) \\
&- \nu\delta\bar{u}\Delta x(\tilde{v}_{j+1} - \tilde{v}_{j-1})\,, \\
0 &= \tilde{u}_{j+\frac{1}{2}} - \tilde{u}_{j-\frac{1}{2}}\,.
\end{aligned}
$$

The general solution for this system yields

$$
\begin{aligned}
\tilde{u}_j &= a_0^z\,, \\
\tilde{v}_j &= b_0^z \frac{1}{2}(\kappa_5^{j-N} + \kappa_6^j) + c_0^z \frac{\kappa_5^{j-N} - \kappa_6^j}{1 - \frac{\kappa_6}{\kappa_5}}\,, \\
\tilde{p}_j &= -\beta a_0^z x_j + d_0^z\,,
\end{aligned}
$$

where

$$
\begin{aligned}
\kappa_5 &= \frac{1 + \beta\Delta x^2/2 + \sqrt{(1+\beta\Delta x^2/2)^2 - (1+\delta\bar{u}\Delta x)^2}}{1 - \delta\bar{u}\Delta x}\,, \\
\kappa_6 &= \frac{1 + \beta\Delta x^2/2 - \sqrt{(1+\beta\Delta x^2/2)^2 - (1+\delta\bar{u}\Delta x)^2}}{1 - \delta\bar{u}\Delta x}\,.
\end{aligned}
$$

For $\omega = 0$, we still have four undetermined constants a_0^z, b_0^z, c_0^z, d_0^z. However, as for the continuous case, one of the conditions on $\tilde{u}$ must be substituted by a condition on $\tilde{p}$. By formulating the boundary conditions on the form (14), this is achieved for the corresponding Fourier- and Laplace-transformed boundary conditions (18).

From this we obtain a linear system for the unknown coefficients, and the determinant for this coefficient matrix is one. Therefore, the coefficients are bounded by the boundary data.

Now we have to show that the solution itself stays bounded. For $\Delta x, \Delta y \to 0$ the solution tends to the continuous solution, with the exception, that s now corresponds to $\frac{1}{\Delta t}\frac{3}{2}\frac{(z-1)(z-\frac{1}{3})}{z^2}$, where one needs to keep in mind that $z = e^{s\Delta t}$. For $\Delta t \to 0$ this term tends to s, and for all $|z| > 1$ it is in the right half-plane in the complex domain. We can therefore use the analysis of the continuous case. For $\Delta x, \Delta y$ fixed, we use the estimate (23) above to arrive at the statement of the lemma. We again see that $\tilde{u}$ and $\tilde{v}$ are bounded, and $\tilde{p}$ is proportional to $\left|2\frac{sin\frac{\eta}{2}}{\eta}\omega\right| + |\beta|$

■

We use the notation $\mathbf{v}_{j,k} = (u_{j-\frac{1}{2}}, , v_{j,k-\frac{1}{2}})^T$ and $\mathbf{v}_B = (w_0, u_1, v_0, v_1)^T$, and define the norms by

$$
\|\mathbf{v}^n\|_h^2 = \sum_{k=0}^{M-1}\left(\sum_{j=0}^{N+1} |u^n_{j-\frac{1}{2},k}|^2 + \sum_{j=0}^{N} |v^n_{j,k-\frac{1}{2}}|^2\right)\Delta x\Delta y\,,
$$

$$
\begin{aligned}
||\mathbf{v}_B(t_n,\cdot)||_h^2 = \sum_{k=0}^{M-1}\Big(& |w_0(t_n,y_k)|^2 + |u_1(t_n,y_k)|^2 + \\
& |v_0(t_n,y_{k-\frac{1}{2}})|^2 + |v_1(t_n,y_{k-\frac{1}{2}})|^2\Big)\Delta y\,, \\
||p^n||_h^2 = \sum_{k=0}^{M-1}\sum_{j=0}^{N} & |p^n_{j,k}|^2\Delta x\Delta y\,.
\end{aligned}
$$

From this we obtain a similar estimate for the discrete solution itself:

Theorem 3.1 *Assume that the boundary data w_0, u_1, v_0, v_1 are 2π-periodic in y and $\sum_{k=0}^{M-1} w_0(t_n, y_k)\Delta y = 0$, but otherwise arbitrary. Then the second order scheme (13) with boundary conditions (14) has a unique solution and there is an estimate*

$$\sum_{n=1}^{\infty} e^{-2\eta t_n} \|\mathbf{v}^n\|_h^2 \Delta t \le const \sum_{n=1}^{\infty} e^{-2\eta t_n} \left(|q_0(t_n)|^2 + \|\mathbf{v}_B(t_n,\cdot)\|_h^2\right) \Delta t \,,$$

$$\begin{aligned}\sum_{n=1}^{\infty} e^{-2\eta t_n} \|p^n\|_h^2 \Delta t \le\ & const \sum_{n=1}^{\infty} e^{-2\eta t_n} \Big(|q_0(t_n)|^2 + \|\mathbf{v}_B(t_n,\cdot)\|_h^2 \\ & + \|D_{+,y}\mathbf{v}_B(t_n,\cdot)\|_h^2 + \|(\frac{3}{2}D_{+,t} - \frac{1}{2}D_{-,t})\mathbf{v}_B(t_n,\cdot)\|_h^2\Big) \Delta t \,, \\ & \forall \eta > 0.\end{aligned}$$

Proof: The theorem follows with a discrete version of Parseval's identity and the fact that multiplication by $\frac{2i\sin(\eta/2)}{\eta}\omega$ in the transformed space corresponds to the introduction of the difference operator $D_{+,y}$ in physical space, and multiplication by $\beta\nu$ corresponds to the operator $\frac{3}{2}D_{+,t} - \frac{1}{2}D_{-,t}$, which is the BDF2 discretization of a derivative in time.

■

4 Numerical Examples

In this section we will present some numerical computations. We will first study the time dependent Stokes equations, which is a special case of the linearized Navier-Stokes equations, i.e. $\bar{u} = \bar{v} = 0$, and we assume periodic solutions in the y-direction. The computational domain is $\Omega = \{0 \le x \le 1, 0 \le y \le 2\pi\}$. In order to obtain a simple analytic solution, we add forcing functions in the equations. The problem we solve is

$$\begin{aligned} u_t + p_x - \nu(u_{xx} + u_{yy}) &= \sin(x)\cos(y)\cos(t)\,, \\ v_t + p_y - \nu(v_{xx} + v_{yy}) &= -\cos(x)\sin(y)\cos(t) \\ & \quad -4\nu\cos(x)\sin(y)\sin(t)\,, \\ u_x + v_y &= 0\,, \end{aligned} \tag{24}$$

with the exact solution

$$\begin{aligned} u^* &= \sin(x)\cos(y)\sin(t)\,, \\ v^* &= -\cos(x)\sin(y)\sin(t)\,, \\ p^* &= 2\nu\cos(x)\cos(y)\sin(t)\,. \end{aligned} \tag{25}$$

We discretize the (24) according to the second order scheme (13), with $\bar{u} = \bar{v} = 0$. Note that the scheme (13) is a two-step backward differentiation method and require solutions for u and v on two previous time levels. Therefore, we

	Number of time steps				
	100	200	300	400	500
$eu_{10} = \|\|u_{10} - u^*\|\|$	2.2e-4	4.5e-4	6.7e-4	8.9e-4	1.1e-3
$ev_{10} = \|\|v_{10} - v^*\|\|$	2.8e-3	5.7e-3	8.5e-3	1.1e-2	1.4e-2
$ep_{10} = \|\|p_{10} - p^*\|\|$	6.1e-2	8.7e-2	1.1e-1	1.4e-1	1.6e-1
$eu_{20} = \|\|u_{20} - u^*\|\|$	5.1e-5	1.1e-4	1.6e-4	2.1e-4	2.6e-4
$ev_{20} = \|\|v_{20} - v^*\|\|$	7.0e-4	1.4e-3	2.1e-3	2.8e-3	3.4e-3
$ep_{20} = \|\|p_{20} - p^*\|\|$	1.4e-2	2.0e-2	2.6e-2	3.2e-2	3.7e-2
$eu_{40} = \|\|u_{40} - u^*\|\|$	1.3e-5	2.6e-5	3.9e-5	5.2e-5	6.4e-5
$ev_{40} = \|\|v_{40} - v^*\|\|$	1.7e-4	3.5e-4	5.2e-4	6.9e-4	8.4e-4
$ep_{40} = \|\|p_{40} - p^*\|\|$	3.5e-3	5.0e-3	6.5e-3	7.8e-3	9.2e-3
eu_{10}/eu_{20}	4.3	4.2	4.2	4.2	4.2
ev_{10}/ev_{20}	4.1	4.1	4.1	4.1	4.1
ep_{10}/ep_{20}	4.3	4.3	4.3	4.3	4.3
eu_{20}/eu_{40}	4.1	4.1	4.1	4.1	4.1
ev_{20}/ev_{40}	4.0	4.0	4.0	4.0	4.0
ep_{20}/ep_{40}	4.1	4.1	4.1	4.1	4.1
eu_{10}/eu_{40}	17.4	17.3	17.2	17.2	17.2
ev_{10}/ev_{40}	16.3	16.3	16.3	16.3	16.3
ep_{10}/ep_{40}	17.4	17.4	17.4	17.4	17.4

Table 1: Numerical experiments with periodic solutions for the time dependent Stokes equations, i.e. $\bar{u} = \bar{v} = 0$, with $\nu = 1$, $\Delta t = 1.0e - 3$ and $N = M = [10, 20, 40]$.

need a different starting procedure to compute u^1 and v^1, for example a single-step method, i.e. the backward Euler scheme. However, since the aim here is to investigate the behavior of the scheme (13), we have used two given initial solutions as starting procedure in all our numerical examples. To solve the algebraic system of equations we use a sparse direct solver.

In Table 1 the errors after 100, 200, 300, 400 and 500 time steps are shown for the grid sizes $N \times M = 10 \times 10$, 20×20, 40×40. For this computation we had $\nu = 1$. A small time step, $\Delta t = 1.0e-3$, was chosen in order to demonstrate the correct order of accuracy in space. The convergence factors are also shown in Table 1, and we can see that we achieved second-order accuracy both for the velocity components and the pressure.

To demonstrate that the estimates derived for periodic solutions also holds for the non-periodic case, we solve the following two problems for the time dependent Stokes equations. The first example is the same as the previous one (24), but on the domain $\Omega = \{0 \leq x, y \leq 6\}$. The second one is flow in a straight channel $\Omega = \{0 \le x \le 1, -1 \le y \le 1\}$:

$$\begin{aligned} u_t + p_x - \nu(u_{xx} + u_{yy}) &= 0\,, \\ v_t + p_y - \nu(v_{xx} + v_{yy}) &= 0\,, \\ u_x + v_y &= 0\,. \end{aligned} \tag{26}$$

For this problem, we derive an analytic solution with the ansatz

$$\begin{aligned} u^* &= U(y)e^{\alpha x - \omega t}\,, \\ v^* &= V(y)e^{\alpha x - \omega t}\,, \\ p^* &= P(y)e^{\alpha x - \omega t}\,, \end{aligned} \tag{27}$$

and the boundary conditions $U(-1) = U(1) = V(-1) = V(1) = 0$, by solving the transcendental equation. We obtain

$$\begin{aligned} U(y) &= c_1 \sin(\alpha y) + \frac{2\kappa}{\alpha} c_2 \sin(\kappa y)\,, \\ V(y) &= c_1 \cos(\alpha y) + 2c_2 \cos(\kappa y)\,, \\ P(y) &= \frac{\omega}{\alpha} c_1 \sin(\alpha y)\,, \end{aligned}$$

where

$$\kappa = \frac{\sqrt{\nu^2 + \alpha^2 \nu \omega}}{\nu}\,,$$

and for $\nu = 1$ and $\alpha = 1$ the remaining constants are given by

$$\begin{aligned} \omega &= 11.6347883720355431\,, \\ c_1 &= 0.9229302839450678\,, \\ c_2 &= 0.2722128679701572\,. \end{aligned}$$

For these problems we use the following boundary conditions:
<u>$x = x_0$:</u>

$$\frac{1}{2}\big(u^n_{-\frac{1}{2},k} + u^n_{\frac{1}{2},k}\big) - \frac{1}{2L}\sum_{k=0}^{M}\big(u^n_{-\frac{1}{2},k} + u^n_{\frac{1}{2},k}\big)\frac{\Delta y}{c_k} = w_0(y_k, t_n)\,, \tag{28}$$

$$v^n_{0,k-\frac{1}{2}} = v_0(y_{k-\frac{1}{2}}, t_n)\,, \tag{29}$$

$$\sum_{k=0}^{M} p^n_{0,k}\frac{\Delta y}{c_k} = q_0(t_n)\,, \tag{30}$$

<u>$x = x_N$:</u>

$$\frac{1}{2}(u^n_{N-\frac{1}{2},k} + u^n_{N+\frac{1}{2},k}) = u_1(y_k, t_n)\,, \tag{31}$$

$$v^n_{N,k-\frac{1}{2}} = v_1(y_{k-\frac{1}{2}}, t_n)\,, \tag{32}$$

$\underline{y = y_0}$:

$$u^n_{j-\frac{1}{2},0} = u^0(x_{j-\frac{1}{2}}, t_n)\,, \tag{33}$$

$$\frac{1}{2}(v^n_{j,-\frac{1}{2}} + v^n_{j,\frac{1}{2}}) = v^0(x_j, t_n)\,, \tag{34}$$

$\underline{y = y_M}$:

$$u^n_{j-\frac{1}{2},M} = u^1(x_{j-\frac{1}{2}}, t_n)\,, \tag{35}$$

$$\frac{1}{2}(v^n_{j,M-\frac{1}{2}} + v^n_{j,M+\frac{1}{2}}) = v^1(x_j, t_n)\,, \tag{36}$$

where $L = y_M - y_0$ and $c_0 = c_M = 2$, $c_k = 1$, $k = 1, \ldots, M-1$. Here, we use the solution to the problems (24) and (26), respectively, as the given data w_0, v_0, q_0, u_1, v_1 at $x = x_0$ and x_N, respectively, and u^0, v^0, u^1, v^1 at $y = y_0$ and $y = y_M$, respectively.

Both test problems are discretized according to the scheme (13) in the interior with $\nu = 1$ and the time step $\Delta t = 1.0e-3$. The errors and the convergence factors are shown for respectively computation in Table 2 and Table 3. We can see that for both problem, we obtain second order accuracy for the pressure and the velocity components.

Next, we turn to the linearized Navier-Stokes equations. The first problem demonstrates the ability of the scheme (13) with the boundary conditions (28)-(36) to produce second-order accurate solutions. The problem we solve is

$$\begin{aligned} u_t + \bar{u}u_x + \bar{v}u_y + p_x - \nu(u_{xx} + u_{yy}) &= f_1(x,y,t)\,, \\ v_t + \bar{u}v_x + \bar{v}v_y + p_y - \nu(v_{xx} + v_{yy}) &= f_2(x,y,t)\,, \\ u_x + v_y &= 0\,, \end{aligned} \tag{37}$$

where

$$\begin{aligned} f_1 &= \bar{u}\cos(x)\cos(y)\sin(t) - \bar{v}\sin(x)\sin(y)\sin(t) \\ &+ \sin(x)\cos(y)\cos(t)\,, \\ f_2 &= \bar{u}\sin(x)\sin(y)\sin(t) - \bar{v}\cos(x)\cos(y)\sin(t) \\ &- \cos(x)\sin(y)\cos(t) - 4\nu\cos(x)\sin(y)\sin(t)\,, \end{aligned}$$

in the domain $\Omega = \{0 \leq x, y \leq 1, 6\}$. The exact solution is (25), i.e. the same as for the first test problem for time dependent Stokes equations.

The errors and the convergence factors of the numerical example is shown in Table 4. For this computation we had $\nu = 1$, $\bar{u} = \bar{v} = 1$ and the time step $\Delta t = 1.0e-3$. We can see that we achieve second-order accuracy for both the velocity components and the pressure.

The last problem is constructed to demonstrate that the scheme (13) with boundary conditions (28)-(36) also produces second-order accuracy for the case

	Number of time steps				
	100	200	300	400	500
$eu_{10} = \|\|u_{10} - u^*\|\|$	3.4e-3	7.1e-3	1.1e-2	1.5e-2	2.0e-2
$ev_{10} = \|\|v_{10} - v^*\|\|$	3.4e-3	7.1e-3	1.1e-2	1.5e-2	2.0e-2
$ep_{10} = \|\|p_{10} - p^*\|\|$	3.2e-2	3.6e-2	4.5e-2	5.7e-2	7.1e-2
$eu_{20} = \|\|u_{20} - u^*\|\|$	6.8e-4	1.5e-3	2.4e-3	3.4e-3	4.5e-3
$ev_{20} = \|\|v_{20} - v^*\|\|$	6.8e-4	1.5e-3	2.4e-3	3.4e-3	4.5e-3
$ep_{20} = \|\|p_{20} - p^*\|\|$	7.6e-3	8.7e-3	1.1e-2	1.3e-2	1.6e-2
$eu_{40} = \|\|u_{40} - u^*\|\|$	1.5e-4	3.4e-4	5.7e-4	8.1e-4	1.1e-3
$ev_{40} = \|\|v_{40} - v^*\|\|$	1.5e-4	3.4e-4	5.7e-4	8.1e-4	1.1e-3
$ep_{40} = \|\|p_{40} - p^*\|\|$	1.9e-3	2.3e-3	2.7e-3	3.3e-3	4.0e-3
eu_{10}/eu_{20}	5.0	4.8	4.6	4.5	4.4
ev_{10}/ev_{20}	5.0	4.8	4.6	4.5	4.4
ep_{10}/ep_{20}	4.3	4.2	4.3	4.4	4.4
eu_{20}/eu_{40}	4.4	4.3	4.2	4.2	4.2
ev_{20}/ev_{40}	4.4	4.3	4.2	4.2	4.2
ep_{20}/ep_{40}	3.9	3.8	3.9	3.9	4.0
eu_{10}/eu_{40}	22.2	20.8	19.7	19.0	18.5
ev_{10}/ev_{40}	22.2	20.8	19.7	19.0	18.5
ep_{10}/ep_{40}	16.7	16.1	16.5	17.1	17.7

Table 2: Numerical experiments with non-periodic solutions for the time dependent Stokes equations (24), i.e. $\bar{u} = \bar{v} = 0$, with $\nu = 1$, $\Delta t = 1.0e-3$ and $N = M = [10, 20, 40]$.

when $\bar{u}$ and $\bar{v}$ are functions that depend on x, y and t. We consider the equations

$$\begin{aligned} u_t + \bar{u}(x,y,t)\,u_x + \bar{v}(x,y,t)\,u_y - \nu(u_{xx} + u_{yy}) + p_x &= f_1(x,y,t)\,, \\ v_t + \bar{u}(x,y,t)\,v_x + \bar{v}(x,y,t)\,v_y - \nu(v_{xx} + v_{yy}) + p_y &= f_2(x,y,t)\,, \\ u_x + v_y &= 0\,, \end{aligned} \tag{38}$$

where

$$\begin{aligned} f_1 &= \sin(x)\cos(x)\sin^2(t) + \sin(x)\cos(y)\cos(t)\,, \\ f_2 &= \sin(y)\cos(y)\sin^2(t) - 4\nu\cos(x)\sin(y)\sin(t) - \cos(x)\sin(y)\cos(t)\,, \end{aligned}$$

and

$$\begin{aligned} \bar{u} &= \sin(x)\cos(y)\sin(t)\,, \\ \bar{v} &= -\cos(x)\sin(y)\sin(t)\,, \end{aligned}$$

in the domain $\Omega = \{0 \leq x, y \leq 6\}$. Also in this case, the analytical solution is (25).

In Table 5 the errors and the convergence factors of the numerical example is shown. For this computation we had $\nu = 1$ and the time step $\Delta t = 1.0e-3$. As for the previous experiments, we can see that we achieve second-order accuracy for both the velocity components and the pressure.

	Number of time steps				
	100	200	300	400	500
$eu_{10} = \|\|u_{10} - u^*\|\|$	4.5e-3	1.4e-3	4.5e-4	1.4e-4	4.4e-5
$ev_{10} = \|\|v_{10} - v^*\|\|$	2.1e-2	6.7e-3	2.1e-3	6.6e-4	2.0e-4
$ep_{10} = \|\|p_{10} - p^*\|\|$	2.2e-1	6.9e-2	2.1e-2	6.7e-3	2.1e-3
$eu_{20} = \|\|u_{20} - u^*\|\|$	1.2e-3	3.9e-4	1.2e-4	3.9e-5	1.2e-5
$ev_{20} = \|\|v_{20} - v^*\|\|$	4.8e-3	1.5e-3	4.7e-4	1.5e-4	4.6e-5
$ep_{20} = \|\|p_{20} - p^*\|\|$	5.6e-2	1.8e-2	5.5e-3	1.7e-3	5.4e-4
$eu_{40} = \|\|u_{40} - u^*\|\|$	3.2e-4	1.0e-4	3.2e-5	9.9e-6	3.1e-6
$ev_{40} = \|\|v_{40} - v^*\|\|$	1.2e-3	3.6e-4	1.1e-4	3.6e-5	1.1e-5
$ep_{40} = \|\|p_{40} - p^*\|\|$	1.5e-2	4.6e-3	1.4e-3	4.5e-4	1.4e-4
eu_{10}/eu_{20}	3.6	3.7	3.7	3.7	3.7
ev_{10}/ev_{20}	4.4	4.4	4.4	4.4	4.4
ep_{10}/ep_{20}	3.9	3.9	3.9	3.9	3.9
eu_{20}/eu_{40}	3.9	3.9	3.9	3.9	3.9
ev_{20}/ev_{40}	4.1	4.1	4.1	4.1	4.1
ep_{20}/ep_{40}	3.8	3.8	3.8	3.8	3.8
eu_{10}/eu_{40}	14.2	14.3	14.3	14.3	14.3
ev_{10}/ev_{40}	18.4	18.4	18.4	18.4	18.4
ep_{10}/ep_{40}	14.8	14.8	14.8	14.8	14.8

Table 3: Numerical experiments with non-periodic solutions for the time dependent Stokes equations (26), i.e. $\bar{u} = \bar{v} = 0$, with $\nu = 1$, $\Delta t = 1.0e - 3$ and $N = M = [10, 20, 40]$.

5 Conclusion

We have investigated the linearized incompressible Navier-Stokes equations in two dimensions. Under prescribed boundary conditions we have shown that the problem is non-singular and have derived estimates for the solution. The same type of estimates have been derived for a second order finite difference staggered grid approximation in space and a second order implicit-explicit backward difference method in time. The estimates have been derived for the case of periodic solutions in one spatial direction on a simple geometry. With a series of numerical experiments on more complicated geometries and without the restriction of periodicity in one direction we could show that the theoretical results hold for more general cases and verify the second order accuracy of the discretization method.

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	Number of time steps				
	100	200	300	400	500
$eu_{10} = \|\|u_{10} - u^*\|\|$	3.5e-3	7.7e-3	1.3e-2	1.8e-2	2.4e-2
$ev_{10} = \|\|v_{10} - v^*\|\|$	3.5e-3	7.7e-3	1.3e-2	1.8e-2	2.4e-2
$ep_{10} = \|\|p_{10} - p^*\|\|$	5.3e-2	8.1e-2	1.1e-1	1.4e-1	1.7e-1
$eu_{20} = \|\|u_{20} - u^*\|\|$	7.1e-4	1.7e-3	2.9e-3	4.2e-3	5.7e-3
$ev_{20} = \|\|v_{20} - v^*\|\|$	7.1e-4	1.7e-3	2.9e-3	4.2e-3	5.7e-3
$ep_{20} = \|\|p_{20} - p^*\|\|$	1.2e-2	1.8e-2	2.5e-2	3.1e-2	3.7e-2
$eu_{40} = \|\|u_{40} - u^*\|\|$	1.6e-4	3.9e-4	6.9e-4	1.0e-3	1.4e-3
$ev_{40} = \|\|v_{40} - v^*\|\|$	1.6e-4	3.9e-4	6.9e-4	1.0e-3	1.4e-3
$ep_{40} = \|\|p_{40} - p^*\|\|$	3.0e-3	4.5e-3	6.0e-3	7.6e-3	9.1e-3
eu_{10}/eu_{20}	4.9	4.6	4.4	4.3	4.2
ev_{10}/ev_{20}	4.9	4.6	4.4	4.3	4.2
ep_{10}/ep_{20}	4.4	4.4	4.5	4.5	4.5
eu_{20}/eu_{40}	4.3	4.2	4.2	4.1	4.1
ev_{20}/ev_{40}	4.3	4.2	4.2	4.1	4.1
ep_{20}/ep_{40}	4.0	4.1	4.1	4.1	4.1
eu_{10}/eu_{40}	21.6	19.6	18.4	17.8	17.4
ev_{10}/ev_{40}	21.6	19.6	18.4	17.8	17.4
ep_{10}/ep_{40}	17.9	18.0	18.2	18.4	18.6

Table 4: Numerical experiments with non-periodic solutions for the time dependent linearized Navier-Stokes equations (37), where $\bar{u} = \bar{v} = 1$, $\nu = 1$, $\Delta t = 1.0e-3$ and $N = M = [10, 20, 40]$.

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	Number of time steps				
	100	200	300	400	500
$eu_{10} = \|\|u_{10} - u^*\|\|$	3.4e-3	7.1e-3	1.1e-2	1.5e-2	2.0e-2
$ev_{10} = \|\|v_{10} - v^*\|\|$	3.4e-3	7.1e-3	1.1e-2	1.5e-2	2.0e-2
$ep_{10} = \|\|p_{10} - p^*\|\|$	3.3e-2	3.6e-2	4.5e-2	5.7e-2	7.1e-2
$eu_{20} = \|\|u_{20} - u^*\|\|$	6.8e-4	1.5e-3	2.4e-3	3.4e-3	4.5e-3
$ev_{20} = \|\|v_{20} - v^*\|\|$	6.8e-4	1.5e-3	2.4e-3	3.4e-3	4.5e-3
$ep_{20} = \|\|p_{20} - p^*\|\|$	7.7e-3	8.8e-3	1.1e-2	1.3e-2	1.6e-2
$eu_{40} = \|\|u_{40} - u^*\|\|$	1.5e-4	3.4e-4	5.7e-4	8.1e-4	1.1e-3
$ev_{40} = \|\|v_{40} - v^*\|\|$	1.5e-4	3.4e-4	5.7e-4	8.1e-4	1.1e-3
$ep_{40} = \|\|p_{40} - p^*\|\|$	2.0e-3	2.3e-3	2.8e-3	3.5e-3	4.2e-3
eu_{10}/eu_{20}	5.0	4.8	4.6	4.5	4.4
ev_{10}/ev_{20}	5.0	4.8	4.6	4.5	4.4
ep_{10}/ep_{20}	4.2	4.1	4.2	4.2	4.3
eu_{20}/eu_{40}	4.4	4.3	4.2	4.2	4.2
ev_{20}/ev_{40}	4.4	4.3	4.2	4.2	4.2
ep_{20}/ep_{40}	3.9	3.8	3.8	3.9	3.9
eu_{10}/eu_{40}	22.2	20.8	19.7	19.0	18.5
ev_{10}/ev_{40}	22.2	20.8	19.7	19.0	18.5
ep_{10}/ep_{40}	16.5	15.6	15.8	16.3	16.8

Table 5: Numerical experiments with non-periodic solutions for the time dependent linearized Navier-Stokes equations (38), where $\bar{u}$ and $\bar{v}$ are functions of x, y and t, and $\nu = 1$, $\Delta t = 1.0e - 3$ and $N = M = [10, 20, 40]$.

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